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NONLINEAR DIFFERENTIAL EQUATIONS AND ALGEBRAIC SYSTEMS

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ABSTRACT. In this paper we obtain the general solution of scalar, first-order differential equations. The method is variation of parameters with asymptotic series and the theory of partial differential equations.

The result gives us a form like a differential quotient requiring only that a limit be taken. Like the familiar expression for the solution of linear, first order, ordinary equations, it is the same in all cases.

KEY WORDS AND PHRASES. Riccati Equations, Abel Equations, Cauchy-Kowalewski Theorem, Cauchy-Kowalewski System, Universal Cauchy-Kowalewski System.

AMS (MOS) SUBJECT CLASSIFICATION (1970) CODES. 34A05, 34A10, 34A15.

1. INTRODUCTION.

We present a unified treatment for the general scalar, first-order, ordinary differential equation

$$y' = G(x,y), \; G \varepsilon C^I.$$

Particular examples are linear equations, Riccati equations and Abel equations.

2. PRELIMINARIES.

We begin with the differential system

$$\left\{\begin{array}{l} V_1' = f(V_1,V_2) = -V_1V_2 \\ V_2' = h(V_1,V_2) = V_1 - V_2 \\ V \neq 0 \end{array}\right\} \tag{2.1}$$

with general solution $V_1 = V_1(x,c_1,c_2)$, $V_2 = V_2(x,c_1,c_2)$. Here c_1,c_2 are arbitrary constants.

Now let $x = x(t)$. Then we get

$$\left\{\begin{array}{l} \dot{V}_1 = U_1\dot{x} \qquad \cdot = \frac{d}{dt} \\ V_2 = U_2\dot{x} \\ U_1 = f(V_1,V_2),\ U_2 = h(V_1,V_2) \\ V_1 \neq 0 \end{array}\right\} \tag{2.2}$$

We are now ready to present the algebraic system referred to in the title.

3. THE CAUCHY-KOWALEWSKI SYSTEM.

Let $w_1 = w_1(t,\varepsilon)$, $w_2 = w_2(t,\varepsilon)$ be two functions of t and ε (at present unknown).

The functions V_1,V_2 have been given by (2.1). Finally two more unknown functions $K(w_1,w_2,t,\varepsilon)$ and $L(w_1,w_2,t,\varepsilon)$ will be defined by partial differential equations later. They will contain another variable, λ. It will be possible to substitute an arbitrary $G(w_1,t)$ for λ to solve specific equations.

DEFINITION. The system of algebraic equations

$$\left\{\begin{array}{ll} \text{(a)} & w_1 - K(w_1,w_2,t,\varepsilon)V_1 = 0 \\ \text{(b)} & w_2^2 - L(w_1,w_2,t,\varepsilon) - V_2 = 0 \\ \text{(c)} & x = w_1 + tw_2. \\ & w_1 \neq 0 \end{array}\right\} \tag{3.1}$$

is called the Cauchy-Kowalewski system, for a specific $G(w_1,t)$. Using λ we will get a universal system.

Under suitable conditions on the functions K and L, we can solve it for $w_1 = w_1(t,\varepsilon)$ and $w_2 = w_2(t,\varepsilon)$. We proceed by defining these functions as solutions of appropriate partial differential equations. We will derive these functions $L(w_1,w_2,t,\varepsilon,\lambda)$ and $K(w_1,w_2,t,\varepsilon,\lambda)$ and regard them as fixed like universal constants.

4. THE FIRST FUNCTION K IN THE CAUCHY-KOWALEWSKI SYSTEM.

We differentiate 3(a-b) with respect to t to get expressions for $\dot{w}_1, \dot{w}_2$. Denoting the expression for $\dot{w}_1$ by R we get

$$\dot{w}_1 = R \tag{4.1}$$

To simplify notation, let $K = \alpha$ in (4.1) and get

$$\dot{w}_1 = R = \frac{A_1L_2 + A_2L_3 + A_3}{-A_2L_1 + A_4L_2 + A_5} \tag{4.1a}$$

Some of the A_i, $i = 1,\ldots,5$ are given explicitly later. These are not partial derivatives. By contrast,

$$L_1 = \frac{\partial L}{\partial w_1} \quad \text{etc.}$$

Now let $z = L - w_2^2$ and note that from (2.1), 3(a-b) we have $f = \frac{w_1}{\alpha} z$, $h = \frac{w_1}{\alpha} + z$ in the new notation.

The following equation is of fundamental importance. We arbitrarily set

$$A_2 = 2w_1w_2\alpha_1 - w_1th\alpha_1 + w_1\alpha_3 + tf\alpha^2 = \varepsilon \tag{4.2}$$

where $K = \alpha = \alpha(L,w_1,w_2,t,\varepsilon)$ and $\alpha_1 = \frac{\partial\alpha}{\partial L}$, etc., for real $\varepsilon > 0$.

By the Cauchy-Kowalewski theorem [See e.g. (2.1)] let $\alpha_o = \alpha_o(L,w_1,w_2,t,\varepsilon)$ be an analytic solution of (4.2). Further, we will write

$$\overline{A}_i = A_i(\alpha_o), \qquad i = 1,2,3,4,5.$$

Let $\alpha_o = \sum_{n=0}^{\infty} c_n\varepsilon^n$ where $c_n = c_n(L,w_1,w_2,t)$ are analytic. Before imposing conditions on c_o we give the following definitions.

DEFINITION. $\underset{\varepsilon\to 0}{L}[(\frac{w_1}{\alpha} + z)(w_1\alpha_{o4} - w_1w_2\alpha_{o2} + w_2\alpha_o)] = S_1(L,w_1,w_2,t)$.

Two more of the $\overline{A}_i$ will now be given explicitly.

$$\overline{A}_1 = (\frac{w_1}{\alpha_o} + z)(w_1\alpha_{o4} - w_1w_2\alpha_{o2} + w_2\alpha_o)$$

$$\overline{A}_4 = w_1\alpha_{o2} + \alpha_o^2 f - \alpha_o - w_1h\alpha_{o1}$$

DEFINITION. $\underset{\varepsilon\to 0}{L}\ \overline{A}_1 = \Delta$.

DEFINITION. $\underset{\varepsilon\to 0}{L}\ (\overline{A}_1 - G(w_1,t)\overline{A}_4) = S_2(L,w_1,w_2,t)$.

The conditions on c_o can be stated now as follows:

(1) $c_o \not\equiv 0$, (2) $S_1(L,w_1,w_2,t) \not\equiv 0$, (3) $\Delta \not\equiv 0$.

Substituting $\alpha_o = \sum_{n=0}^{\infty} c_n\varepsilon^n$ in (4.2) we get

$$2w_1w_2c_{o1} - w_1t(\frac{w_1}{c_o} + z)c_{o1} + w_1c_{03} + tw_1zc_o = 0 \tag{4.3}$$

of which some solutions are given

$$H[\beta(c_o,z,w_1),w_2 + \frac{1}{t} P(c_o,w_1,\beta(c_o,z,w_1))] = \text{constant} \tag{4.3a}$$

where

(1) H is arbitrary

(2) β satisfies the partial differential equation

$$c_o z\beta_1 + (\frac{w_1}{c_o} + z)\beta_2 = 0$$

$$(w_1\beta_3 + c_o\beta_1 \not\equiv 0)$$

(3) P is defined as follows: first solve $\beta(c_o,w_1,z) = a$ for $z = Q(c_o,w_1,a)$. Then set

$$P = \int \frac{d\ c_o}{c_o Q(c_o,w_1,a)} .$$

THEOREM 1. The function H can be chosen analytic in (4.3a) so that conditions (2.1), (2.2), (3.1) hold for c_o.

PROOF. Let $\gamma = w_2 + \frac{1}{t} P$ and then (4.3a) becomes $H(\beta,\gamma) = \text{constant}$. The partial derivatives of c_o are computed from (4.3a) and from them we see that $H_\gamma \not\equiv 0$ implies that $\frac{\partial c_o}{\partial t} \not\equiv 0$, so condition (2.1) holds. Further, $\Delta = \underset{\varepsilon \to 0}{L} \bar{A}_1 = 0$ implies $(\frac{P}{t} + w_2)H_\gamma = 0$. So $H_\gamma \not\equiv 0$ implies $\Delta \not\equiv 0$. Thus (2.1), (2.2) hold if merely $H_\gamma \not\equiv 0$. Now $S_1 = 0$ implies that $tw_2(w_1\beta_3 + c_o\beta_1)H_\beta + H_\gamma = 0$. Since $w_1\beta_3 + c_o\beta_1 \not\equiv 0$, we can choose H so that $S_1 \not\equiv 0$. This completes the proof.

Summarizing the results of this section, $K = \alpha = \alpha_o$ can be defined as the solution of (4.2) where H is analytic, $c_o \not\equiv 0$, $S_1 \not\equiv 0$, and $\Delta \not\equiv 0$. To solve (3.1) however, we must define L.

5. SOLUTION OF THE CAUCHY-KOWALEWSKI SYSTEM.

To solve the system (3.1), we must now define the function $L(w_1,w_2,t,\varepsilon)$.

Setting $\dot{w} = G$, $\alpha = \alpha_o$ and $\bar{A}_2 = \varepsilon$, (4.2) in (4.1a) suggests defining L by

$$\varepsilon G L_1 + (\bar{A}_1 - G\bar{A}_4)L_2 + \varepsilon L_3 = G\bar{A}_5 - \bar{A}_3.$$

$L_1 = \frac{\partial L}{\partial w_1}$, etc. This does not seem to be feasible. Instead, letting ε tend to zero leads to

$$L_2 = \frac{\partial L}{\partial w_2} = \frac{G\bar{A}_5 - \bar{A}_3}{\bar{A}_1 - G\bar{A}_4} \qquad (5.1)$$

This will be used to define L.

Let λ be a new variable and consider

$$L_2 = \frac{\lambda\bar{A}_5 - \bar{A}_3}{\bar{A}_1 - \lambda\bar{A}_4} \qquad (5.2)$$

Note that the right side of (5.2) is analytic where $w_1 \neq 0$ and $\bar{A}_1 - \lambda\bar{A}_4 \neq 0$. So let $L = \bar{L}(w_1,w_2,t,\varepsilon,\lambda) = P_1(w_2) + P_2(w_1,w_2,t,\varepsilon,\lambda)$ be an analytic solution on (5.2) and assume that none of the expressions Δ, S_1, c_o vanish when $L \equiv P_1(w_2)$.

Now since the value of $\frac{\partial}{\partial w_2}(\bar{L}(w_1,w_2,t,\varepsilon,\lambda)$ for $\lambda = G(w_1,t)$ is the same as $\frac{\partial}{\partial w_2}(\bar{L}(w_1,w_2,t,\varepsilon,G(w_1,t))$ we see that $\bar{\bar{L}}(w_1,w_2,t,\varepsilon) \equiv \bar{L}(w_1,w_2,t,\varepsilon,G(w_1,t))$ is a solution of (5.1) for any G. Moreover $\bar{\bar{L}} \in C^I$ since G is continuous and $\bar{L}$ is analytic. Let $K_G = \alpha_o(\bar{\bar{L}},w_1,w_2,t)$ and $L = \bar{\bar{L}}$.

We now prove the solvability near suitable points of the Cauchy-Kowalewski system. The variable λ gives our functions the universal character referred to previously.

LEMMA I. Let (a,b,c) be such that $S_1(P_1(b)a,b,c) \neq 0$. Then, for small t, the Jacobian of (3.1) is nonzero at (a,b,c,ε).

PROOF. If the Jacobian of (3.1) = 0, then

$$-\overline{A}_2 L_1 + \overline{A}_4 L_2 + \overline{A}_5 = 0 \tag{5.3}$$

The subsidiary equations of (5.3) are:

$$\frac{dw_1}{-\overline{A}_2} = \frac{dw_2}{\overline{A}_4} = \frac{dL}{-\overline{A}_5}, \quad \text{so that} \quad \frac{dL}{dw_2} = \frac{-\overline{A}_5}{\overline{A}_4}$$

But from (5.1), $\dfrac{dL}{dw_2} = \dfrac{G\overline{A}_5 - \overline{A}_3}{\overline{A}_1 - G\overline{A}_4}$.

Thus $\overline{A}_1\overline{A}_5 - \overline{A}_3\overline{A}_4 = 0$.

But $\overline{A}_1\overline{A}_5 - \overline{A}_3\overline{A}_4 = (w_1\alpha_{o4} - w_1w_2\alpha_{o2} + w_2\alpha_o)(\frac{w_1}{\alpha_o} + z)\varepsilon$. So

$\underset{\varepsilon\to 0}{L} (w_1\alpha_{o4} - w_1w_2\alpha_{o2} + w_2\alpha_o)(\frac{w_1}{\alpha_o} + z) = 0$. However

$\underset{\varepsilon\to 0}{L} (w_1\alpha_{o4} - w_1w_2\alpha_{o2} + w_2\alpha_o)(\frac{w_1}{\alpha_o} + z) = S_1(P_1(w_2),w_1,w_2,t) \neq 0$ and the proof is complete.

We next consider continuity in order to apply the implicit function theorem to (3.1). We first observe that $\underset{\varepsilon\to 0}{L} \overline{A}_1 \neq 0$. If $\underset{\varepsilon\to 0}{L} \overline{A}_4 = 0$, then $\underset{\varepsilon\to 0}{L} (\overline{A}_1 - G\overline{A}_4) \neq 0$.

Now consider the case where $\underset{\varepsilon\to 0}{L} \overline{A}_4 \neq 0$, but $\underset{\varepsilon\to 0}{L} (\overline{A}_1 - G\overline{A}_4) = 0$.

LEMMA II. There is at most one function G such that $\underset{\varepsilon\to 0}{L}(\overline{A}_1 - G\overline{A}_4) = 0$.

PROOF. $\overline{\overline{L}}(w_1,w_2,t,\varepsilon) = \overline{L}(w_1,w_2,t,G(w_1,t)) = P_1(w_2) + \varepsilon P_2(w_1,w_2,t,\varepsilon,G(w_1,t))$. So it and its partials with respect to w_1,w_2,t do not contain G as $\varepsilon\to 0$. Since

$$\alpha_o = \sum_{n=0}^{\infty} c_n(L,w_1,w_2,t) = c_o(L,w_1,w_2,t) + c_1(L,w_1,w_2,t) + c_2(L,w_1,w_2,t)\varepsilon^2 + \ldots,$$

the same holds for it.

Thus $\underset{\varepsilon\to 0}{L}\ \bar{A}_1$ and $\underset{\varepsilon\to 0}{L}\ \bar{A}_4$ are <u>independent</u> <u>of</u> G.

So $G = \dfrac{\underset{\varepsilon\to 0}{L}\ \bar{A}_1}{\underset{\varepsilon\to 0}{L}\ \bar{A}_4}$. This completes the proof.

In the sequel, we ignore this possible exception and assume that $\underset{\varepsilon\to 0}{L}(\bar{A}_1 - G\bar{A}_4) \not\equiv 0$ for any G.

LEMMA III. If (a,b,c) is such that $S_2(P_1(b),a,b,c) \neq 0$, there is an $\varepsilon > 0$ such that the left sides of (3.1) are C^I at (a,b,c,ε).

PROOF. Based on analytic properties of $V_1, V_2, \bar{\bar{L}}, K_G$ and the nonvanishing of S_2, we will not give details.

Choosing constant values for w_1, w_2 in (3.1), we can get $c_1(\varepsilon), c_2(\varepsilon)$ so that left sides vanishes and apply the implicit function theorem to (3.1). Then we solve for $w_1(t,\varepsilon)$ and $w_2(t,\varepsilon)$. Here c_1, c_2 come from equation (2.1) of section 2.

6. <u>THE PRINCIPAL DIFFERENTIAL EQUATION</u>.

We now consider the differential equation

$$\frac{dy}{dx} = y' = g(x,y) \tag{6.1}$$

DEFINITION. $W_1(t) = \underset{\varepsilon\to 0}{L}\ w_1(t,\varepsilon)$.

It will be shown that $W_1(t)$ satisfies (6.1). Of course we change y,x to W_1,t respectively.

We begin this process with

THEOREM II. Let $S_1 \neq 0$ at $(\bar{w}_1, \bar{w}_2, \bar{t})$. Then $\frac{d}{dt} w_1(t,\varepsilon) \to G(\bar{w}_1, \bar{t})$ as $\varepsilon \to 0$.

PROOF. $\bar{\bar{L}} = P_1(w_2) + \varepsilon P_2(w_1, w_2, t, G(w_1,t))$ so that $\frac{\partial \bar{\bar{L}}}{\partial w_1} \to 0$ as $\varepsilon \to 0$ and also $\frac{\partial \bar{\bar{L}}}{\partial t} \to 0$ as $\varepsilon \to 0$. Thus $\bar{\bar{L}}_1, \bar{\bar{L}}_3 \to 0$ as $\varepsilon \to 0$.

Now (5.1) $\bar{A}_1L_2 + \bar{A}_3 = G(\bar{A}_4L_2 + \bar{A}_5)$.

Also $\bar{A}_4L_2 + \bar{A}_5 = \dfrac{\bar{A}_1\bar{A}_5 - \bar{A}_3\bar{A}_4}{\bar{A}_1 - G\bar{A}_4} = \dfrac{S_1\varepsilon}{\bar{A}_1 - G\bar{A}_4}$.

Thus $R = \dfrac{\bar{A}_1L_2 + \bar{A}_2L_3 + \bar{A}_3}{-\bar{A}_2L_1 + \bar{A}_4L_2 + \bar{A}_5} = \dfrac{\varepsilon L_3 + \bar{A}_1L_2 + \bar{A}_3}{-\varepsilon L_1 + \bar{A}_4L_2 + \bar{A}_5}$.

So $\dot{w}_1 = \dfrac{\varepsilon L_3 + G(\bar{A}_4L_2 + \bar{A}_5)}{-\varepsilon L_1 + (\bar{A}_4L_2 + \bar{A}_5)} = \dfrac{(\bar{A}_1 - G\bar{A}_4)L_3 + GS_1}{-(\bar{A}_1 - G\bar{A}_4)L_1 + S_1}$.

Therefore $\dot{w}_1 \to \dfrac{GS_1}{S_1}$ as $\varepsilon \to 0$ and $S_1 \neq 0$. This completes the proof.

By the last theorem, $\underset{\varepsilon\to 0}{L} \frac{d}{dt} w_1(t,\varepsilon) = \underset{\varepsilon\to 0}{L} G(w_1(t,\varepsilon),t) = G(\underset{\varepsilon\to 0}{L} w_1(t,\varepsilon),t) =$ $G(W_1(t),t)$.

But also it is true [2: P.461] that

$$\underset{\varepsilon\to 0}{L} \frac{d}{dt} w_1(t,\varepsilon) = \frac{d}{dt}(\underset{\varepsilon\to 0}{L} w_1(t,\varepsilon)) = W_1'(t).$$

So $$W_1'(t) = G(W_1(t),t) \qquad (6.2)$$

7. PARTICULAR AND GENERAL SOLUTIONS OF $y' = G(x,y)$.

7(a) PARTICULAR SOLUTIONS. Let $J(w_1,t) \in C^I$,

$L^*(w_1,w_2,t) = \bar{L}(w_1,w_2,t,\varepsilon,J(w_1,t))$ and $\alpha^*(w_1,w_2,t) = \alpha_o(L^*,w_1,w_2,t)$.

Let Q be the set of points in (w_1,w_2,t)-space where

(1) $w_1 \neq 0$ (2) $c_o \neq 0$ (3) $S_1 \neq 0$ (4) $S_2 \neq 0$.

Let $\bar{Q}$ be the projection of Q on the (w_1,t) plane.

The Universal Cauchy-Kowalewski System

DEFINITION. $\bar{\alpha}(w_1,w_2,t,\varepsilon,\lambda) = \alpha_o(\bar{L},w_1,w_2,t)$.

DEFINITION. $F_1 \equiv w_1 - \bar{\alpha}V_1(w_1 + tw_2,c_1,c_2)$.

DEFINITION. $F_2 \equiv w_2^2 - \overline{L}(w_1,w_2,t,\varepsilon,\lambda) - V_2(w_1 + tw_2,c_1,c_2)$.

DEFINITION. $F_3 \equiv \overline{A}_1 - J(w_1,t)\overline{A}_4$ with λ replaced by $J(w_1,t)$.

DEFINITION. The system $\left\{\begin{matrix} F_1 = 0 \\ F_2 = 0 \\ F_3 \neq 0 \end{matrix}\right\}$ is also called the Universal Cauchy-Kowalewski System.

We refer to it in the following

THEOREM III. Let $P \in \overline{Q}$. There is a region in which the solution through P of $\dot{w}_1 = J(w_1,t)$ is determined as follows:

(1) In F_1,F_2 replace λ by $J(w_1,t)$ and c_1,c_2 by suitable functions of ε.

(2) Equate the results in (2.1) to zero.

(3) Solve the resulting system for $w_1(t,\varepsilon)$ and $w_2(t,\varepsilon)$.

(4) Take the limit of $w_1(t,\varepsilon)$ as $\varepsilon \to 0$.

PROOF. Let $P = (a,t_o)$, $P \in \overline{Q}$. Since $c_o(P_1(b),a,b,t_o) \neq 0$, there is an ε such that $\alpha^*(a,b,t_o\varepsilon) \neq 0$. Let $(\overline{V}_1,\overline{V}_2)$ be a solution of (2.1) such that

$$\left\{\begin{aligned} \overline{V}_1(a + t_ob) &= \frac{a}{\alpha^*(a,b,t_o,\varepsilon)} \\ \overline{V}_2(a + t_ob) &= b^2 - L^*(a,b,t_o,\varepsilon). \end{aligned}\right\}$$

Solve the system:

$$\left\{\begin{aligned} &(1)\quad V_1(a + t_ob,c_1,c_2) - \frac{a}{\alpha^*(a,b,t_o,\varepsilon)} = 0 \\ &(2)\quad V_2(a + t_ob,c_1,c_2) - b^2 + L^*(a,b,t_o,\varepsilon) = 0 \end{aligned}\right\}$$

to get suitable $c_1 = c_1(\varepsilon)$, $c_2 = c_2(\varepsilon)$.

Since $S_1 \neq 0$ our system has nonzero Jacobian. We solve for $w_1(t,\varepsilon)$ and get the result.

7(b) GENERAL SOLUTIONS. Alternatively, eliminating w_2 from the Universal Cauchy-Kowalewski System we get

$$X(w_1, t, \varepsilon, \lambda, c_1, c_2) = 0 \tag{7.1}$$

where c_1, c_2 are constants.

The general solution of a specific equation is obtained as follows:

(1) Replace λ by $G(w_1, t)$ in (7.1).

(2) Take the limit as $\varepsilon \to 0$ of the result.

X is derived from L and K and is like the familiar differential quotient in generality.

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