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A NOTE ON CHEN'S BASIC EQUALITY FOR SUBMANIFOLDS IN A SASAKIAN SPACE FORM

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It is proved that a Riemannian manifold M isometrically immersed in a Sasakian space form $\tilde{M}(c)$ of constant φ-sectional curvature $c<1$, with the structure vector field ξ tangent to M, satisfies Chen's basic equality if and only if it is a 3-dimensional minimal invariant submanifold.

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1. Introduction. Let $\tilde{M}$ be an m-dimensional almost contact manifold endowed with an almost contact structure (φ,ξ,η), that is, φ be a $(1,1)$-tensor field, ξ be a vector field, and η be a 1-form, such that $\varphi^2=-I+\eta\otimes\xi$ and $\eta(\xi)=1$. Then, $\varphi(\xi)=0$, $\eta\circ\varphi=0$, and m is an odd positive integer. An almost contact structure is said to be *normal*, if in the product manifold $\tilde{M}\times\mathbb{R}$ the induced almost complex structure J defined by $J(X,\lambda d/dt)=(\varphi X-\lambda\xi,\eta(X)d/dt)$ is integrable, where X is tangent to $\tilde{M}$, t is the coordinate of $\mathbb{R}$, and λ is a smooth function on $\tilde{M}\times\mathbb{R}$. The condition for an almost contact structure to be *normal* is equivalent to the vanishing of the torsion tensor $[\varphi,\varphi]+2d\eta\otimes\xi$, where $[\varphi,\varphi]$ is the Nijenhuis tensor of φ.

Let g be a compatible Riemannian metric with the structure (φ,ξ,η), that is, $g(\varphi X,\varphi Y)=g(X,Y)-\eta(X)\eta(Y)$ or equivalently, $g(X,\varphi Y)=-g(\varphi X,Y)$ and $g(X,\xi)=\eta(X)$ for all $X,Y\in T\tilde{M}$. Then, $\tilde{M}$ becomes an almost contact metric manifold equipped with the almost contact metric structure (φ,ξ,η,g). Moreover, if $g(X,\varphi Y)=d\eta(X,Y)$, then $\tilde{M}$ is said to have a *contact metric structure* (φ,ξ,η,g), and $\tilde{M}$ is called a *contact metric manifold*. A normal contact metric structure in $\tilde{M}$ is a *Sasakian structure* and $\tilde{M}$ is a *Sasakian manifold*. A necessary and sufficient condition for an almost contact metric structure to be a Sasakian structure is

$$(\tilde{\nabla}_X\varphi)Y=g(X,Y)\xi-\eta(Y)X,\quad X,Y\in T\tilde{M},\tag{1.1}$$

where $\tilde{\nabla}$ is the Levi-Civita connection of the Riemannian metric g. The manifolds $\mathbb{R}^{2n+1}$ and S^{2n+1} are equipped with standard Sasakian structures. The sectional curvature $\tilde{K}(X\wedge\varphi X)$ of a plane section spanned by a unit vector X orthogonal to ξ is called a *φ-sectional curvature*. If $\tilde{M}$ has a constant

φ-sectional curvature c, then it is called a *Sasakian space form* and is denoted by $\tilde{M}(c)$. For more details, we refer to [2].

Let M be an n-dimensional submanifold immersed in an almost contact metric manifold $\tilde{M}(\varphi,\xi,\eta,g)$. Also let g denote the induced metric on M. We denote by h the second fundamental form of M and by A_N the shape operator associated to any vector N in the normal bundle $T^\perp M$. Then $g(h(X,Y),N) = g(A_N X,Y)$ for all $X,Y \in TM$ and $N \in T^\perp M$. The mean curvature vector is given by $nH = \operatorname{trace}(h)$, and the submanifold M is *minimal* if $H = 0$.

For a vector field X in M, we put $\varphi X = PX + FX$, where $PX \in TM$ and $FX \in T^\perp M$. Thus, P is an endomorphism of the tangent bundle of M and satisfies $g(X,PY) = -g(PX,Y)$ for all $X,Y \in TM$. From now on, let the structure vector field ξ be tangent to M. Then we write the orthogonal direct decomposition $TM = \mathfrak{D} \oplus \{\xi\}$. Let $\{e_1,\dots,e_n\}$ be an orthonormal basis of the tangent space T_pM. We can define the squared norm of P by $\|P\|^2 = \sum_{i,j=1}^n g(e_i,Pe_j)^2$. For a plane section $\pi \subset T_pM$, we denote the functions $\alpha(\pi)$ and $\beta(\pi)$ of tangent space T_pM into $[0,1]$ by $\alpha(\pi) = (g(X,PY))^2$ and $\beta(\pi) = (\eta(X))^2 + (\eta(Y))^2$, where π is spanned by any orthonormal vectors X and Y.

The scalar curvature τ at $p \in M$ is given by $\tau = \sum_{i<j} K(e_i \wedge e_j)$, where $K(e_i \wedge e_j)$ is the sectional curvature of the plane section spanned by e_i and e_j. The well-known Chen's invariant δ_M on M is defined by

$$\delta_M = \tau - \inf K, \tag{1.2}$$

where $(\inf K)(p) = \inf\{K(\pi) \mid \pi \text{ is a plane section} \subset T_pM\}$. For a submanifold M in a real space form $\mathbb{R}^m(c)$, Chen [4] gave the following inequality:

$$\delta_M \le \frac{n^2(n-2)}{2(n-1)}\|H\|^2 + \frac{1}{2}(n+1)(n-2)c. \tag{1.3}$$

He also established in [5] the similar basic inequalities for submanifolds in a complex space form. For an n-dimensional submanifold M in a Sasakian space form $\tilde{M}(c)$ tangential to the structure vector field ξ in [7], the authors established the following Chen's basic inequality.

Theorem 1.1. *Let M be an n-dimensional $(n \ge 3)$ Riemannian manifold isometrically immersed in a Sasakian space form $\tilde{M}(c)$ of constant φ-sectional curvature $c < 1$ with the structure vector field ξ tangent to M. Then,*

$$\delta_M \le \frac{n^2(n-2)}{2(n-1)}\|H\|^2 + \frac{1}{8}\{n(n-3)c + 3n^2 - n - 8\} \tag{1.4}$$

with equality holding if and only if M admits a quasi-anti-invariant structure of rank $(n-2)$.

For certain inequalities concerned with the invariant $\delta(n_1,\dots,n_k)$, which is a generalization of δ_M, we also refer to [6].

In this note, we prove the following obstruction to the Chen's basic equality.

Theorem 1.2. *Let M be an n-dimensional Riemannian manifold isometrically immersed in an m-dimensional Sasakian space form $\tilde{M}(c)$ of a constant φ-sectional curvature $c<1$ with the structure vector field ξ tangent to M. Then, M satisfies the Chen's basic equality*

$$\delta_M = \frac{n^2(n-2)}{2(n-1)}\|H\|^2 + \frac{1}{8}\{n(n-3)c+3n^2-n-8\}, \tag{1.5}$$

if and only if M is a 3-dimensional minimal invariant submanifold. Hence, Chen's basic equality (1.5) becomes

$$\delta_M = 2. \tag{1.6}$$

2. Proof of Theorem 1.2. First, we recall the following theorem [3].

Theorem 2.1. *Let $\tilde{M}$ be an m-dimensional Sasakian space form $\tilde{M}(c)$. Let M be an n-dimensional $(n \geq 3)$ submanifold isometrically immersed in $\tilde{M}$ such that $\xi \in TM$. For each plane section $\pi \subset \mathcal{D}_p$, $p \in M$,*

$$\begin{aligned}\tau - K(\pi) \leq{}& \frac{n^2(n-2)}{2(n-1)}\|H\|^2 + \frac{1}{8}\{n(n-3)c+3n^2-n-8\}\\ &+\frac{c-1}{8}\{3\|P\|^2-6\alpha(\pi)\}.\end{aligned} \tag{2.1}$$

The equality in (2.1) holds at $p \in M$ if and only if there exist an orthonormal basis $\{e_1,\dots,e_n\}$ of T_pM and an orthonormal basis $\{e_{n+1},\dots,e_m\}$ of $T_p^{\perp}M$ such that (a) $e_n=\xi$, (b) $\pi=\operatorname{Span}\{e_1,e_2\}$, *and* (c) *the shape operators $A_r \equiv A_{e_r}$, $r=n+1,\dots,m$, take the following forms:*

$$\begin{gathered}A_{n+1} = \begin{pmatrix} h_{11}^{n+1} & 0 & 0 & \cdots & 0\\ 0 & -h_{11}^{n+1} & 0 & \cdots & 0\\ 0 & 0 & 0 & \cdots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & \cdots & 0\end{pmatrix},\\ A_r = \begin{pmatrix} h_{11}^{r} & h_{12}^{r} & 0 & \cdots & 0\\ h_{12}^{r} & -h_{11}^{r} & 0 & \cdots & 0\\ 0 & 0 & 0 & \cdots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & \cdots & 0\end{pmatrix}, \quad r = n+2,\dots,m.\end{gathered} \tag{2.2}$$

A submanifold M of an almost contact metric manifold $\tilde{M}$ with $\xi \in TM$ is called a *semi-invariant submanifold* [1] of $\tilde{M}$ if the distributions $\mathcal{D}^1 = TM \cap \varphi(TM)$ and $\mathcal{D}^0 = TM \cap \varphi(T^{\perp}M)$ satisfy $TM = \mathcal{D}^1 \oplus \mathcal{D}^0 \oplus \{\xi\}$. In fact, the condition $TM = \mathcal{D}^1 \oplus \mathcal{D}^0 \oplus \{\xi\}$ implies that the endomorphism P is an *f-structure* [9] on M with a $\operatorname{rank}(P) = \dim(\mathcal{D}^1)$. A semi-invariant submanifold of an almost contact metric manifold becomes an *invariant* or an *anti-invariant submanifold* according as the anti-invariant distribution $\mathcal{D}^0$ is $\{0\}$ (i.e., $F = 0$) or the invariant distribution $\mathcal{D}^1$ is $\{0\}$ (i.e., $P = 0$) [1].

For each point $p \in M$, we put [3]

$$\delta_M^{\mathcal{D}}(p) = \tau(p) - \left(\inf{}_{\mathcal{D}} K\right)(p) = \inf\left\{K(\pi) \mid \text{ plane sections } \pi \subset \mathcal{D}_p\right\}. \tag{2.3}$$

For $c < 1$, we prove the following result.

Theorem 2.2. *Let M be an n-dimensional $(n \geq 3)$ submanifold isometrically immersed in a Sasakian space form $\tilde{M}(c)$ such that the structure vector field ξ is tangent to M. If $c < 1$, then*

$$\delta_M^{\mathcal{D}} \leq \frac{n^2(n-2)}{2(n-1)}\|H\|^2 + \frac{1}{8}\left\{n(n-3)c + 3n^2 - n - 8\right\}. \tag{2.4}$$

The equality case in (2.4) holds if and only if M is a 3-dimensional minimal invariant submanifold.

Proof. Since $c < 1$, in order to estimate δ_M, we minimize $\|P\|^2 - 2\alpha(\pi)$ in (2.1). For an orthonormal basis $\{e_1, \ldots, e_n = \xi\}$ of T_pM with $\pi = \operatorname{span}\{e_1, e_2\}$, we write

$$\|P\|^2 - 2\alpha(\pi) = \sum_{i,j=3}^{n} g\left(e_i, \varphi e_j\right)^2 + 2\sum_{j=3}^{n}\left\{g\left(e_1, \varphi e_j\right)^2 + g\left(e_2, \varphi e_j\right)^2\right\}. \tag{2.5}$$

Thus, the minimum value of $\|P\|^2 - 2\alpha(\pi)$ is 0, provided that

$$\operatorname{span}\left\{\varphi e_j \mid j = 3, \ldots, n\right\} \tag{2.6}$$

is orthogonal to the tangent space T_pM. Thus, we have (2.4) with equality case holding if and only if M is a semi-invariant such that $\operatorname{rank}(P) = 2$. This means that

$$TM = \mathcal{D}^1 \oplus \mathcal{D}^0 \oplus \{\xi\} \tag{2.7}$$

with the $\dim(\mathcal{D}^1) = 2$. From (2.2), we see that M is minimal.

Next, from [8, Proposition 5.2], we have

$$A_{FX}Y - A_{FY}X = \eta(X)Y - \eta(Y)X, \quad X, Y \in \mathfrak{D}^0 \oplus \{\xi\}. \tag{2.8}$$

For $X \in \mathfrak{D}^0$ and using (2.8), we have

$$g(X,X) = -g(A_{FX}\xi, X), \tag{2.9}$$

which in view of (2.2) becomes zero. Thus $\mathfrak{D}^0 = \{0\}$, and M becomes invariant. This completes the proof. □

From (1.2) and (2.3), it follows that $\delta_M^{\mathfrak{D}}(p) \leq \delta_M(p)$. Hence in view of Theorem 2.2, we get the proof of Theorem 1.2.

Remark 2.3. In Theorem 1.1, the phrase "M admits a quasi-anti-invariant structure of rank$(n-2)$" is identical with the statement "M is a semi-invariant submanifold with rank$(P) = 2$ or equivalently $\dim(\mathfrak{D}^1) = 2$, where $\mathfrak{D}^1$ is the invariant distribution." Thus, nothing is stated here about the dimension of the anti-invariant distribution $\mathfrak{D}^0$. But, in the proof of Theorem 2.2, we observe that M becomes minimal and consequently invariant, which makes $\dim(\mathfrak{D}^0) = 0$ and $\dim(M) = 3$.

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