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KY FAN INEQUALITY AND BOUNDS FOR DIFFERENCES OF MEANS

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We prove an equivalent relation between Ky Fan-type inequalities and certain bounds for the differences of means. We also generalize a result of Alzer et al. (2001).

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1. Introduction. Let $P_{n,r}(\mathbf{x})$ be the generalized weighted power means: $P_{n,r}(\mathbf{x}) = (\sum_{i=1}^{n}\omega_i x_i^r)^{1/r}$, where $\omega_i > 0$, $1 \le i \le n$ with $\sum_{i=1}^{n}\omega_i = 1$ and $\mathbf{x} = (x_1, x_2, \ldots, x_n)$. Here, $P_{n,0}(\mathbf{x}) = \prod_{i=1}^{n} x_i^{\omega_i}$ denotes the limit of $P_{n,r}(\mathbf{x})$ as $r \to 0^+$, which can be proved by noting that if $p(r) = \ln(\sum_{i=1}^{n}\omega_i x_i^r)$, then $p'(0) = \ln(\prod_{i=1}^{n} x_i^{\omega_i}) = \ln(P_{n,0}(\mathbf{x}))$. We write $P_{n,r}$ for $P_{n,r}(\mathbf{x})$ when there is no risk of confusion.

In this paper, we assume that $0 < x_1 \le x_2 \le \cdots \le x_n$. With any given $\mathbf{x}$, we associate $\mathbf{x}' = (1-x_1, 1-x_2, \ldots, 1-x_n)$ and write $A_n = P_{n,1}$, $G_n = P_{n,0}$, and $H_n = P_{n,-1}$. When $1 - x_i \ge 0$ for all i, we define $A_n' = P_{n,1}(\mathbf{x}')$ and similarly for G_n' and H_n'. We also let $\sigma_n = \sum_{i=1}^{n}\omega_i[x_i - A_n]^2$.

The following counterpart of the arithmetic mean-geometric mean inequality, due to Ky Fan, was first published by Beckenbach and Bellman [7].

Theorem 1.1. *For* $x_i \in (0, 1/2]$,

$$\frac{A_n'}{G_n'} \le \frac{A_n}{G_n} \tag{1.1}$$

with equality holding if and only if $x_1 = \cdots = x_n$.

In this paper, we consider the validity of the following additive Ky Fan-type inequalities (with $x_1 < x_n < 1$):

$$\frac{x_1}{1-x_1} < \frac{P_{n,r}' - P_{n,s}'}{P_{n,r} - P_{n,s}} < \frac{x_n}{1-x_n}. \tag{1.2}$$

Note that by a change of variables $x_i \to 1 - x_i$, the left-hand side inequality is equivalent to the right-hand side inequality in (1.2). We can deduce (see [9]) Theorem 1.1 from the case $r = 1$, $s = 0$, and $x_n \le 1/2$ in (1.2), which is a result of Alzer [5]. Gao [9] later proved the validity of (1.2) for $r = 1$, $-1 \le s < 1$, and $x_n \le 1/2$.

What is worth mentioning is a nice result of Mercer [12] who showed that the validity of $r = 1$ and $s = 0$ in (1.2) is a consequence of a result of Cartwright and Field [8] who established the validity of $r = 1$ and $s = 0$ for the following bounds for the differences between power means ($r > s$):

$$\frac{r-s}{2x_1}\sigma_n \ge P_{n,r} - P_{n,s} \ge \frac{r-s}{2x_n}\sigma_n, \tag{1.3}$$

where the constant $(r-s)/2$ is the best possible (see [10]).

We point out that inequalities (1.2) and (1.3) do not hold for all $r > s$. We refer the reader to the survey article [2] and the references therein for an account of Ky Fan's inequality, and to [4, 5, 10, 11] for other interesting refinements and extensions of (1.3).

Mercer's result reveals a close relation between (1.3) and (1.2), and it is our main goal in the paper to prove that the validities of (1.3) and (1.2) are equivalent for fixed r and s. As a consequence of this result, we give a characterization of the validity of (1.3) for $r = 1$ or $s = 1$. A solution of an open problem from [11] is also given.

Among the numerous sharpenings of Ky Fan's inequality in the literature, we have the following inequalities connecting the three classical means (with $\omega_i = 1/n$ here):

$$\left(\frac{H_n}{H'_n}\right)^{n-1}\frac{A_n}{A'_n} \le \left(\frac{G_n}{G'_n}\right)^{n} \le \left(\frac{A_n}{A'_n}\right)^{n-1}\frac{H_n}{H'_n}. \tag{1.4}$$

The right-hand side inequality of (1.4) is due to W. L. Wang and P. F. Wang [14] and the left-hand side inequality was recently proved by Alzer et al. [6].

It is natural to ask whether we can extend the above inequality to the weighted case, and using the same idea as in [6], we show that this is indeed true in Section 5.

2. The main theorem

Theorem 2.1. *For fixed $r > s$, the following inequalities are equivalent:* (i) *inequality (1.2) for $x_n \le 1/2$;* (ii) *inequality (1.2);* (iii) *inequality (1.3).*

Proof. (iii)⇒(ii) follows from a similar argument as given in [12], (ii)⇒(i) is trivial, so it suffices to show that (i)⇒(iii).

Fix $r > s$ assuming that (1.2) holds for $x_n \le 1/2$. Without loss of generality, we can assume that $x_1 < x_n$. For a given $\mathbf{x} = (x_1, x_2, \dots, x_n)$, let $\mathbf{y} = (\epsilon x_1, \epsilon x_2, \dots, \epsilon x_n)$. We can choose ϵ small so that $\epsilon x_n \le 1/2$. Now, applying the right-hand side inequality (1.2) for $\mathbf{y}$, we get

$$x_n\left(P_{n,r}(\mathbf{x}) - P_{n,s}(\mathbf{x})\right) > \frac{1-\epsilon x_n}{\epsilon^2}\left(P_{n,r}(\mathbf{y}') - P_{n,s}(\mathbf{y}')\right). \tag{2.1}$$

Let $f(\epsilon) = P_{n,r}(\mathbf{y}') - P_{n,s}(\mathbf{y}')$, then $f'(0) = 0$ and $f''(0) = (r-s)\sigma_n$. Thus, by letting ϵ tend to 0, it is easy to verify that the limit of the expression on the right-hand side of (2.1) is $(r-s)\sigma_n/2$. We can consider the left-hand side of (1.2) by a similar argument and this completes the proof. □

3. An application of Theorem 2.1

Lemma 3.1. *If inequality (1.3) holds for $r > s$, then $0 \le r+s \le 3$.*

Proof. Let $n = 2$, and write $\omega_1 = 1-q$, $\omega_2 = q$, $x_1 = 1$, and $x_2 = 1+t$ with $t \ge -1$. Let

$$D(t;r,s,q) = \frac{r-s}{2}\sum_{i=1}^{2} w_i\left[x_i - A_2\right]^2 - P_{2,r} + P_{2,s}. \tag{3.1}$$

For $t \ge 0$, $D(t;r,s,q) \ge 0$ implies the validity of the left-hand side inequality of (1.3) while for $-1 \le t \le 0$, $D(t;r,s,q) \le 0$ implies the validity of the right-hand side inequality of (1.3).

Using the Taylor series expansion of $D(t;r,s,q)$ around $t = 0$, it is readily seen that $D(0;r,s,q) = D^{(1)}(0;r,s,q) = D^{(2)}(0;r,s,q) = 0$. Thus, by the Lagrangian remainder term of the Taylor expansion,

$$D(t;r,s,q) = \frac{D^{(3)}(\theta t;r,s,q)}{3!}t^3 \tag{3.2}$$

with $0 < \theta < 1$.

Since

$$\lim_{t\to 0^+} D^{(3)}(\theta t;r,s,q) = D^{(3)}(0;r,s,q), \tag{3.3}$$

a necessary condition for (1.3) to hold is $D^{(3)}(0;r,s,q) \ge 0$ for $0 \le q \le 1$. The calculation yields

$$D^{(3)}(0;r,s,q) = (r-s)q(q-1)\big((3-2r-2s)q - (3-r-s)\big). \tag{3.4}$$

It is easy to check that this is equivalent to $0 \le r+s \le 3$. □

Theorem 3.2. *Let $r > s$. If $r = 1$, inequality (1.3) holds if and only if $-1 \le s < 1$. If $s = 1$, inequality (1.3) holds if and only if $1 < r \le 2$.*

Proof. A result of Gao [9] shows the validity of (1.2) for $r = 1$, $-1 \le s < 1$, $x_n \le 1/2$, and a similar result of his [10] shows the validity of (1.2) for $s = 1$, $1 < r \le 2$, $x_n \le 1/2$. Thus, it follows from Theorem 2.1 that (1.3) holds for $r = 1$, $-1 \le s < 1$, and $s = 1$, $1 < r \le 2$. This proves the "if" part of the statement, and the "only if" part follows from the previous lemma. □

We note here that a special case of Theorem 3.2 answers an open problem of Mercer [11], namely, we have shown that

$$\frac{1}{x_1}\sigma_n \geq A_n - H_n \geq \frac{1}{x_n}\sigma_n. \tag{3.5}$$

4. Two lemmas

Lemma 4.1. *Let x, b, u, and v be real numbers with $0 < x \leq b$, $u \geq 1$, $v \geq 0$, and $u+v \geq 2$, then $f(u,v,x,b) \leq 0$, where*

$$f(u,v,x,b) = \frac{u+v-1}{ux+vb} + \frac{1}{x^2(u/x+v/b)} - \frac{1}{x} - \frac{u+v-2}{b^2(u+v)^2}v(x-b) \tag{4.1}$$

with equality holding if and only if $x = b$ or $v = 0$ or $u = v = 1$.

Proof. Let $x < b$, $u > 1$, and $v > 1$. We have

$$\begin{aligned} f(u,v,x,b) &= v(b-x)\left(-\frac{(u-1)b+(v-1)x}{x(bv+ux)(bu+vx)} + \frac{(u-1)+(v-1)}{b^2(u+v)^2}\right) \\ &< \frac{v(b-x)}{xb^2(u+v)^2}\left[\left((u-1)+(v-1)\right)x-(u-1)b-(v-1)x\right] \\ &= -\frac{v(u-1)(b-x)^2}{xb^2(u+v)^2} < 0 \end{aligned} \tag{4.2}$$

since $b^2(u+v)^2 > (bv+ux)(bu+vx)$. Thus, we conclude that $f(u,v,x,b) \leq 0$ for $0 < x \leq b$, $u \geq 1$, $v \geq 0$, and $u+v \geq 2$. □

Lemma 4.2. *Let x, a, b, u, v, and s be real numbers with $0 < x \leq a \leq b$, $u \geq 1$, $v \geq 1$, $u+v \geq 3$, and $0 \leq s \leq v$, then*

$$\begin{aligned} &\frac{u+v-1}{ux+sa+(v-s)b} + \frac{1}{x^2(u/x+s/a+(v-s)/b)} - \frac{1}{x} \\ &\quad - \frac{u+v-2}{b^2(u+v)^2}\left(s(x-a)+(v-s)(x-b)\right) \leq 0 \end{aligned} \tag{4.3}$$

with equality holding if and only if one of the following cases is true: (1) $x = a = b$; (2) $s = 0$ *and* $x = b$; (3) $s = v$ *and* $x = a$.

Proof. Let $M = \{(s,a) \in R^2 \mid 0 \leq s \leq v,\ x \leq a \leq b\}$. Furthermore, we define $H(s,a)$ as the expression on the left-hand side of (4.3), where $(s,a) \in M$. It suffices to show that $H(s,a) < 0$. We denote the absolute minimum of H by $m = (s_0,a_0)$. If m is an interior point of M, then we obtain

$$0 = \frac{1}{s}\frac{\partial H}{\partial a} - \frac{1}{a-b}\frac{\partial H}{\partial s}\bigg|_{(s,a)=(s_0,a_0)} = \frac{b-a}{x^4a^2b\left(u/x+s/a+(v-s)/b\right)^2} > 0. \tag{4.4}$$

Hence, m is a boundary point of M, so we get

$$m \in \{(s_0,x),(s_0,b),(0,a_0),(v,a_0)\}. \tag{4.5}$$

Using Lemma 4.1, we obtain

$$\begin{aligned} H(s_0,x) &= f(u+s_0,v-s_0,x,b) \le 0,\\ H(s_0,b) &= H(0,a_0) = f(u,v,x,b) \le 0,\\ H(v,a_0) &= f(u,v,x,a_0) - \frac{v(u+v-2)(a_0-x)(b^2-a_0^2)}{a_0^2b^2(u+v)^2} \le 0. \end{aligned} \tag{4.6}$$

Thus, we get that if $(s,a) \in M$, then $H(s,a) \le 0$. The conditions for equality can be easily checked using Lemma 4.1. □

5. A sharpening of Ky Fan's inequality. In this section, we prove the following theorem.

Theorem 5.1. *For* $0 < x_1 \le \cdots \le x_n$, $q = \min\{\omega_i\}$,

$$\frac{1-2q}{2x_1^2}\sigma_n \ge (1-q)\ln A_n + q\ln H_n - \ln G_n \ge \frac{1-2q}{2x_n^2}\sigma_n, \tag{5.1}$$

$$\frac{1-2q}{2x_1^2}\sigma_n \ge \ln G_n - q\ln A_n - (1-q)\ln H_n \ge \frac{1-2q}{2x_n^2}\sigma_n \tag{5.2}$$

with equality holding if and only if $q = 1/2$ *or* $x_1 = \cdots = x_n$.

Proof. The proof uses the ideas in [6]. We prove the right-hand side inequality of (5.1); the proofs for other inequalities are similar. Fix $0 < x = x_1$, $x_n = b$ with $x_1 < x_n$, $n \ge 2$; we define

$$f_n(\mathbf{x}_n,q) = (1-q)\ln A_n + q\ln H_n - \ln G_n - \frac{1-2q}{2x_n^2}\sigma_n, \tag{5.3}$$

where we regard A_n, G_n, and H_n as functions of $\mathbf{x}_n = (x_1,\dots,x_n)$.

We then have

$$g_n(x_2,\dots,x_{n-1}) := \frac{1}{\omega_1}\frac{\partial f_n}{\partial x_1} = \frac{1-q}{A_n} + \frac{qH_n}{x_1^2} - \frac{1}{x_1} - \frac{1-2q}{x_n^2}(x_1 - A_n). \tag{5.4}$$

We want to show that $g_n \le 0$. Let $D = \{(x_2,\dots,x_{n-1}) \in R^{n-2} \mid 0 < x \le x_2 \le \cdots \le x_{n-1} \le b\}$. Let $\mathbf{a} = (a_2,\dots,a_{n-1}) \in D$ be the point in which the absolute minimum of g_n is reached. Next, we show that

$$\mathbf{a} = (x,\dots,x,a,\dots,a,b,\dots,b) \quad \text{with } x < a < b, \tag{5.5}$$

where the numbers x, a, and b appear r, s, and t times, respectively, with $r,s,t \ge 0$ and $r+s+t = n-2$.

Suppose not, this implies that two components of **a** have different values and are interior points of D. We denote these values by a_k and a_l. Partial differentiation leads to

$$\frac{B}{a_i^2}+C=0 \tag{5.6}$$

for $i=k,l$, where

$$B=q\frac{H_n^2}{x_1^2}, \qquad C=-\frac{1-q}{A_n^2}+\frac{1-2q}{x_n^2}. \tag{5.7}$$

Since $z\mapsto B/z^2+C$ is strictly monotonic for $z>0$, then (5.6) yields $a_k=a_l$. This contradicts our assumption that $a_k\neq a_l$. Thus, (5.5) is valid and it suffices to show that $g_n\leq 0$ for the case $n=2,3$.

When $n=2$, by setting $x_1=x$, $x_2=b$, $\omega_1/q=u$, and $\omega_2/q=v$, we can identify g_2 as (4.1), and the result follows from Lemma 4.1.

When $n=3$, by setting $x_1=x$, $x_2=a$, $x_3=b$, $\omega_1/q=u$, $\omega_2/q=s$, and $\omega_3/q=v-s$, we can identify g_3 as (4.3), and the result follows from Lemma 4.2.

Thus, we have shown that $g_n=(1/\omega_1)\partial f_n/\partial x_1\leq 0$ with equality holding if and only if $n=1$ or $n=2$, $q=1/2$. By letting x_1 tend to x_2, we have

$$f_n(\mathbf{x}_n,q)\geq f_{n-1}(\mathbf{x}_{n-1},q)\geq f_{n-1}(\mathbf{x}_{n-1},q'), \tag{5.8}$$

where $\mathbf{x}_{n-1}=(x_2,\dots,x_n)$ with weights $\omega_1+\omega_2,\dots,\omega_{n-1},\omega_n$ and $q'=\min\{\omega_1+\omega_2,\dots,\omega_n\}$. Here, we have used the following inequality, which is a consequence of (3.5) (see [9]):

$$\ln A_n-\ln H_n\geq\frac{1}{x_n^2}\sigma_n. \tag{5.9}$$

It then follows by induction that $f_n\geq f_{n-1}\geq\cdots\geq f_2=0$ when $q=1/2$ in f_2 or else $f_n\geq f_{n-1}\geq\cdots\geq f_1=0$, and this completes the proof. □

We note that the above theorem gives a sharpening of Sierpiński's inequality [13], originally stated for the unweighted case ($\omega_i=1/n$) as

$$H_n^{n-1}A_n\leq G_n\leq A_n^{n-1}H_n. \tag{5.10}$$

The following corollary gives refinements of (1.4).

Corollary 5.2. *For* $0 < x_1 \le \cdots \le x_n < 1$, $q = \min\{\omega_i\}$,

$$\begin{gathered} \left(\frac{A_n'^{(1-q)}H_n'^{q}}{G_n'}\right)^{(1-x_1)^2/x_1^2} \ge \frac{A_n^{1-q}H_n^{q}}{G_n} \ge \left(\frac{A_n'^{(1-q)}H_n'^{q}}{G_n'}\right)^{(1-x_n)^2/x_n^2}, \\ \left(\frac{G_n'}{A_n'^{q}H_n'^{(1-q)}}\right)^{(1-x_1)^2/x_1^2} \ge \frac{G_n}{A_n^{q}H_n^{1-q}} \ge \left(\frac{G_n'}{A_n'^{q}H_n'^{(1-q)}}\right)^{(1-x_n)^2/x_n^2}, \end{gathered} \tag{5.11}$$

with equality holding if and only if $x_1 = x_2 = \cdots = x_n$ *or* $q = 1/2$.

Proof. This is a direct consequence of Theorem 5.1, following from a similar argument as in [12]. □

6. Concluding remarks.

We note that if for $x_n \le 1/2$, we have

$$\left(\frac{x_1}{1-x_1}\right)^{\beta} < \frac{P'_{n,r}-P'_{n,s}}{P_{n,r}-P_{n,s}} < \left(\frac{x_n}{1-x_n}\right)^{\alpha}, \tag{6.1}$$

then $\beta \ge 1$ and $\alpha \le 1$; otherwise, by letting ϵ tend to 0 in (2.1), we get contradictions.

It was conjectured that an additive companion of (1.4) is true (see [1])

$$n(G_n - G_n') \le (n-1)(A_n - A_n') + H_n - H_n'. \tag{6.2}$$

In [3], Alzer asked if the above conjecture is true and whether there exists a weighted version. Based on what we have got in this paper, it is natural to give the following conjecture of the weighed version of (6.2).

Conjecture 6.1. *For* $0 < x_1 \le \cdots \le x_n \le 1/2$ *and* $q = \min\{\omega_i\}$,

$$G_n - G_n' \le (1-q)(A_n - A_n') + q(H_n - H_n'). \tag{6.3}$$

Recently, Alzer et al. [6] asked the following question: what is the largest number $\alpha = \alpha(n)$ and what is the smallest number $\beta = \beta(n)$ such that

$$\alpha(A_n - A_n') + (1-\alpha)(H_n - H_n') \le G_n - G_n' \le \beta(A_n - A_n') + (1-\beta)(H_n - H_n') \tag{6.4}$$

for all $x_i \in (0, 1/2]$ $(i = 1, \ldots, n)$?

We note here that $\alpha \le 0$ since the left-hand side inequality above can be written as

$$\alpha A_n + (1-\alpha)H_n - G_n \le \alpha A_n' + (1-\alpha)H_n' - G_n'. \tag{6.5}$$

By a similar argument as in the proof of Theorem 2.1, replacing $(x_1,\dots,x_n)$ by $(\epsilon x_1,\dots,\epsilon x_n)$ and letting ϵ tend to 0 in (6.5), we find that (6.5) implies that

$$\alpha A_n + (1-\alpha)H_n - G_n \le 0 \tag{6.6}$$

for any $\mathbf{x}$. If we further let x_1 tend to 0 in (6.6), we get

$$\alpha A_n \le 0 \tag{6.7}$$

which implies that $\alpha \le 0$.

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