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DUAL INTEGRAL EQUATIONS—REVISITED

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Dual integral equations with trigonometric kernel are reinvestigated here for a solution. The behaviour of one of the integrals at the end points of the interval complementary to the one in which it is defined plays the key role in determining the solution of the dual integral equations. The solution of the dual integral equations is then applied to find an exact solution of the water wave scattering problems.

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1. Introduction. Boundary value problems with mixed boundary conditions arising in different branches of mathematical physics can be reduced to dual integral equations. A mixed boundary condition is the one in which one condition is prescribed at one part of the boundary while some other condition is prescribed at the remaining part of the boundary. The solution of the dual integral equations essentially depends on the behaviour of one of the integrals at the end points of the interval complementary to the one in which it is defined [1, 4]. This behaviour is dictated by the physics of the problem.

In the present paper, we consider the following dual integral equations:

$$\begin{aligned}\int_0^\infty A_j(k)L(k,y)\,dk &= -R_j\exp(-Ky), \quad y\in G_j,\\ \int_0^\infty kA_j(k)L(k,y)\,dk &= iK(1-R_j)\exp(-Ky), \quad y\in B_j,\end{aligned} \tag{1.1}$$

where

$$\begin{aligned}L(k,y) &= k\cos ky - K\sin ky,\\ G_j &= (0,\infty)-B_j,\end{aligned} \tag{1.2}$$

$A(k)$ is an unknown function, and R is an unknown constant. This integral equation arises in the well-known problem of scattering water waves by a vertical barrier under the assumption of linearised theory [5, 6, 7, 8]. The vertical barrier may be (i) partially immersed in deep water, (ii) completely submerged and extending infinitely downwards in deep water, (iii) a vertical wall with a gap, or (iv) a submerged plate. The solution of (1.1) has been obtained here by noting the behaviour of the second equation of (1.1) at the end points of the interval G_j, which can be determined from physical consideration. Equation

(1.1) was then reduced to a singular integral equation whose kernel involves Cauchy and logarithmic type singularity. The solution of this singular integral equation is known (cf. [3, 4, 6, 8]). The solution of (1.1) was then obtained by utilizing the solution of aforesaid singular integral equation. Knowing the solution of (1.1), the solution of the corresponding scattering problems was obtained in a closed form. In Section 2, we consider the genesis of dual integral equation (1.1), and in Section 3, we find the solution of (1.1) and hence the solution of the corresponding scattering problems.

2. Genesis of the dual integral equations. The two-dimensional problem of the scattering of surface waves by a vertical barrier present in deep water under the assumption of linearised theory consists in solving mixed two-dimensional boundary value problem given as follows: ϕ_j satisfies

$$\nabla^2\phi_j = 0 \quad \text{in } -\infty < x < \infty,\ y \ge 0, \tag{2.1}$$

the free surface condition

$$K\phi_j + \phi_{jy} = 0 \quad \text{on } y = 0,\ K = \frac{\sigma^2}{g},\ \text{a constant}, \tag{2.2}$$

the condition on the barrier,

$$\frac{\partial \phi_j}{\partial x} = 0 \quad \text{on } x = 0\ y \in B_j,\ j = 1,2,3,4. \tag{2.3}$$

Here, B_j represents the vertical barrier. (i) For $j = 1$, the barrier is partially immersed to a depth a_1 below the mean free surface $y = 0$ so that $B_1 = (0, a_1)$. (ii) For $j = 2$, the vertical barrier is completely submerged and extends infinitely downwards, so $B_2 = (a_2, \infty)$. (iii) For $j = 3$, the vertical barrier is in the form of a wall with a gap, so $B_3 = (0, a_3) + (a_4, \infty)$. (iv) For $j = 4$, the barrier is in the form of a plate submerged in deep water, so $B_4 = (a_5, a_6)$. The bottom condition is given by

$$\nabla\phi_j \longrightarrow 0 \quad \text{as } y \longrightarrow \infty. \tag{2.4}$$

At the sharp edges of the barrier, we must have

$$r^{1/2}\nabla\phi_j \quad \text{bounded as } r \longrightarrow 0, \tag{2.5}$$

where r denotes the distance from sharp edges a_j of the barrier, $j = 1, \dots, 6$

$$\phi_j \sim \begin{cases} R_j \exp(-Ky - iKx) + \exp(-Ky + iKx) & \text{as } x \longrightarrow -\infty, \\ T_j \exp(-Ky + iKx) & \text{as } x \longrightarrow \infty, \end{cases} \tag{2.6}$$

where T_j, R_j are unknown complex constant. The function ϕ_j, $j = 1,2,3,4$, represents the velocity potential for two-dimensional irrotational motion corresponding to various scattering problems. The function $\exp(-Ky + iKx)$ (dropping the time dependent factor $\exp(-i\sigma t)$ where σ is the circular frequency $K = \sigma^2/g$, g being acceleration due to gravity) represents the wave propagating from the negative x-direction incident upon the barrier B_j. The complex constants R_j and T_j are the reflection and transmission coefficients, respectively.

By Havelock expansion of water wave potential, a suitable representation of ϕ_j satisfying (2.1), (2.2), (2.4), and (2.6) is

$$\phi_j = \begin{cases} R_j \exp(-Ky - iKx) + \exp(-Ky + iKx) \\ \quad + \displaystyle\int_0^\infty B_j(k) L(k,y) \exp(kx)\, dk, & x < 0, \\ T_j \exp(-Ky + iKx) + \displaystyle\int_0^\infty A_j(k) L(k,y) \exp(-kx)\, dk, & x > 0, \end{cases} \tag{2.7}$$

where (cf. [8])

$$T_j + R_j = 1, \qquad A_j(k) = -B_j(k). \tag{2.8}$$

By condition (2.3), using (2.7) we have

$$\int_0^\infty k A_j(k) L(k,y)\, dk = iK(1 - R_j) \exp(-ky), \quad y \in B_j. \tag{2.9}$$

Also, ϕ_j is continuous across the gap G_j below/above/between the barrier so that

$$\phi_j(+0,y) = \phi_j(-0,y), \quad y \in G_j. \tag{2.10}$$

Using (2.7), we have

$$\int_0^\infty A_j(k) L(k,y)\, dk = R_j \exp(-ky), \quad y \in G_j. \tag{2.11}$$

Here, $G_1 = (a_1,\infty)$, $G_2 = (0,a_2)$, $G_3 = (a_3,a_4)$, and $G_4 = (0,a_5) + (a_6,\infty)$. Equations (2.9) and (2.11) give the required integral equations. In the following section, we determine the solution of (1.1).

3. The solution of (1.1).

Let

$$iK(1 - R_j) \exp(-Ky) - \int_0^\infty k A_j(k) L(k,y)\, dk = \begin{cases} 0, & y \in B_j, \\ h_j(y), & y \in G_j, \end{cases} \tag{3.1}$$

where $h_j(y)$ is the unknown function. In view of (2.9), (2.3), and (2.4),

$$h_1(y) \sim \begin{cases} O\left(|y-a_1|^{-1/2}\right) & \text{as } y \longrightarrow a_1, \\ \longrightarrow 0 & \text{as } y \longrightarrow \infty, \end{cases} \tag{3.2}$$

$$h_2(y) \sim \begin{cases} O\left(|y-a_2|^{-1/2}\right) & \text{as } y \longrightarrow a_2, \\ \text{bounded} & \text{as } y \longrightarrow 0, \end{cases} \tag{3.3}$$

$$h_3(y) \sim \left\{ O\left(|y-a_i|^{-1/2}\right) \quad \text{as } y \longrightarrow a_i,\ i = 3,4, \right. \tag{3.4}$$

$$h_4(y) \sim \begin{cases} O\left(|y-a_i|^{-1/2}\right) & \text{as } y \longrightarrow a_i,\ i = 5,6, \\ \longrightarrow 0 & \text{as } y \longrightarrow \infty, \\ \text{bounded} & \text{as } y \longrightarrow 0. \end{cases} \tag{3.5}$$

By Havelocks' expansion theorem [8], we have from (3.1)

$$i(1-R_j) = 2\int_{G_j} h_j(t)\exp(-Kt)\,dt, \tag{3.6}$$

$$kA_j(k) = \frac{2}{\pi}\frac{1}{K^2+k^2}\int_{G_j} h_j(t)L(k,t)\,dt. \tag{3.7}$$

Substituting $A_j(k)$ from (3.7) into (2.11), we have

$$\frac{2}{\pi}\int_{G_j} h_j(t)\int_0^\infty \frac{L(k,t)L(k,y)}{k(K^2+k^2)}\,dk\,dt = R_j\exp(-Ky), \quad y \in G_j. \tag{3.8}$$

Simplifying (3.8) and applying $(d/dy+K)$, we have

$$\int_{G_j} h_j(t)\left[K\ln\left|\frac{y-t}{y+t}\right| + \frac{1}{y+t} + \frac{1}{y-t}\right]dt = 0, \quad y \in G_j. \tag{3.9}$$

This is a singular integral equation in $h_j(t)$, whose kernel involves a combination of Cauchy type and logarithmic singularity. An appropriate solution of (3.9) can be obtained by considering the behaviour of $h_j(t)$ at the end points of G_j, which is given in (3.2), (3.3), (3.4), and (3.5) for various configurations of the barrier. Hence (3.6) and (3.7) show that the behaviour of $h_j(t)$ at the end points of G_j plays the key role in determining the solution of (1.1).

Now, considering (3.2), (3.3), (3.4), and (3.5), we find $h_j(t)$ for $j = 1,2,3,4$ and hence $A_j(k)$ and R_j for $j = 1,2,3,4$.

(1) Knowing (3.2), $h_1(t)$ is given by (cf. [8])

$$h_1(t) = C_1 \frac{d}{dy}\left\{\exp(-ky)\int_a^y \frac{t\exp(Kt)}{(t^2-a^2)^{1/2}}dt\right\}, \quad y \in G_1, \tag{3.10}$$

where C_1 is a constant. Substituting $h_1(t)$ in (3.6) and (3.7), we have

$$A_1(k) = \frac{-a_1C_1}{K^2+k^2}J_1(ka), \qquad R_1 = 1 + ia_1C_1K_1(Ka). \tag{3.11}$$

To find C_1, $A_1(k)$ and R_1 are substituted in the first equation of (1.1) to get

$$C_1 = \frac{1}{a_1\triangle_1}, \quad \triangle_1 = \pi I_1(Ka_1) - iK_1(Ka_1). \tag{3.12}$$

So that

$$A_1(k) = -\frac{J_1(ka_1)}{\triangle_1(K^2+k^2)}, \qquad R = \frac{\pi I_1(ka_1)}{\triangle_1}. \tag{3.13}$$

(2) For $j = 2$,

$$h_2(y) = C_2\frac{d}{dy}\left\{\exp(-ky)\int_b^y \frac{\exp(Kv)}{(b^2-v^2)^{1/2}}dv\right\} \quad \text{(cf. [6])}, \tag{3.14}$$

where C_2 is a constant. Substituting in (3.6) and (3.7)

$$A_2(k) = \frac{-C_2}{K^2+k^2}J_0(ka_2), \qquad R_2 = 1 + i\pi C_2 I_0(Ka_2). \tag{3.15}$$

The constant C_2 is determined by substituting $A_2(k)$, R_2 in first equation of (1.1). On simplification, this gives

$$C_2 = -\frac{1}{K_0(Ka_2) + i\pi I_0(Ka_2)}. \tag{3.16}$$

(3) For $j = 3$ (cf. [3]),

$$h_3(y) = \frac{d}{dy}\exp(-Ky)\int_{a_4}^{y} C_3\exp(Ku)\lambda(u)\,du, \tag{3.17}$$

where

$$\lambda(u) = \frac{u}{R(u)}\left\{\delta - \frac{2}{\pi}F_1(a_3,a_4,u)\right\}, \tag{3.18}$$

C_3 is a constant,

$$
\begin{gathered}
F_1(a_3,a_4,u) = \int_0^{a_3} \frac{R(v)}{v^2-u^2}\,dv, \\
R(u) = |a_3^2-u^2|^{1/2}|a_4^2-u^2|^{1/2}, \\
\delta = \frac{K^{-1}\exp(Ka)+(2/\pi)\alpha_2(-K,F_1)}{\alpha_2(-K)}, \\
\alpha_i(K) = \alpha_i(K,1), \qquad \alpha_i(K,F_1) = \int_{t_i} \frac{uF_1(a_3,a_4,u)}{R(u)}\,du, \\
t_i = \begin{cases} (-a_3,a_3), & i=1, \\ (a_3,a_4), & i=2, \\ (a_4,\infty), & i=3, \end{cases}
\end{gathered}
\tag{3.19}
$$

and hence (3.6) and (3.7) give

$$
\begin{gathered}
A_3(k) = \frac{2}{\pi}\frac{C_3}{k(K^2+k^2)}\left\{-\sin ka + k\int_{a_3}^{a_4} \lambda(u)\cos ku\,du\right\}, \\
R_3 = C_3 I, \\
I = \delta\{\alpha_1(K)-\alpha_3(K)\} - \frac{2}{\pi}\{\alpha_1(K,F_1)-\alpha_3(K,F_1)\}.
\end{gathered}
\tag{3.20}
$$

To find C_3, substitute $A_3(k)$ and R_3 in the first equation of (1.1) to get

$$
C_3 = \frac{i}{J+iI}, \tag{3.21}
$$

where

$$
J = K^{-1}\exp(ka) + \delta\alpha_2(K) - \alpha_2(K,F_1). \tag{3.22}
$$

(4) For $j=4$ (cf. [2]),

$$
h_4(y) = \begin{cases} \dfrac{d}{dy}\left\{\exp(-Ky)\displaystyle\int_{a_5}^{y} \exp(Ku)P(u)\,du\right\}, & y < a_5, \\ \dfrac{d}{dy}\left\{-\exp(-Ky)\displaystyle\int_{a_6}^{y} \exp(Ku)P(u)\,du\right\}, & y < a_6, \end{cases} \tag{3.23}
$$

where

$$
P(u) = \frac{C_4}{R_0(u)}(d_0^2-u^2), \tag{3.24}
$$

C_4 and d_0^2 are constants,

$$
R_0(u) = |u^2-a_5^2|^{1/2}|u^2-a_6^2|^{1/2}, \tag{3.25}
$$

and (3.6) and (3.7) give

$$A_4(k) = \frac{J(k)}{K^2+k^2}C_4, \quad J(k) = \int_a^b \frac{(d_0^2-u^2)}{R_0(u)}\sin ku\,du, \tag{3.26}$$

$$R_4 = 1 - iC_4(\alpha_0-\beta_0). \tag{3.27}$$

To determine C_4 and d_0^2, we substitute $A_4(k)$ in the first equation of (1.1) to get the relations

$$-R_4 = C_4\gamma_0, \tag{3.28}$$

$$-R_4 = C_4\left\{\gamma_0 - \int_{a_5}^{a_6}\frac{(d_0^2-x^2)}{R_0(x)}\exp(Kx)dx\right\}, \tag{3.29}$$

which yield

$$\int_{a_5}^{a_6}\frac{(d_0^2-x^2)}{R_0(x)}\exp(Kx)dx = 0. \tag{3.30}$$

This determines d_0^2. Equating (3.26) and (3.28), we have

$$C_4 = \frac{i}{\triangle_4}, \quad \triangle_4 = \alpha_0-\beta_0-i\gamma_0, \tag{3.31}$$

where

$$\begin{aligned} \alpha_0 &= \int_{a_{-5}}^{a_5}\frac{(d_0^2-x^2)}{R_0(x)}\exp(Kx)dx, \\ \beta_0 &= \int_{a_6}^{\infty}\frac{(d_0^2-x^2)}{R_0(x)}\exp(Kx)dx, \\ \gamma_0 &= \int_{a_5}^{a_6}\frac{(d_0^2-x^2)}{R_0(x)}\exp(Kx)dx. \end{aligned} \tag{3.32}$$

Thus, knowing $A_j(k)$ and R_j, the corresponding $\phi_j(x,y)$ for $j=1,2,3,4$ are known from (2.7).

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