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# ON THE $L_w^2$-BOUNDEDNESS OF SOLUTIONS FOR PRODUCTS OF QUASI-INTEGRO DIFFERENTIAL EQUATIONS

**SOBHY EL-SAYED IBRAHIM**

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Given a general quasidifferential expressions $\tau_1, \tau_2, \dots, \tau_n$ each of order $n$ with complex coefficients and their formal adjoints are $\tau_1^+, \tau_2^+, \dots, \tau_n^+$ on $[0,b)$, respectively, we show under suitable conditions on the function $F$ that all solutions of the product of the quasi-integrodifferential equation $[\prod_{j=1}^n \tau_j]y = wF(t, y, \int_0^t g(t, s, y, y', \dots, y^{(n^2-1)}(s))ds)$ on $[0,b)$, $0 < b \le \infty$; $t, s \ge 0$, are bounded and $L_w^2$-bounded on $[0,b)$. These results are extensions of those by the author (1994), Wong (1975), Yang (1984), and Zettl (1970, 1975).

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**1. Introduction.** In [8, 11, 15] Wong and Zettl proved that all solutions of a perturbed linear differential equation belong to $L^2(0,\infty)$ assuming the fact that all solutions of the unperturbed equation possess the same property. In [6] the author extends their results for a general quasidifferential expression $\tau$ of arbitrary order $n$ with complex coefficients, and considered the property of boundedness of solutions of a general quasidifferential equation $\tau[y] - \lambda w y = w f(t, y)$, where $\lambda \in \mathbb{C}$, on $[0,b)$, $f(t,s)$ satisfies

$$\left| f(t,y) \right| \le e_1(t) + r_1(t)|y|^{\sigma}, \quad t \in [0,b) \text{ for some } \sigma \in [0,1], \tag{1.1}$$

where $e_1(t)$ and $r_1(t)$ are nonnegative continuous functions on $[0,b)$.

Our objective in this paper is to extend the results in [4, 6, 8, 9, 11, 15] to more general class of quasi-integrodifferential equation in the form

$$\left[\prod_{j=1}^{n} \tau_j\right] y = wF\left(t, y, \int_0^t g\left(t, s, y, y', \dots, y^{(n^2-1)}(s)\right) ds\right) \quad \text{on } [0,b), \tag{1.2}$$

where $0 < b \le \infty$; $t, s \ge 0$. Also, we prove under suitable condition on the function $F$ that, if all solutions of the equations $(\prod_{j=1}^n \tau_j)y = 0$ and $(\prod_{j=1}^n \tau_j^+)z = 0$ belong to $L_w^2(0,b)$, then all solutions of (1.2) also belong to $L_w^2(0,b)$, where $\tau_j^+$ is the formal adjoint of $\tau_j$, $j = 1, 2, \dots, n$.

We deal throughout this paper with a quasidifferential expression $\tau_j$ each of arbitrary order $n$ defined by Shin-Zettl matrices (see [4, 13]) on the interval $I = [0,b)$. The left-hand end point of $I$ is assumed to be regular but the right-hand end point may be regular or singular.

## 2. Notation and preliminaries.

The domain and range of a linear operator $T$ acting in a Hilbert space $H$ will be denoted by $D(T)$ and $R(T)$, respectively and $N(T)$ will denote its null space. The nullity of $T$, written $\text{nul}(T)$, is the dimension of $N(T)$ and the deficiency of $T$, written $\text{def}(T)$, is the codimension of $R(T)$ in $H$; thus if $T$ is densely defined and $R(T)$ is closed, then $\text{def}(T) = \text{nul}(T^*)$. The Fredholm domain of $T$ is (in the notation of [2]) the open subset $\triangle_3(T)$ of $\mathbb{C}$ consisting of those values of $\lambda \in \mathbb{C}$ which are such that $(T-\lambda I)$ is a Fredholm operator, where $I$ is the identity operator in $H$. Thus $\lambda \in \triangle_3(T)$ if and only if $(T-\lambda I)$ has a closed range and a finite nullity and deficiency.

A closed operator $A$ in a Hilbert space $H$ has property $(C)$, if it has closed range and $\lambda = 0$ is not an eigenvalue, that is, there is some positive number $r$ such that $\|Ax\| \geq r\|x\|$ for all $x \in D(A)$.

Note that, property $(C)$ is equivalent to $\lambda = 0$ being a regular type point of $A$. This in turn is equivalent to the existence of $A^{-1}$ as a bounded operator on the range of $A$ (which need not be all of $H$).

Given two operators $A$ and $B$, both acting in a Hilbert space $H$, we wish to consider the product operator $AB$. This is defined as follows

$$D(AB) := \{x \in D(B) \mid Bx \in D(A)\}, \quad (AB)x = A(Bx), \quad \forall x \in D(AB). \tag{2.1}$$

It may happen in general that $D(AB)$ contains only the null element of $H$. However, in the case of many differential operators, the domains of the product will be dense in $H$.

The next result gives conditions under which the deficiency of a product is the sum of the deficiencies of the factors.

**Lemma 2.1** (cf. [4, Theorem A] and [12]). *Let $A$ and $B$ be closed operators with dense domains in a Hilbert space $H$. Suppose that $\lambda = 0$ is a regular type point for both operators and $\text{def}\,A$ and $\text{def}\,B$ are finite. Then $AB$ is a closed operator with dense domain, has $\lambda = 0$ as a regular type point and*

$$\text{def}\,AB = \text{def}\,A + \text{def}\,B. \tag{2.2}$$

Evidently, Lemma 2.1 extends to the product of any finite number of operators $A_1, A_2, \ldots, A_n$.

We now turn to the quasidifferential expressions defined in terms of a Shin-Zettl matrix $F$ on an interval $I$. The set $Z_n(I)$ of Shin-Zettl matrices on $I$ consists of $n \times n$ matrices $P = \{p_{rs}\}$, $1 \leq r,s \leq n$, whose entries are complex-valued functions on $I$ which satisfy the following conditions:

$$\begin{aligned} &p_{rs} \in L^1_{\text{loc}}(I) \quad (1 \le r, s \le n,\ n \ge 2),\\ &p_{rs} \ne 0 \quad \text{a.e. on } I\ (1 \le r \le n-1),\\ &p_{rs} = 0 \quad \text{a.e. on } I\ (2 \le r+1 < s \le n). \end{aligned} \tag{2.3}$$

For $P \in Z_n(I)$, the quasiderivatives associated with $P$ are defined by

$$\begin{aligned} &y^{[0]} := y,\\ &y^{[r]} := (p_{r,r+1})^{-1}\left\{(y^{[r-1]})' - \sum_{s=1}^{r} p_{rs} y^{[s-1]}\right\} \quad (1 \le r \le n-1),\\ &y^{[n]} := (y^{[n-1]})' - \sum_{s=1}^{n} p_{ns} y^{[s-1]}, \end{aligned} \tag{2.4}$$

where the prime $'$ denotes differentiation.

The quasidifferential expression $\tau$ associated with $P$ is given by

$$\tau[y] := i^n y^{[n]} \quad (n \ge 2), \tag{2.5}$$

this being defined on the set

$$V(\tau) := \{y : y^{[r-1]} \in AC_{\text{loc}}(I),\ r = 1,\dots,n\}, \tag{2.6}$$

where $L^1_{\text{loc}}(I)$ and $AC_{\text{loc}}(I)$ denote, respectively, the spaces of complex-valued Lebesgue measurable functions on $I$ which are locally integrable and locally absolutely continuous on every compact subinterval of $I$.

The formal adjoint $\tau^+$ of $\tau$ defined by the matrix $P^+ \in Z_n(I)$ is given by

$$\begin{aligned} &\tau^+[z] := i^n z^{[n]} \quad \forall y \in V(\tau^+),\\ &V(\tau^+) := \{z : z_+^{[r-1]} \in AC_{\text{loc}}(I),\ r = 1,\dots,n\}, \end{aligned} \tag{2.7}$$

where $z_+^{[r-1]}$, $r = 1,2,\dots,n$, are the quasiderivatives associated with the matrix $P^+$,

$$P^+ = \{p^+_{rs}\} = (-1)^{r+s+1}\overline{p}_{n-s+1,n-r+1} \quad \text{for each } r,s;\ 1 \le r,s \le n. \tag{2.8}$$

Note that $(P^+)^+ = P$ and so $(\tau^+)^+ = \tau$. We refer to [2, 3, 6, 7, 13] for a full account of the above and subsequent results on quasidifferential expressions.

Let the interval $I$ have end points $a$, $b$ $(-\infty \le a < b \le \infty)$, and let $w : I \to \mathbb{R}$ be a nonnegative weight function with $w \in L^1_{\text{loc}}(I)$ and $w(x) > 0$ (for almost all $x \in I$). Then $H = L^2_w(I)$ denotes the Hilbert function space of equivalence classes of Lebesgue measurable functions such that $\int_I w|f|^2 < \infty$; the inner-product is defined by

$$(f,g) := \int_I w(x)f(x)\overline{g(x)}dx \quad (f,g \in L^2_w(I)). \tag{2.9}$$

The equation

$$\tau[y] - \lambda w y = 0 \quad (\lambda \in \mathbb{C}) \text{ on } I \tag{2.10}$$

is said to be regular at the left end point $a \in \mathbb{R}$, if for all $X \in (a,b)$, $a \in \mathbb{R}$; $w, p_{rs} \in L^1[a,X]$, $(r,s = 1,\dots,n)$. Otherwise (2.10) is said to be singular at $a$. If (2.10) is regular at both end points, then it is said to be regular; in this case we have,

$$a,b \in \mathbb{R}, \quad w, p_{rs} \in L^1(a,b), \quad (r,s = 1,\dots,n). \tag{2.11}$$

We will be concerned with the case when $a$ is a regular end point of (2.10), the end point $b$ being allowed to be either regular or singular. Note that, in view of (2.8), an end point of $I$ is regular for (2.10), if and only if it is regular for the equation

$$\tau^+[z] - \overline{\lambda} w z = 0 \quad (\lambda \in \mathbb{C}) \text{ on } I. \tag{2.12}$$

Note that, at a regular end point $a$, $y^{[r-1]}(a)(z_+^{[r-1]}(a))$, $r = 1,\dots,n$, is defined for all $u \in V(\tau)$ $(v \in V(\tau^+))$. Set

$$\begin{aligned} D(\tau) &:= \{y : y \in V(\tau),\ y, w^{-1}\tau[y] \in L^2_w(a,b)\}, \\ D(\tau^+) &:= \{z : z \in V(\tau^+),\ z, w^{-1}\tau^+[z] \in L^2_w(a,b)\}. \end{aligned} \tag{2.13}$$

The subspaces $D(\tau)$ and $D(\tau^+)$ of $L^2_w(a,b)$ are domains of the so-called maximal operators $T(\tau)$ and $T(\tau^+)$, respectively, defined by

$$\begin{aligned} T(\tau)y &:= w^{-1}\tau[y] \quad (y \in D(\tau)), \\ T(\tau^+)z &:= w^{-1}\tau^+[z], \quad (z \in D(\tau^+)). \end{aligned} \tag{2.14}$$

For the regular problem, the minimal operators $T_0(\tau)$ and $T_0(\tau^+)$ are the restrictions of $w^{-1}\tau[y]$ and $w^{-1}\tau^+[z]$ to the subspaces

$$\begin{aligned} D_0(\tau) &:= \{y : y \in D(\tau),\ y^{[r-1]}(a) = y^{[r-1]}(b) = 0,\ r = 1,\dots,n\}, \\ D_0(\tau^+) &:= \{z : z \in D(\tau^+),\ z_+^{[r-1]}(a) = z_+^{[r-1]}(b) = 0,\ r = 1,\dots,n\}, \end{aligned} \tag{2.15}$$

respectively. The subspaces $D_0(\tau)$ and $D_0(\tau^+)$ are dense in $L_w^2(a,b)$, and $T_0(\tau)$ and $T_0(\tau^+)$ are closed operators (see [2, 3, 6] and [13, Section 3]).

In the singular problem, we first introduce the operators $T_0'(\tau)$ and $T_0'(\tau^+)$; $T_0'(\tau)$ being the restriction of $w^{-1}\tau[\cdot]$ to the subspace

$$D_0'(\tau) := \{y : y \in D(\tau),\ \operatorname{supp} y \subset (a,b)\} \tag{2.16}$$

and with $T_0'(\tau^+)$ defined similarly. These operators are densely defined and closable in $L_w^2(a,b)$, and we defined the minimal operators $T_0(\tau)$, $T_0(\tau^+)$ to be their respective closures (see [2] and [13, Section 5]). We denote the domains of $T_0(\tau)$ and $T_0(\tau^+)$ by $D_0(\tau)$ and $D_0(\tau^+)$, respectively. It can be shown that

$$\begin{aligned} y \in D_0(\tau) &\Longrightarrow y^{[r-1]}(a) = 0 \quad (r = 1,\dots,n), \\ z \in D_0(\tau^+) &\Longrightarrow z_+^{[r-1]}(a) = 0 \quad (r = 1,\dots,n), \end{aligned} \tag{2.17}$$

because we are assuming that $a$ is a regular end point. Moreover, in both regular and singular problems, we have

$$T_0^*(\tau) = T(\tau^+), \qquad T^*(\tau) = T_0(\tau^+), \tag{2.18}$$

see [13, Section 5] in the case when $\tau = \tau^+$ and compare it with treatment in [2, Section III.10.3] and [3] in general case.

**3. Some technical lemmas.** The proof of the general theorem is based on the results in this section. We start by listing some properties and results of quasi-differential expressions $\tau_1, \tau_2, \dots, \tau_n$ each of order $n$. For proofs, the reader is referred to [4, 12, 13, 14].

$$\begin{gathered} (\tau_1 + \tau_2)^+ = \tau_1^+ + \tau_2^+, \\ (\tau_1\tau_2)^+ = \tau_2^+\tau_1^+, \qquad (\lambda\tau)^+ = \overline{\lambda}\tau^+, \quad \text{for } \lambda \text{ is a complex number.} \end{gathered} \tag{3.1}$$

A consequence of properties (3.1) is that if $\tau^+ = \tau$, then $P(\tau)^+ = P(\tau^+)$ for $P$ is any polynomial with complex coefficients. Also we note that the leading coefficients of a product is the product of the leading coefficients. Hence the product of regular differential expressions is regular.

**Lemma 3.1** (cf. [4, Theorem 1]). *Suppose that $\tau_j$ is a regular differential expression on the interval $[0,b]$ such that the minimal operator $T_0(\tau_j)$ has property $(C)$ for $j=1,2,\dots,n$. Then*

(i) *the product operator $\prod_{j=1}^{n}[T_0(\tau_j)]$ is closed and has dense domain, property $(C)$, and*

$$\operatorname{def}\left[\prod_{j=1}^{n} T_0(\tau_j)\right] = \sum_{j=1}^{n} \operatorname{def}[T_0(\tau_j)]; \tag{3.2}$$

(ii) *the operators $T_0(\tau_1\tau_2\cdots\tau_n)$ and $\prod_{j=1}^{n}[T_0(\tau_j)]$ are not equal in general, that is, $[T_0(\tau_1\tau_2\cdots\tau_n)] \subseteq \prod_{j=1}^{n}[T_0(\tau_j)]$.*

**Lemma 3.2** (cf. [4, Theorem 2]). *Let $\tau_1,\tau_2,\dots,\tau_n$ be regular differential expressions on $[0,b]$. Suppose that $T_0(\tau_j)$ satisfies property $(C)$ for $j=1,2,\dots,n$. Then*

$$T_0(\tau_1\tau_2\cdots\tau_n) = \prod_{j=1}^{n} T_0(\tau_j) \tag{3.3}$$

*if and only if the following partial separation condition is satisfied:*

$$\{f \in L_w^2(a,b),\ f^{[s-1]} \in AC_{\text{loc}}[a,b)\}, \tag{3.4}$$

*where $s$ is the order of product expression $(\tau_1\tau_2\cdots\tau_n)$ and $(\tau_1\tau_2\cdots\tau_n)^+ f \in L_w^2(a,b)$ together imply that $(\prod_{j=1}^{k}(\tau_j^+))f \in L_w^2(a,b)$, $k=1,\dots,n-1$.*

*Furthermore, $T_0(\tau_1\tau_2\cdots\tau_n) = \prod_{j=1}^{n} T_0(\tau_j)$ if and only if*

$$\operatorname{def}[T_0(\tau_1\tau_2\cdots\tau_n)] = \sum_{j=1}^{n} \operatorname{def}[T_0(\tau_j)], \tag{3.5}$$

*then the product $(\tau_1\tau_2\cdots\tau_n)$ is partially separated expressions in $L_w^2(0,b)$ whenever property (3.4) holds.*

**Lemma 3.3** (cf. [4, Corollary 1]). *Let $\tau_j$ be a regular differential expression on $[0,b]$ for $j=1,\dots,n$. If all solutions of the differential equations $(\tau_j)y=0$ and $(\tau_j^+)z=0$ on $[0,b]$ are in $L_w^2(0,b)$ for $j=1,\dots,n$, then all solutions of $(\tau_1\tau_2\cdots\tau_n)y=0$ and $(\tau_1\tau_2\cdots\tau_n)^+z=0$ are in $L_w^2(0,b)$.*

The special case of Lemma 3.3 when $\tau_j=\tau$ for $j=1,2,\dots,n$ and $\tau$ is symmetric was established in [14]. In this case, it is easy to see that the converse also holds. If all solutions of $\tau^n u=0$ are in $L_w^2(0,b)$, then all solutions of $\tau y=0$ must be in $L_w^2(0,b)$. In general, if all solutions of $(\tau_1\tau_2\cdots\tau_n)y=0$

are in $L_w^2(0,b)$, then all solutions of $\tau_n y = 0$ are in $L_w^2(0,b)$ since these also the solutions of $(\tau_1\tau_2\cdots\tau_n)y = 0$. If all solutions of the adjoints equation $(\tau_1\tau_2\cdots\tau_n)^+ z = 0$ are also in $L_w^2(0,b)$, then it follows similarly that all solutions of $\tau_1^+ z = 0$ are in $L_w^2(0,b)$. So, in particular, for $n = 2$ we have established the following lemma.

**Lemma 3.4.** *Suppose that $\tau_1$, $\tau_2$, and $\tau_1\tau_2$ are all regular expressions on $[0,b]$. Then the product is in the maximal deficiency case at $b$ if and only if both $\tau_1$, $\tau_2$ are in the maximal deficiency case at $b$, see* [4, Corollary 2] *for more details.*

Denote by $S(\tau)$ and $S(\tau^+)$ the sets of all solutions of the equations

$$\left(\prod_{j=1}^{n}\tau_j\right)y = 0, \qquad \left(\prod_{j=1}^{n}\tau_j^+\right)z = 0, \tag{3.6}$$

respectively. Let $\phi_k(t)$, $k = 1,2,\ldots,n^2$, denote the solutions of the homogeneous equation $(\prod_{j=1}^{n}\tau_j)y = 0$ determined by the initial conditions

$$\phi_k^{[r]}(t_0) = \delta_{k,r+1} \quad \forall t_0 \in [0,b] \tag{3.7}$$

(where $k = 1,2,\ldots,n^2$; $r = 0,1,\ldots,n^2-1$). Let $\phi_k^+(t)$, $k = 1,2,\ldots,n^2$, denote the solutions of the homogeneous equation $(\prod_{j=1}^{n}\tau_j^+)z = 0$ determined by the initial conditions

$$(\phi_k^+)^{[r]}(t_0) = (-1)^{k+r}\delta_{k,n^2-r} \quad \forall t_0 \in [0,b], \tag{3.8}$$

where $k = 1,2,\ldots,n^2$; $r = 0,1,\ldots,n^2-1$.

**Remark 3.5.** If all solutions $\phi_k(t)$, $\phi_k^+(t)$, $k = 1,2,\ldots,n^2$, of $(\prod_{j=1}^{n}\tau_j)y = 0$ and $(\prod_{j=1}^{n}\tau_j^+)z = 0$, respectively are bounded ($L_w^2$-bounded) on $[0,b)$, then $S(\tau)$ and $S(\tau^+)$ are bounded ($L^2$-bounded) and hence $S(\tau)\cup S(\tau^+)$ is bounded ($L^2$-bounded) on $[0,b)$; see [6] and [7, Lemmas 3.4 and 3.5].

The next lemma is a form of the variation of parameters formula of a general quasidifferential equation, see [6, Section 3] and [7, 13].

**Lemma 3.6.** *For $f$ locally integrable, the solution $\phi$ of the quasidifferential equation*

$$\left(\prod_{j=1}^{n}\tau_j\right)y = wf \quad \text{on } [0,b) \tag{3.9}$$

*satisfying*

$$\phi^{[r]}(t_0) = \alpha_{r+1} \quad \forall t_0 \in [0,b),\ r = 0,1,\ldots,n^2-1 \tag{3.10}$$

*is given by*

$$\phi(t) = \sum_{j=1}^{n^2} \alpha_j \phi_j(t) + \frac{1}{i^{n^2}} \sum_{j,k=1}^{n^2} \zeta^{jk} \phi_j(t) \int_{t_0}^{t} \overline{\phi_k^+(s)} f(s) w(s)\, ds \tag{3.11}$$

*for some* $\alpha_1, \alpha_2, \ldots, \alpha_{n^2} \in \mathbb{C}$*, where* $\phi_j(t)$ *and* $\phi_k^+(t)$*,* $j,k = 1,2,\ldots,n^2$*, are solutions of the equations in (3.6), respectively,* $\zeta^{jk}$ *is a constant which is independent of* $t$*.*

In the sequel, we will require the following nonlinear integral inequality which generalizes those integral inequalities used in [1, 5, 9, 10].

**Lemma 3.7** (cf. [5, 10]). *Let* $u(t)$*,* $v(t)$*,* $f(t,s)$*,* $g_i(t,s)$*, and* $h_i(t,s)$ $(i = 1,2,\ldots,n^2)$ *be nonnegative continuous functions defined on the interval* $I$ *and* $I \times I$*, respectively, here* $I = (0,c)$*,* $0 < c \le \infty$*, with their ranges in* $\mathbb{R}^+$*. Let* $v(t)$ *be nondecreasing on* $I$*, and* $f(t,s)$*,* $g_i(t,s)$*, and* $h_i(t,s)$*,* $(i = 1,2,\ldots,n^2)$ *be nondecreasing in* $t$ *for each* $s \in I$ *fixed. Suppose that the inequality*

$$u(t) \le v(t) + \int_0^t f(t,s)u(s)\,ds + \sum_{j=1}^{n^2} \int_0^t g_i(t,s) \left[ \int_0^s h_i(t,s)[u(\tau)]^{\sigma} d\tau \right] ds \tag{3.12}$$

*holds for all* $t \in I$*, where* $\sigma \in (0,1]$ *is constant. Then*

(i) *if* $0 < \sigma < 1$*,*

$$u(t) \le \left[ [v(t)F(t)]^{1-\sigma} + (1-\sigma) \sum_{i=1}^{n^2} G_i(t)F(t) \int_0^t h_i(t,s)\,ds \right]^{1/(1-\sigma)}, \quad t \in I, \tag{3.13}$$

(ii) *if* $\sigma = 1$*,*

$$u(t) \le v(t) \exp \int_0^t \left[ f(t,s) + \sum_{i=1}^{n^2} G_i(t)F(t)h_i(t,s) \right] ds, \tag{3.14}$$

*where*

$$F(t) = \exp \int_0^t f(t,s)\,ds, \qquad G_i(t) = \int_0^t g_i(s)\,ds, \quad i = 1,2,\ldots,n^2. \tag{3.15}$$

**Corollary 3.8** (cf. [9, 10]). *Let* $u(t)$*,* $f_1(t)$*,* $f_2(t)$*,* $g_1(t,s)$*, and* $g_2(t,s)$ *be nonnegative continuous functions defined on the intervals* $I = [0,b)$ *and* $I \times I$*,* *respectively. Suppose that the inequality*

$$u(t) \le C + \int_0^t f_1(s)u(s)ds + \int_0^t f_2(s)u^\sigma(s)ds + \int_0^t \left( \int_0^s g_1(s,x)u(x)dx \right) + \int_0^t (g_2(s,x)u^\sigma(x))ds \tag{3.16}$$

*holds for all* $t \in [0,b)$, *where* $\sigma \in (0,1]$ *and* $C$ *is constant. Then*

$$u(t) \le \Bigg[ C^{(1-p)} + (1-\sigma) \times \int_0^t \left[ f_2(s) + \int_0^s g_2(s,x)dx \right] \times \exp\left[ (1-\sigma) \int_0^s \left[ f_1(\tau) + \int_0^s g_1(\tau,x)dx \right] d\tau \right] ds \Bigg]^{1/(1-p)} \times \exp\left( \int_0^t \left[ f_1(s) + \int_0^s g_1(s,x)dx \right] ds \right). \tag{3.17}$$

**4. Boundedness of solutions.** In this section, we consider the question of determining conditions under which all solutions of (1.2) are bounded and $L_w^2$-bounded.

Suppose there exist nonnegative continuous functions $e_1(t)$, $e_2(t)$, $e_3(t)$, $r_1(t)$, $r_2(t)$, $K_0(t,s)$, and $K_i(t-s)$ on $[0,b)$, $0 < b \le \infty$; $i = 1,2,\dots,n^2-1$ such that,

$$\begin{aligned} &\left| F\left( t, y, \int_0^t g\left( t,s,y,y',\dots,y^{(n^2-1)}(s) \right) ds \right) \right| \\ &\quad \le e_1(t) + r_1(t)|y(t)|^\sigma + r_2(t) \left[ \int_0^t [e_2(t) + e_3(s) + K_0(t,s)|y(s)|^\sigma] ds \right. \\ &\qquad \left. + \left| \int_0^t \sum_{i=1}^{n^2-1} K_i(t-s) y^{(i)}(s) ds \right| \right], \end{aligned} \tag{4.1}$$

for $t,s \ge 0$ and some $\sigma \in [0,1]$; see [5, 9, 10].

**Theorem 4.1.** *Suppose that (4.1) is satisfied with* $\sigma = 1$, $S(\tau) \cup S(\tau^+)$ *is bounded on* $[0,b)$, *and that*

(a) $k_i^{(\ell)}(0) = 0$ *for all* $\ell = 0,1,\dots,i-1$; $i = 1,2,\dots,n^2-1$,

(b) $e_1(t)$, $r_1(t)$, *and* $r_2 k_i^{(\ell)}(t) \in L_w^1(0,b)$, $\ell = 0,1,\dots,i-1$; $i = 1,2,\dots,n^2-1$,

(c) *the following integrals are bounded at* $t \to b$,

(i) $\int_0^t r_2(s)(\int_0^s [e_2(x)+e_3(x)]dx)w(s)ds$,
(ii) $\int_0^t r_2(s)(\int_0^s K_0(s,x)dx)w(s)ds$,
(iii) $\int_0^t r_2(s)(\sum_{i=1}^{n^2-1}\int_0^s |(\partial^i/\partial x^i)K_i(s-x)|dx)w(s)ds$.

*Then all solutions of (1.2) are also bounded on* $[0,b)$.

**Proof.** Note that (4.1) implies that all solutions are defined on $[0,b)$. Let $\{\phi_1(t),\dots,\phi_{n^2}(t)\}$ and $\{\phi_1^+(t),\dots,\phi_{n^2}^+(t)\}$ be two sets of linearly independent solutions of the equations in (3.6), respectively, and let $\phi(t)$ be any solution of (1.2) on $[0,b)$, then by Lemma 3.6, we have

$$\phi(t)=\sum_{j=1}^{n^2}\alpha_j\phi_j(t)+\frac{1}{i^{n^2}}\sum_{j,k=1}^{n^2}\zeta^{jk}\phi_j(t)\int_0^t\overline{\phi_k^+(s)}F(s)w(s)ds. \tag{4.2}$$

Hence,

$$\begin{aligned}|\phi(t)| \le &\sum_{j=1}^{n^2}|\alpha_j||\phi_j(t)|\\ &+\sum_{j,k=1}^{n^2}|\zeta^{jk}||\phi_j(t)|\\ &\quad\times\int_0^t|\overline{\phi_k^+(s)}|\Bigg[e_1(s)+r_1(s)|\phi(s)|\\ &\qquad\qquad+r_2(s)\Bigg[\int_0^s[e_2(s)+e_3(x)+K_0(s,x)|\phi(x)|]dx\\ &\qquad\qquad\qquad+\left|\int_0^s\sum_{i=1}^{n^2-1}K_i(s-x)\phi^{(i)}(x)dx\right|\Bigg]\Bigg]w(s)ds.\end{aligned} \tag{4.3}$$

Since $\phi_k^+(t)$ is bounded on $[0,b)$, $k=1,\dots,n^2$, and $e_1(t)\in L_w^1(0,b)$, then $\phi_k^+(t)e_1(t)\in L_w^1(0,b)$, $k=1,2,\dots,n^2$. Setting

$$C_k=\int_0^t|\overline{\phi_k^+(s)}|e_1(s)w(s)ds, \tag{4.4}$$

and integrating the last integral in (4.3) by parts, we have

$$\begin{aligned}&\sum_{i=1}^{n^2-1}\int_0^s K_i(s-x)\phi^{(i)}(x)dx\\ &\quad=\sum_{\ell=0}^{i-1}(-1)^{\ell+1}K_i^{(\ell)}(s)\phi^{(i-1-\ell)}(0)+(-1)^i\int_0^s\frac{\partial^i}{\partial x^i}K_i(s-x)\phi(x)dx,\end{aligned} \tag{4.5}$$

where $K_i^{(\ell)}(0) = 0$ for all $\ell = 0, 1, \dots, i-1$; $i = 1, 2, \dots, n^2 - 1$. Then (4.3) becomes

$$\begin{aligned} |\phi(t)| \le{}& \sum_{j=1}^{n^2} (C_j + |\alpha_j|)|\phi_j(t)| \\ &+ \sum_{j,k=1}^{n^2} |\zeta^{jk}||\phi_j(t)| \\ &\times \int_0^t |\overline{\phi_k^+(s)}| \Bigg[ r_1(s)|\phi(s)| \\ &\qquad + r_2(s)\Bigg( \int_0^s [e_2(s) + e_3(x) + K_0(s,x)|\phi(x)|]dx \\ &\qquad\qquad + \sum_{i=1}^{n^2-1} \sum_{\ell=0}^{i-1} |K_i^{(\ell)}(s)||\phi^{(i-1-\ell)}(0)| \\ &\qquad\qquad + \sum_{i=1}^{n^2-1} \int_0^s \left| \frac{\partial^i}{\partial x^i} K_i(s-x) \right| |\phi(x)|dx \Bigg)\Bigg] w(s)ds, \end{aligned} \tag{4.6}$$

where $|\phi^{(i-1-\ell)}(0)| \le \beta$ for all $\ell = 0, 1, \dots, i-1$; $i = 0, \dots, n^2 - 1$. Let

$$\begin{aligned} h(t) ={}& \sum_{j=1}^{n^2} (C_j + |\alpha_j|)|\phi_j(t)| \\ &+ \sum_{j,k=1}^{n^2} |\zeta^{jk}||\phi_j(t)| \int_0^t |\overline{\phi_k^+(s)}| \Bigg[ r_2(s)\Bigg[ \int_0^s [e_2(s) + e_3(x)]dx \\ &\qquad + \sum_{i=1}^{n^2-1} \sum_{\ell=0}^{i-1} |K_i^{(\ell)}(s)|\beta \Bigg]\Bigg] w(s)ds. \end{aligned} \tag{4.7}$$

Then (4.6) becomes

$$\begin{aligned} |\phi(t)| \le{}& h(t) \\ &+ \sum_{j,k=1}^{n^2} |\zeta^{jk}||\phi_j(t)| \\ &\times \int_0^t |\overline{\phi_k^+(s)}| \Bigg[ r_1(s)|\phi(s)| \\ &\qquad + r_2(s)\Bigg[ \int_0^s K_0(s,x)|\phi(x)|dx \\ &\qquad\qquad + \sum_{i=1}^{n^2-1} \int_0^s \left| \frac{\partial^i}{\partial x^i} K_i(s-x) \right| |\phi(x)|dx \Bigg]\Bigg] w(s)ds. \end{aligned} \tag{4.8}$$

From our assumptions and conditions (i) and (ii), it follows that $h(t)$ is bounded on $[0,b)$. Applying Lemma 3.7 with $\sigma = 1$, we obtain

$$\begin{aligned}
&|\phi(t)| \\
&\le h(t)\exp\Bigg\{\sum_{j,k=1}^{n^2}|\zeta^{jk}||\phi_j(t)| \\
&\qquad \times\int_0^t|\overline{\phi_k^+(s)}|\Bigg[r_1(s)+r_2(s)\Bigg[\int_0^s\Bigg[K_0(s,x) \\
&\qquad\qquad +\sum_{i=1}^{n^2-1}\left|\frac{\partial^i}{\partial x^i}K_i(s-x)\right|\Bigg]dx\Bigg]\Bigg]w(s)ds\Bigg\},
\end{aligned} \tag{4.9}$$

and hence our assumptions and conditions (i), (ii), and (iii) yield that $\phi(t)$ is bounded on $[0,b)$. □

**Theorem 4.2.** *Suppose that $S(\tau)\cup S(\tau^+)\subset L^2_w(0,b)$ with $\sigma = 1$, and that*

(i) *$r_1(t)$ and $r_2(t)$ are bounded on $[0,b)$,*

(ii) *$e_1(s)$ and $K_i^{(\ell)}(s)\in L^2_w(0,b)$ for all $\ell = 0,1,\dots,i-1$; $i = 1,2,\dots,n^2-1$,*

(iii) *$\int_0^t[\int_0^s[e_2(s)+e_3(x)]dx]^2w(s)ds < \infty$,*

(iv) *$\int_0^t[\int_0^s[(1/w)(K_0^2(s,x))+[\sum_{i=1}^{n^2-1}|(\partial^i/\partial x^i)K_i(s-x)|]^2]dx]w(s)ds < \infty$.*

*Then all solutions of (1.2) are in $L^2_w(0,b)$.*

**Proof.** The proof is the same up to (4.5), since $\phi_k^+(s)$, $e_1(s)\in L^2_w(0,b)$ (see Lemma 3.3), then $\phi_k^+(s)e_1(s)\in L^1_w(0,b)$, $k = 1,2,\dots,n^2$, for all $s\in(0,b)$. By using (4.4) and applying the Cauchy-Schwartz inequality to the integral in (4.6), we have

$$\begin{aligned}
|\phi(t)| \le{}& \sum_{j=1}^{n^2}(C_j+|\alpha_j|)|\phi_j(t)| \\
&+\sum_{j,k=1}^{n^2}|\zeta^{jk}||\phi_j(t)| \\
&\times\int_0^t|\overline{\phi_k^+(s)}|\Bigg[\left(\int_0^t|\overline{\phi_k^+(s)}|^2r_1^2(s)w(s)ds\right)^{1/2}\left(\int_0^t|\phi(s)|^2w(s)ds\right)^{1/2} \\
&\qquad+\left(\int_0^t|\overline{\phi_k^+(s)}|^2r_2^2(s)w(s)ds\right)^{1/2} \\
&\qquad\times\Bigg\{\left(\int_0^t\left[\int_0^s[e_2(s)+e_3(x)]dx\right]^2w(s)ds\right)^{1/2} \\
&\qquad\quad+\left(\int_0^t\left[\int_0^sK_0(s,x)|\phi(x)|dx\right]^2w(s)ds\right)^{1/2}
\end{aligned}$$

$$+\left(\int_0^t\left[\sum_{i=1}^{n^2-1}\sum_{\ell=0}^{i-1}\left|K_i^{[\ell]}(s)\beta\right|\right]^2 w(s)ds\right)^{1/2}$$
$$+\left(\int_0^t\left[\sum_{i=1}^{n^2-1}\left|\frac{\partial^i}{\partial x^i}K_i(s-x)\phi(x)\right|dx\right]^2 w(s)ds\right)^{1/2}\Bigg\}\Bigg]. \tag{4.10}$$

Since $r_1(s), r_2(s)$ are bounded on $[0,b)$ and $\phi_k^+(s)\in L_w^2(0,b)$, then $\phi_k^+(s)r_1(s)$, $\phi_k^+(s)r_2(s)\in L_w^2(0,b)$; $k=1,2,\dots,n^2$ for all $s\in[0,b)$ and hence there exist positive constants $\zeta_1$, $\zeta_2$ such that

$$\|\phi_k^+(s)r_i(s)\|_{L_w^2(0,b)}\le \xi_i \quad \forall k=1,2,\dots,n^2;\ i=1,2. \tag{4.11}$$

Therefore (4.10) becomes

$$\begin{aligned}
|\phi(t)|&\\
\le &\sum_{j=1}^{n^2}(C_j+|\alpha_j|)|\phi_j(t)|\\
&+\sum_{j,k=1}^{n^2}|\zeta^{jk}||\phi_j(t)|\Bigg[\xi_1\left(\int_0^t|\phi(s)|^2w(s)ds\right)^{1/2}\\
&\qquad+\xi_2\Bigg\{\left(\int_0^t\left[\int_0^s[e_2(s)+e_3(x)]dx\right]^2w(s)ds\right)^{1/2}\\
&\qquad+\left(\int_0^t\left[\int_0^s K_0(s,x)|\phi(x)|dx\right]^2w(s)ds\right)^{1/2}\\
&\qquad+\left(\int_0^t\left[\sum_{i=1}^{n^2-1}\sum_{\ell=0}^{i-1}\left|K_i^{(\ell)}(s)\beta\right|\right]^2w(s)ds\right)^{1/2}\\
&\qquad+\left(\int_0^t\left[\sum_{i=1}^{n^2-1}\left|\frac{\partial^i}{\partial x^i}K_i(s-x)\phi(x)\right|dx\right]^2w(s)ds\right)^{1/2}\Bigg\}\Bigg].
\end{aligned} \tag{4.12}$$

Let

$$\begin{aligned}
h(t)=&\sum_{j=1}^{n^2}(C_j+|\alpha_j|)|\phi_j(t)|\\
&+\sum_{j,k=1}^{n^2}|\zeta^{jk}||\phi_j(t)|\Bigg[\xi_2\left(\int_0^t\left[\int_0^s[e_2(s)+e_3(x)]dx\right]^2w(s)ds\right)^{1/2}\\
&\qquad+\left(\int_0^t\left[\sum_{i=1}^{n^2-1}\sum_{\ell=0}^{i-1}\left|K_i^{(\ell)}(s)\beta\right|\right]^2w(s)ds\right)^{1/2}\Bigg],
\end{aligned} \tag{4.13}$$

then

$$
\begin{aligned}
|\phi(t)| \leq h(t) + \sum_{j,k=1}^{n^2} |\zeta^{jk}|\,|\phi_j(t)| \\
\times \Bigg[ \xi_1 \left( \int_0^t |\phi(s)|^2 w(s)\,ds \right)^{1/2} \\
+ \xi_2 \Bigg\{ \left( \int_0^t \left[ \int_0^s K_0(s,x)\,|\phi(x)|\,dx \right]^2 w(s)\,ds \right)^{1/2} \\
+ \left( \int_0^t \left[ \sum_{i=1}^{n^2-1} \left| \frac{\partial^i}{\partial x^i} K_i(s-x)\phi(x) \right| dx \right]^2 w(s)\,ds \right)^{1/2} \Bigg\} \Bigg].
\end{aligned}
\tag{4.14}
$$

Applying the Cauchy-Schwartz inequality and squaring both sides of (4.1), we have

$$
\begin{aligned}
|\phi(t)|^2 \leq 2h^2(t) \\
+ 4 \sum_{j,k=1}^{n^2} |\zeta^{jk}|\,|\phi_j(t)|^2 \\
\times \Bigg[ \xi_1^2 \left( \int_0^t |\phi(s)|^2 w(s)\,ds \right) \\
+ \xi_2^2 \int_0^t \left( \int_0^s \frac{1}{w} \left( K_0^2(s,x) + \left[ \sum_{i=1}^{n^2-1} \left| \frac{\partial^i}{\partial x^i} K_i(s-x) \right| dx \right]^2 \right) dx \right) \\
\times \left( \int_0^t |\phi(s)|^2 w(s)\,ds \right) \Bigg].
\end{aligned}
\tag{4.15}
$$

If $u(t) = \int_0^t |\phi(s)|^2 w(s)\,ds$, then

$$
\begin{aligned}
u(t) \leq 2 \int_0^t h^2(s) w(s)\,ds \\
+ 4\xi_1^2 \sum_{j,k=1}^{n^2} |\zeta^{jk}| \int_0^t |\phi_j(s)|^2 w(s)\,ds \\
+ 4\xi_2^2 \sum_{j,k=1}^{n^2} \zeta^{jk} \int_0^t |\phi_j(s)|^2 \\
\times \Bigg[ \int_0^s \left( \int_0^\tau \frac{1}{w} \left( K_0^2(s,x) + \left[ \sum_{i=1}^{n^2-1} \left| \frac{\partial^i}{\partial x^i} K_i(s-x) \right| \right]^2 \right) dx \right) \\
\times u(\tau) w(\tau)\,d\tau \Bigg] w(s)\,ds.
\end{aligned}
\tag{4.16}
$$

From conditions (ii) and (iii), it follows that the integral $\int_0^t h^2(s)w(s)ds$ will be finite and by using Lemma 3.7, we obtain

$$u(t) \le \left(2\int_0^t h^2(s)w(s)ds\right) \times \exp\Bigg\{4\xi_1^2 \sum_{j,k=1}^{n^2} |\zeta^{jk}| \int_0^t |\phi_j(s)|^2 w(s)ds + 4\xi_1^2 \sum_{j,k=1}^{n^2} \zeta^{jk} \int_0^t |\phi_j(s)|^2 \times \left[\int_0^s \left(\int_0^\tau \frac{1}{w}\left(K_0^2(s,x) + \left[\sum_{i=1}^{n^2-1} \left|\frac{\partial^i}{\partial x^i} K_i(s-x)\right|\right]^2\right)dx\right) \times w(\tau)d\tau\right] w(s)ds\Bigg\}. \tag{4.17}$$

Hence our assumption and condition (iv) yield that $\phi(t) \in L_w^2(0,b)$. □

Next, we consider (4.1) with $0 \le \sigma < 1$, and we have the following results.

**Theorem 4.3.** *Suppose that $S(\tau)\cup S(\tau^+)$ is bounded on $[0,b)$ and that*

(i) *$e_1(s)$ and $r_1(s) \in L_w^2(0,b)$ for all $s \in [0,b)$,*
(ii) *$\int_0^t r_2(s)(\int_0^s [e_2(s)+e_3(x)]dx)w(s)ds < \infty$,*
(iii) *$\int_0^t r_2(s)(\sum_{i=1}^{n^2-1}\sum_{\ell=0}^{i-1} |K_i^{(\ell)}(s)|)w(s)ds < \infty$,*
(iv) *$\int_0^t r_2(s)(\int_0^s K_0(s,x)dx)w(s)ds < \infty$,*
(v) *$\int_0^t r_2(s)(\sum_{i=1}^{n^2-1}\int_0^s |(\partial^i/\partial x^i)K_i(s-x)|dx)w(s)ds < \infty$.*

*Then all solutions of (1.2) are bounded in $[0,b)$.*

**Proof.** For $0 \le \sigma < 1$, the proof is the same up to (4.6). In this case (4.6) becomes

$$|\phi(t)| \le \sum_{j=1}^{n^2} (C_j + |\alpha_j|)|\phi_j(t)| + \sum_{j,k=1}^{n^2} |\zeta^{jk}||\phi_j(t)| \times \int_0^t |\overline{\phi_k^+(s)}| \Big[ r_1(s)|\phi(s)|^\sigma$$

$$+r_2(s)\bigg(\int_0^s [e_2(s)+e_3(x)+K_0(s,x)|\phi(x)|^\sigma]dx + \sum_{i=1}^{n^2-1}\sum_{\ell=0}^{i-1}|K_i^{(\ell)}(s)|\beta + \sum_{i=1}^{n^2-1}\int_0^s \left|\frac{\partial^i}{\partial x^i}K_i^{(i)}(s-x)\right| |\phi(x)|dx\bigg)\bigg]w(s)ds. \quad (4.18)$$

Let,

$$h(t) = \sum_{j=1}^{n^2}(C_j+|\alpha_j|)|\phi_j(t)| + \sum_{j,k=1}^{n^2}|\zeta^{jk}||\phi_j(t)|\int_0^t |\overline{\phi_k^+(s)}|\bigg[r_2(s)\bigg[\int_0^s [e_2(s)+e_3(x)]dx + \sum_{i=1}^{n^2-1}\sum_{\ell=0}^{i=1}|K_i^{(\ell)}(s)|\beta\bigg]\bigg]w(s)ds. \quad (4.19)$$

Then,

$$|\phi(t)| \le h(t)+\sum_{j,k=1}^{n^2}|\zeta^{jk}||\phi_j(t)| \times\int_0^t |\overline{\phi_k^+(s)}|\bigg[r_1(s)|\phi(s)|^\sigma + r_2(s)\bigg[\int_0^s k_0(s,x)|\phi(x)|^\sigma dx + \sum_{i=1}^{n^2-1}\int_0^s\left|\frac{\partial^i}{\partial x^i}K_i^{(i)}(s-x)\right||\phi(x)|dx\bigg]\bigg]w(s)ds. \quad (4.20)$$

By hypothesis, there exist positive constants $\xi_1$ and $\xi_2$ such that,

$$|\phi_j(t)| \le \xi_1, \quad |\phi_k^+(t)| \le \xi_2 \quad \forall j,k=1,2,\dots,n^2, \quad (4.21)$$

and from conditions (i), (ii), and (iii), it follows that $h(t)$ is bounded on $[0,b)$, that is, there exists a positive constant $\xi_3$ such that $h(t)\le\xi_3$ for all $t\in[0,b)$.

Then,

$$\begin{aligned}|\phi(t)| \le \xi_3 + n^2\xi_1\xi_2\Bigg[&\int_0^t r_1(s)|\phi(s)|^\sigma w(s)ds \\ &+\int_0^t r_2(s)\left(\int_0^s k_0(s,x)|\phi(x)|^\sigma dx\right)w(s)ds \\ &+\sum_{i=1}^{n^2-1}\int_0^t\left(r_2(s)\int_0^s\left|\frac{\partial^i}{\partial x^i}K_i(s-x)\right||\phi(x)|dx\right)w(s)ds\Bigg].\end{aligned} \tag{4.22}$$

Applying Corollary 3.8 with $f_1(x)=0$, we have

$$\begin{aligned}|\phi(t)| \le{}& \exp\left(\int_0^t r_2(s)\left(\sum_{i=1}^{n^2-1}\int_0^s\left|\frac{\partial^i}{\partial x^i}K_i(s-x)\right|dx\right)w(s)ds\right) \\ &\times\Bigg\{\Bigg[\xi_3^{(1-\sigma)}+(1-\sigma) \\ &\times\int_0^t\left(r_1(s)+\int_0^s r_2(x)k_0(s,x)dx\right) \\ &\times\exp(1-\sigma)\left(\int_0^t\left(r_2(\tau)\sum_{i=1}^{n^2-1}\int_0^\tau\left|\frac{\partial^i}{\partial x^i}K_i(s-x)\right|dx\right)\right. \\ &\left.\times w(\tau)d\tau\right)w(s)ds\Bigg]^{1/(1-\sigma)}\Bigg\}.\end{aligned} \tag{4.23}$$

Hence, from conditions (i), (ii), and (iii), it follows that $\phi(s)$ is bounded on $[0,b)$. □

**Theorem 4.4.** *Suppose that $S(\tau)\cup S(\tau^+)\subset L_w^2(a,b)$, $r_2(s)$ is bounded on $[0,b)$, and the following conditions are satisfied:*

(i) *$e_1(s)\in L_w^2(0,b)$ and $r_1(s)\in L_w^{2/(1-\sigma)}(0,b)$ for all $s\in[0,b)$,*

(ii) *$\int_0^t(\int_0^s[e_2(s)+e_3(x)]dx)^2w(s)ds<\infty$,*

(iii) *$\int_0^t(\sum_{i=1}^{n^2-1}\sum_{\ell=0}^{i-1}|K_i^{(\ell)}(s)|)^2w(s)ds<\infty$,*

(iv) *$[\int_0^s w^{\sigma/(\sigma-2)}K_0^{2/(2-\sigma)}(s,x)dx]^{(2-\sigma)/2}<\infty$,*

(v) *$[\int_0^s w^{-1}|(\partial^i/\partial x^i)K_i(s-x)|^2dx]^{1/2}w(s)ds<\infty$.*

*Then all solutions of (1.2) are in $L_w^2(0,b)$.*

**Proof.** For $0 \le \sigma < 1$, the proof is the same up to (4.18). Applying the Cauchy-Schwartz inequality to the integrals in (4.18), we have that

$$\begin{aligned}
&\int_0^t |\overline{\phi_j^+(s)}|r_1(s)|\phi(s)|^\sigma w(s)ds \\
&\qquad \le \left(\int_0^t |\phi(s)|^2 w(s)ds\right)^{\sigma/2}\left(\int_0^t |\overline{\phi_k^+(s)}r_1(s)|^\mu w(s)ds\right)^{1/\mu}, \\
&\int_0^s K_0(s\cdot x)|\overline{\phi_k^+(s)}|^\sigma dx \\
&\qquad \le \left(\int_0^s |\phi(x)|^2 w(x)dx\right)^{\sigma/2}\left(\int_0^s w^{1-\mu}K_0^\mu(s,x)dx\right)^{1/\mu}, \\
&\int_0^s \left|\frac{\partial^i}{\partial x^i}K_i(s-x)\phi(x)\right|dx \\
&\qquad \le \left(\int_0^s |\phi(x)|^2 w(x)dx\right)^{1/2}\left(\int_0^s w^{-1}\left|\frac{\partial^i}{\partial x^i}K_i(s-x)\right|^2 dx\right)^{1/2},
\end{aligned} \tag{4.24}$$

where $\mu = 2/(2-\sigma)$. Since $\phi_k^+(s) \in L_w^2(0,b)$ (see Lemma 3.3), $k = 1,\dots,n^2$, and $r_1(s) \in L_w^{2/(1-\sigma)}(0,b)$ by hypothesis, we have $\phi_k^+ r_1 \in L_w^\mu(0,b)$, $k = 1,2,\dots,n^2$. Using this fact and (4.18) in (4.16), we obtain

$$\begin{aligned}
|\phi(t)| \le h(t) + \sum_{j,k=1}^{n^2} |\zeta^{jk}||\phi_j(t)| \\
&\times\Bigg[\xi_0\left(\int_0^t |\phi(s)|^2 w(s)ds\right)^{\sigma/2} \\
&\quad + \xi_1\left(\int_0^t |\overline{\phi_k^+(s)}|r_2(s)\left(\int_0^s |\phi(x)|^2 w(x)dx\right)^{\sigma/2} w(s)ds\right) \\
&\quad + n^2\xi_2\left(\int_0^t |\overline{\phi_k^+(s)}|r_2(s)\left(\int_0^s |\phi(x)|^2 w(x)dx\right)^{1/2} w(s)ds\right)\Bigg],
\end{aligned} \tag{4.25}$$

where

$$\begin{aligned}
\xi_0 &= \left(\int_0^t |\overline{\phi_k^+(s)}r_1(s)|^\mu w(s)ds\right)^{1/\mu}, \\
\xi_1 &= \left(\int_0^t w^{1-\mu}K_0^\mu(s,x)dx\right)^{1/\mu}, \\
\xi_2 &= \left(\int_0^t w^{-1}\left|\frac{\partial^i}{\partial x^i}K_i(s-x)\right|^2 dx\right)^{1/2}, \quad \text{for } i = 1,2,\dots,n^2.
\end{aligned} \tag{4.26}$$

Applying the Cauchy-Schwartz inequality again to the integrals in (4.25) and squaring both sides, we have

$$\begin{aligned}
&|\phi(t)|^2 \\
&\quad\le 2h^2(t)+4\sum_{j,k=1}^{n^2}|\zeta^{jk}|^2|\phi_j(t)|^2 \\
&\qquad\qquad \times\Bigg[\xi_0^2\left(\int_0^t|\phi(s)|^2w(s)ds\right)^{\sigma} \\
&\qquad\qquad\quad +\xi_1^2\left(\int_0^t|\overline{\phi_k^+(s)}|^2w(s)ds\right)\left(\int_0^t r_2^2\left(\int_0^s|\phi(x)|^2w(x)dx\right)^{\sigma}\right) \\
&\qquad\qquad\quad +n^4\xi_0^2\left(\int_0^t|\overline{\phi_k^+(s)}|^2w(s)ds\right)\left(\int_0^t r_2^2\left(\int_0^s|\phi(x)|^2w(x)dx\right)^{\sigma} \right.\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \left.\times w(s)ds\right)\Bigg].
\end{aligned} \tag{4.27}$$

Let

$$\begin{aligned}
u(t) &= \int_0^t|\phi(s)|^2w(s)ds, \\
\xi_3 &= \left(\int_0^t|\overline{\phi_k^+(s)}|^2w(s)ds\right)^{1/2}, \quad j=1,2,\dots,n^2,
\end{aligned} \tag{4.28}$$

and integrate (4.27), to obtain

$$\begin{aligned}
u(t) &\le 2\int_0^t h^2(s)w(s)ds \\
&\quad +4\xi_0^2\sum_{j,k=1}^{n^2}|\zeta^{jk}|^2\int_0^t|\phi_j(s)|^2u^{\sigma}(s)w(s)ds \\
&\quad +4\xi_1^2\xi_3^2\sum_{j,k=1}^{n^2}|\zeta^{jk}|^2\int_0^t|\phi_j(s)|^2\left(\int_0^s r_2^2(x)u^{\sigma}(x)w(x)dx\right)w(s)ds \\
&\quad +4n^4\xi_2^2\xi_3^2\sum_{j,k=1}^{n^2}|\zeta^{jk}|^2\int_0^t|\phi_j(s)|^2\left(\int_0^s r_2^2(x)u(x)w(x)dx\right)w(s)ds.
\end{aligned} \tag{4.29}$$

From our assumptions and conditions (ii) and (iii), it follows that the integral $\int_0^t h^2(s)w(s)ds$ is finite, that is, there exists a positive constant $\xi_4$ such that $\|h(t)\|_{L^2_w(0,b)} \le \xi_4$ for all $t\in[0,b)$. Applying Corollary 3.8 with $f_1(x)=0$, we obtain

$$u(t)\le \exp\left(\int_0^t\left(4n^4\xi_2^2\xi_3^2\sum_{j,k=1}^{n^2}|\zeta^{jk}|^2|\phi_j(s)|^2\int_0^s r_2^2(x)w(x)dx\right)w(s)ds\right)$$
$$\times\Bigg[\xi_4^{(1-\sigma)}+(1-\sigma)$$
$$\times\int_0^t 4\sum_{j,k=1}^{n^2}|\zeta^{jk}|^2|\phi_j(s)|^2\left(\xi_0^2+\xi_1^2\xi_3^2\int_0^s r_2^2(x)w(x)dx\right)$$
$$\times\exp\left((1-\sigma)\int_0^s\left[4n^4\xi_2^2\xi_3^2\sum_{j,k=1}^{n^2}|\zeta^{jk}|^2|\phi_j(x)|^2\int_0^x r_2^2(\tau)w(\tau)d\tau\right]\right.$$
$$\left.\times w(x)dx\right)w(s)ds\Bigg]^{1/(1-\sigma)}. \tag{4.30}$$

Since $\phi_j(t)\in L_w^2(0,b)$, $j=1,2,\dots,n^2$, and $r_2(t)$ is bounded on $[0,b)$, then $\phi(t)\in L_w^2(0,b)$ and hence the result. □

**Corollary 4.5.** *Suppose that $S(\tau)\cup S(\tau^+)\subset L_w^2(0,b)\cap L^\infty(0,b)$ and the following conditions are satisfied:*

(i) $e_1(s)\in L_w^2(0,b)$ *and* $r_1(s)\in L_w^p(0,b)$ *for any* $p$, $1\le p\le 2/(1-\sigma)$,

(ii) $r_2(s)$ *and* $K_i^{(\ell)}(s)\in L_w^2(0,b)\cap L^\infty(0,b)$ *for* $\ell=0,\dots,i-1$; $i=1,\dots,n^2-1$,

(iii) $\int_0^t(\int_0^s[e_2(s)+e_3(x)]dx)^2w(s)ds<\infty$,

(iv) $(\int_0^s w^{\sigma/(\sigma-2)}K_0^{2/(2-\sigma)}(s,x)dx)^{(2-\sigma)/2}<\infty$, $0\le\sigma<1$,

(v) $[\int_0^s w^{-1}|(\partial^i/\partial x^i)K_i(s-x)|^2dx]^{1/2}<\infty$, $i=1,\dots,n^2-1$.

*Then all solutions of (1.2) belong to $L_w^2(0,b)\cap L^\infty(0,b)$.*

**Proof.** The proof follows from Theorems 4.3 and 4.4. We refer to [6, 7, 8, 11] for more details. □

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Sobhy El-Sayed Ibrahim: Department of Mathematics, Faculty of Science, Benha University, Benha 13518, Kalubia, Egypt

*E-mail address*: abomohamed@hotmail.com