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ON L_1-CONVERGENCE OF WALSH-FOURIER SERIES

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ABSTRACT. Let G denote the dyadic group, which has as its dual group the Walsh(-Paley) functions. In this paper we formulate a condition for functions in $L_1(G)$ which implies that their Walsh-Fourier series converges in $L_1(G)$-norm. As a corollary we obtain a Dini-Lipschitz-type theorem for $L_1(G)$ convergence and we prove that the assumption on the $L_1(G)$ modulus of continuity in this theorem cannot be weakened. Similar results also hold for functions on the circle group T and their (trigonometric) Fourier series.

Let G be the direct product of countably many groups of order 2. Thus $G = \{x\ ;\ x = (x_i)_0^\infty$ with $x_i \in \{0,1\}$ for each $i \geq 0\}$, and for $x,y \in G$ the sum $x + y$ is obtained by adding the i-th coordinates of

x and y modulo 2 for each $i \geq 0$. The topology of G can be described by means of the (non-archimedean) norm $||\cdot||$ on G, where $||x|| = 2^{-k}$ if $x_0 = \ldots = x_{k-1} = 0$ and $x_k = 1$, and $||0|| = 0$. Also, if we define the subgroups G_k of G by $G_0 = G$ and for $k \geq 1$

$$G_k = \{x \in G \,;\, ||x|| \leq 2^{-k}\} = \{x \in G \,;\, x_0 = \ldots = x_{k-1} = 0\},$$

then the G_k form a basis for the neighborhoods of 0 in G. For $k \geq 0$ we define the cosets $I(n,k)$, $0 \leq n < 2^k$, of G_k as follows. If $0 \leq n < 2^k$, then n can be represented uniquely as

$$n = b_0 2^{k-1} + b_1 2^{k-2} + \ldots + b_{k-1},$$

with $b_i \in \{0,1\}$ for each i. Let $e(n,k) = (b_0, b_1, \ldots, b_{k-1}, 0, 0, \ldots)$ in G and let $I(n,k) = e(n,k) \dotplus G_k$. So, in particular, $I(0,k) = G_k$. Furthermore, in order to simplify the notation, we shall denote $e(1,k)$ by $e(k)$.

Next, let $\hat{G}$ denote the dual group of G. Its elements are the Walsh functions and Paley defined the following enumeration for them. For each $k \geq 0$ and $x = (x_i)_0^\infty \in G$ define $\phi_k(x)$ by $\phi_k(x) = \exp(\pi i x_k)$. If $n \geq 0$ is represented as

$$n = a_0 + a_1 2^1 + \ldots + a_k\, 2^k$$

with $a_i \in \{0,1\}$ for all i, then the n-th Walsh function χ_n is defined by

$$\chi_n(x) = \phi_0^{a_0}(x) \cdot \ldots \cdot \phi_k^{a_k}(x).$$

Let dx denote normalized Haar measure on G. For $f \in L_1(G)$ we define its Walsh-Fourier series by

$$f(x) \sim \sum_{k=0}^{\infty} \hat{f}(k)\, \chi_k(x), \text{ where } \hat{f}(k) = \int_G f(t)\chi_k(t)dt.$$

For the partial sums of this series we have

$$S_n(f;x) = \sum_{k=0}^{n-1} \hat{f}(k)\, \chi_k(x) = \int_G f(x-t)D_n(t)dt = f * D_n(x),$$

where $D_n(t) = \sum_{k=0}^{n-1} \chi_k(t)$ is called the Dirichlet kernel of order n.

The following properties hold.

(D1) If $n \geq 0$ is expressed as $n = 2^k + n'$, with $0 \leq n' < 2^k$, then

$$D_n(x) = D_{2^k}(x) + \phi_k(x)D_{n'}(x).$$

(D2) For each $k \geq 0$ we have

$$D_{2^k}(x) = \begin{bmatrix} 2^k, & \text{if } x \in G_k, \\ 0, & \text{if } x \in G \setminus G_k. \end{bmatrix}$$

(D3) If $f \in L_1(G)$ then $\lim_{k\to\infty} ||S_{2^k}(f)-f||_1 = 0$.

(D4) for each $n \geq 0$ we have $D_n(0) = n$.

(D5) If $k \geq 0$, $1 \leq m < 2^k$ and $0 < n \leq 2^k$, then for each $x \in I(m,k) = e(m,k) + G_k$ we have

$$|D_n(x)| = |D_n(e(m,k))| \leq m^{-1}2^{k+1}.$$

A proof of these properties and additional information on Walsh-Fourier series can be found in [2]. Finally, if f is a function on G and if $y \in G$ the function f_y is defined by $f_y(x) = f(x-y)$.

Theorem 1. Let f be a function in $L_1(G)$ for which $n||f-f_{e(n)}||_1 = o(1)$ as $n \to \infty$. Then $||S_n(f)-f||_1 = o(1)$ as $n \to \infty$.

Proof. Let $n > 0$ be given and assume that $n = 2^k + n'$ with $0 \leq n' < 2^k$. Then

$$||S_n(f)-f||_1 \leq ||S_n(f)-S_{2^k}(f)||_1 + ||S_{2^k}(f)-f||_1.$$

Thus, according to (D1) and (D3) we have

$$||S_n(f)||_1 \leq ||\phi_k D_{n'} * f||_1 + o(1) = A + o(1), \text{ as } n \to \infty.$$

In order to find the appropriate estimate for A we continue as follows.

$$A = \int_G |\sum_{p=0}^{2^k-1} \int_{I(p,k)} \phi_k(t) D_{n'}(t) f(x-t)dt|dx.$$

Clearly, $D_{n'}(t)$ is constant on each set $I(p,k) \subset G$. Also $I(p,k) = I(2p,k+1) \cup I(2p+1,k+1)$ and if $t \in I(2p,k+1)$ then $\phi_k(t) = 1$, whereas if $t \in I(2p+1,k+1)$ then $\phi_k(t) = -1$. Therefore

$$A = \int_G |\sum_{p=0}^{2^k-1} D_{n'}(e(p,k)) \left[\int_{I(2p,k+1)} f(x-t)dt - \int_{I(2p+1,k+1)} f(x-t)dt\right]|dx$$

$$\leq |D_{n'}(e(0,k))| \int_G \int_{G_{k+1}} |f(x-t)-f(x-t-e(1,k+1))|dt\ dx$$

$$+ \sum_{p=1}^{2^k-1} |D_{n'}(e(p,k))| \int_G \int_{G_{k+1}} |f(x-t-e(2p,k+1)) - f(x-t-e(2p+1,k+1))|dt\ dx$$

$$= B + C.$$

According to (D4) we have

$$B \leq n' \int_{G_{k+1}} \int_G |f(x-t)-f(x-t+e(k+1))|dx\ dt$$

$$= n' \int_{G_{k+1}} ||f-f_{e(k+1)}||_1\ dt = o(1) \text{ as } n \to \infty.$$

Finally, if we use (D5) and apply Fubini's Theorem we obtain

$$C \leq \sum_{p=1}^{2^k-1} p^{-1} 2^{k+1} \int_{G_{k+1}} ||f-f_{e(k+1)}||_1\ dt$$

$$= \sum_{p=1}^{2^k-1} p^{-1}\, 2^{k+1} \cdot 2^{-(k+1)} ||f-f_{e(k+1)}||_1$$

$$\leq C_1 \log 2^k ||f-f_{e(k+1)}||_1 = o(1) \text{ as } n \to \infty,$$

according to the assumption of the Theorem. Thus $||S_n(f)-f||_1 = o(1)$ as $n \to \infty$.

Before stating a corollary to Theorem 1 we first introduce some additional terminology.

Definition 1. For $f \in L_1(G)$ and $\delta > 0$ the integral modulus of continuity is defined by

$$\omega_1(\delta;f) = \sup\{||f_y-f||_1;\ ||y|| \leq \delta\}.$$

Since $||e(n)|| = 2^{-(n+1)}$ for each $n \geq 0$ we see immediately that the following holds.

Corollary 1. If $f \in L_1(G)$ and if $\omega_1(\delta;f) = o(|\log \delta|^{-1})$ as $\delta \to 0$, then $||S_n(f)-f||_1 = o(1)$ as $n \to \infty$.

Remark 1. Corollary 1 can be considered as the L_1 analogue of the Dini-Lipschitz test for uniform convergence of Walsh-Fourier series, see [2, Theorem XIII].

We first show that Corollary 1 is weaker than Theorem 1 by giving an example of a function $f \in L_1(G)$ such that (i) $\omega_1(\delta;f) \neq o(|\log \delta|^{-1})$ as $\delta \to 0$ and (ii) $n||f-f_{e(n)}||_1 = o(1)$ as $n \to \infty$.

For each $k \geq 0$ and $x = (x_i)_0^\infty$ in G define the function f_k by $f_k(x) = 1 - x_k$; then $f_k \in L_1(G)$ and $||f_k||_1 = \frac{1}{2}$. Next let

$$f(x) = \sum_{k=0}^{\infty} (k+1)^{-3/2} f_k(x).$$

Clearly, the sum defining $f(x)$ converges for each $x \in G$ and applying the Monotone Convergence Theorem to its partial sums we see that $f \in L_1(G)$. Fix $r \geq 0$. Then for all $k \geq 0$ and $x \in G$ we have

$$f_k(x-e(r)) = \begin{bmatrix} f_k(x), & \text{if } k \neq r, \\ 1-f_k(x), & \text{if } k = r. \end{bmatrix}$$

Thus,

$$f(x-e(r)) = \sum_{k=0}^{\infty} (k+1)^{-3/2} f_k(x) + (r+1)^{-3/2}(1-2f_r(x)).$$

Hence,

$$f(x) - f(x-e(r)) = (r+1)^{-3/2}(2f_r(x)-1),$$

and since $2f_r(x) - 1 = \phi_r(x)$, we obtain

$$||f-f_{e(r)}||_1 = (r+1)^{-3/2}||\phi_r||_1 = o(r^{-1}).$$

Next, let $d(r) = \sum_{k=r}^{\infty} e(k)$. Then for each $k \geq 0$ and $x \in G$ we have

$$f_k(x-d(r)) = \begin{bmatrix} f_k(x), & \text{if } k < r, \\ 1 - f_k(x), & \text{if } k \geq r. \end{bmatrix}$$

Thus

$$f(x) - f(x-d(r)) = \sum_{k=r}^{\infty} (k+1)^{-3/2}(2f_k(x)-1)$$

$$= \sum_{k=r}^{\infty} (k+1)^{-3/2} \phi_k(x).$$

From a well-known inequality for Rademacher functions, see [4, Chapter V, Theorem (8.4)], we obtain

$$||f-f_{d(r)}||_1 \geq A_1(\sum_{k=r}^{\infty} (k+1)^{-3})^{1/2} \geq A_2 r^{-1}$$

for some positive constants A_1 and A_2. Therefore, since $||d(r)|| = 2^{-(r+1)}$, we see that $\omega_1(2^{-r};f) \neq o(r^{-1})$, that is, $\omega_1(\delta;f) \neq o(|\log \delta|^{-1})$ as $\delta \to 0$.

We shall now show that Corollary 1 is the best possible in the following sense.

Theorem 2. There exists a function f in $L_1(G)$ with the following two properties: (i) $\omega_1(\delta;f) = O(|\log \delta|^{-1})$ and (ii) $\{S_n(f)\}$ does not converge in $L_1(G)$.

Proof. In [2, p. 386] it was shown that if

$$n = 2^{n_1} + 2^{n_2} + \dots + 2^{n_\nu} \text{ with } n_1 > n_2 > \dots > n_\nu,$$

then

$$||D_n||_1 = \nu - \sum_{p=1}^{\nu-1} 2^{-n_p}\Big(\sum_{r=p+1}^{\nu} 2^{n_r}\Big).$$

Thus, if $n = 2^{2s} + 2^{2(s-1)} + \dots + 2^2 + 2^0$ for some $s > 0$ then

$$||D_n||_1 = (s+1) - \sum_{p=1}^{s} 2^{-2p}\Big(\sum_{r=0}^{p-1} 2^{2r}\Big)$$

$$> (s+1) - \sum_{p=1}^{s} \frac{1}{2} > \frac{s}{2}.$$

Also, it follows immediately from (D2) that for each $k \geq 0$ we have $||D_{2^k}||_1 = 1$. Next, for each $n \geq 0$, let $\mu_n = \sum_{k=0}^{n-1} a_k 2^k$, with $a_k = 0$ if k is odd and $a_k = 1$ if k is even. Furthermore, for each $n \geq 0$, let

$$P_n(x) = D_{2^{n+1}}(x) \text{ and } Q_n(x) = D_{2^n+\mu_n}(x).$$

Then $||P_n||_1 = 1$ and if n is even then $||Q_n||_1 > n/4$, and the (Walsh) polynomial Q_n is "part" of the polynomial P_n. In order to simplify the notation we shall from here on write k' for 2^k and k" for (k')'. Consider the function

$$f(x) = \sum_{k=1}^{\infty} 2^{-k} \chi_{(k+1)''}(x) P_{k'}(x).$$

Clearly, the series defining f(x) converges for all $x \neq 0$ and $f \in L_1(G)$. For $k \geq 1$ we have

$$Q_{k'} ||S_{(k+1)''+\mu_{(k+1)'}}(f) - S_{(k+1)''}(f)||_1 = ||2^{-k}\chi_{(k+1)''} Q_{k'}||_1 \geq 2^{-k}2^k/4 = \frac{1}{4}$$

Thus, the sequence $\{S_n(f)\}$ does not converge in $L_1(G)$. Next, take any δ with $0 < \delta < 1$ and let ℓ be the natural number for which $2^{-(\ell+1)} < \delta \leq 2^{-\ell}$. Then $||y|| \leq \delta$ implies $y \in G_\ell$. Choose the natural number s so that for all $k \leq s$ the polynomial $\chi_{(k+1)''}(x)P_{k'}(x)$ is of degree $< \ell'$, whereas the polynomial $\chi_{(s+2)''}(x)P_{s'}(x)$ is of degree $\geq \ell'$. The last condition implies that $2(s+2)'' \geq \ell'$, hence, that $(s+2)' \geq \ell - 1$, so that $2^{-s} = O(\ell^{-1})$. Also $y \in G_\ell$ implies that $\chi_n(x+y) = \chi_n(x)$ for all x in G and all n such that $0 \leq n < 2^\ell$. Consequently, we have

$$||f_y - f||_1 = \int_G |f(x-y) - f(x)|dx$$

$$\leq \sum_{k=1}^{s} 2^{-k} \int_G |\chi_{(k+1)''}(x-y)D_{(k+1)''}(x-y) - \chi_{(k+1)''}(x)D_{(k+1)''}(x)|dx$$

$$+ \sum_{k=s+1}^{\infty} 2^{-k} \int_G |\chi_{(k+1)''}(x-y)D_{(k+1)''}(x-y)|dx$$

$$+ \sum_{k=s+1}^{\infty} 2^{-k} \int_G |\chi_{(k+1)''}(x)D_{(k+1)''}(x)|dx$$

$$= 0 + 2 \sum_{k=s+1}^{\infty} 2^{-k} = 0(2^{-s}) = 0(\ell^{-1}).$$

Therefore, if $||y|| \leq \delta$ then $||f_y - f||_1 = 0(|\log 2^{-(\ell+1)}|^{-1}) = 0(|\log \delta|^{-1})$, that is, $\omega_1(\delta;f) = 0(|\log \delta|^{-1})$. This completes the proof of Theorem 2.

Remark 2. As was observed in the abstract, except for some minor modifications the theorems presented thus far also hold for functions defined on the circle group T and their (trigonometric) Fourier series. In this context we have

Theorem 1'. If $f \in L_1(T)$ and if $\log n||f-f_{\pi/n}||_1 = o(1)$ as $n \to \infty$, then $||S_n(f)-f||_1 = o(1)$ as $n \to \infty$.

Theorem 1' can be proved by modifying the proof of a test for uniform convergence of Fourier series due to Salem, see [1, Chapter 4, §5]. Also, in order to see more clearly the similarity between Theorem 1 and Theorem 1' we mention that the condition $n||f-f_{e(n)}||_1 = o(1)$ as $n \to \infty$ in Theorem 1 is equivalent to $\log(||e(n)||)^{-1}||f-f_{e(n)}||_1 = o(1)$ as $n \to \infty$.

Theorem 2'. There exists an $f \in L_1(T)$ such that (i) $\omega_1(\delta;f) = 0(|\log \delta|^{-1})$ as $\delta \to 0$ and (ii) $\{S_n(f)\}$ does not converge in $L_1(T)$.

In order to prove Theorem 2' we use a result of F. Riesz, who showed that for each $n \geq 1$ there exists a trigonometric polynomial P_n of degree $2n$ such that $||P_n||_1 = 1$, and a polynomial Q_n of degree n, which is "part" of P_n and such that $||Q_n||_1 > C \log n$, see [1, Chapter VIII, §22] or [3].

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