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A NOTE ON RIESZ ELEMENTS IN C*-ALGEBRAS

DAVID LEGG

Department of Mathematics
Indiana University-Purdue University
Fort Wayne, Indiana

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ABSTRACT. It is known that every Riesz operator R on a Hilbert space can be written R = Q + C, where C is compact and both Q and CQ - QC are quasinilpotent. This result is extended to a general C*-algebra setting.

1. INTRODUCTION.

In [3], Smyth develops a Riesz theory for elements in a Banach algebra with respect to an ideal of algebraic elements. In [1], Chui, Smith and Ward show that every Riesz operator on a Hilbert space is decomposible into R = Q + C, where C is compact and both Q and CQ - QC are quasinilpotent. In this paper we use Smyth's work to show that the analogous result holds in an arbitrary C*-algebra.

2. DEFINITIONS AND NOTATION.

Let A be a C*-algebra, and let F be a two-sided ideal of algebraic elements of A. An element $T \in A$ is a Riesz element if its coset $T + \overline{F}$ in $A/\overline{F}$ has spectral radius 0. A point $\lambda \in \sigma(T)$ is a finite pole of T if it is isolated in $\sigma(T)$ and the corresponding spectral projection lies in F. Let $E\sigma(T) = \{\lambda \in \sigma(T): \lambda$ is not a finite pole of T$\}$. Smyth has shown that T is a Riesz element if and only if $E\sigma(T) \subseteq \{0\}$, [3, Thm. 5.3]. Smyth also showed that if T is a Riesz element, then $T = Q + U$, where Q is quasinilpotent and $U \in \overline{F}$. [3, Thm. 6.9]. This is a generalization of West's result [4, Thm. 7.5]. We now extend the result of Chui, Smith and Ward [1, Thm. 1] by showing that $UQ - QU$ is quasinilpotent, where $T = Q + U$ is the Smyth decomposition.

3. OUTLINE OF SMYTH'S CONSTRUCTION.

Let T be a Riesz element, and label the elements of $\sigma(T)\setminus E\sigma(T)$ by λ_n, $n = 1, 2, \ldots$, in such a way that $|\lambda_n| \geq |\lambda_{n+1}|$, $\lambda_n \to 0$ as $n \to \infty$. Each λ_n is a finite pole, so each spectral projection P_n is in F. Let $S_n = P_1 + \ldots + P_n$, then find a self-adjoint projection Q_n satisfying $S_n Q_n = Q_n$ and $Q_n S_n = S_n$. Let $V_n = Q_n - Q_{n-1}$, and define $U = \sum \lambda_k V_k$. U is clearly in $\overline{F}$ and $Q = T - U$ is shown to be quasinilpotent.

4. THEOREM 1 $UQ - QU$ is quasinilpotent.

PROOF. For any $S \in A$, let $\tilde{S}$ denote the left regular representation of S. Then by Lemma 6.6 in Smyth [3], we have that $Q_n A$ is an invariant subspace of $\tilde{Q}$. Since $Q_n = Q_n Q_n$, we have $Q_n \in Q_n A$. Hence $\tilde{Q}(Q_n) \in Q_n A$, say $\tilde{Q}(Q_n) = Q_n S$ for some $S \in A$. That is, $QQ_n = Q_n S$. Now let $v \in$ range Q_n, say $v = Q_n x$. Then $Qv = QQ_n x = Q_n Sx$ belongs to range Q_n. Hence we see that range Q_n is an invariant subspace of Q. It follows that Q has an operator matrix representation of the form

$$Q = \left(\begin{array}{ccccc|c} A_{11} & A_{12} & A_{13} & \cdots & & \\ 0 & A_{22} & A_{23} & \cdots & & * \\ 0 & 0 & A_{33} & \cdots & & \\ \vdots & \vdots & \vdots & \ddots & & \\ \hline & & 0 & & & * \end{array}\right)$$

where $A_{ij} = V_i Q V_j$. With respect to this blocking, we have

$$U = \left(\begin{array}{ccccc|c} \lambda_1 I_1 & 0 & 0 & \cdots & & \\ 0 & \lambda_2 I_2 & 0 & \cdots & & \\ 0 & 0 & \lambda_3 I_3 & \cdots & & 0 \\ \vdots & \vdots & \vdots & \ddots & & \\ \hline & & 0 & & & 0 \end{array}\right)$$

Hence

$$UQ - QU = \left(\begin{array}{cccccc|c} 0 & (\lambda_1-\lambda_2)A_{12} & \cdots & (\lambda_1-\lambda_n)A_{1n} & \cdots & & \\ 0 & 0 & & & & & * \\ \vdots & \vdots & \ddots & (\lambda_{n-1}-\lambda_n)A_{n-1,n} & \cdots & & \\ \vdots & \vdots & & 0 & & & \\ \hline & & 0 & & & & 0 \end{array}\right)$$

Now let P be the orthogonal projection onto $\bigcup_n$ range Q_n, and let $A_n = (P - Q_n)(UQ - QU)(P - Q_n)$. It is easy to see that $||A_n|| \le |\lambda_n| ||Q - \text{diag. } Q|| \to 0$ as $n \to \infty$. Hence $UQ - QU - A_n$ converges in the uniform norm to $UQ - QU$ as $n \to \infty$. But $UQ - QU - A_n$ has the form

N	*	*
0	0	*
0	0	0

where N is nilpotent. It follows that $UQ - QU - A_n$ has no non-zero eigenvalues. Thm. 3.1, p. 14 of [2] can now be easily modified to show that $UQ - QU$ has no non-zero eigenvalues. Since $UQ - QU$ belongs to $\overline{F}$, this means $\sigma(UQ - QU) \subseteq \{0\}$, i.e., $UQ - QU$ is quasinilpotent.

REFERENCES

1. Chui, C. K., Smith, P. W., and Ward, J. D., A note on Riesz operators, Proc. Amer. Math. Soc., 60, (1976), 92-94.

2. Gohberg, I. C., and Krein, M. G., Introduction to the theory of linear nonselfadjoint operators, "Nauka," Moscow, 1965; English transl., Transl. Math Monographs, vol. 18, Amer. Math. Soc., Providence, R. I. 1969.

3. Smyth, M. R. F., Riesz theory in Banach algebras, Math Z., 145, (1975), 145-155.

4. West, T. T., The decomposition of Riesz operators, Proc. London Math. Soc., III, Ser. 16, (1966), 737-752.

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