

BOUNDEDNESS FOR MULTILINEAR MARCINKIEWICZ OPERATORS ON CERTAIN HARDY SPACES

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The boundedness for the multilinear Marcinkiewicz operators on certain Hardy and Herz-Hardy spaces are obtained.

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1. Introduction and definitions. Suppose that S^{n-1} is the unit sphere of \mathbb{R}^n ($n \geq 2$) equipped with normalized Lebesgue measure $d\sigma = d\sigma(x')$. Let Ω be homogeneous of degree zero and satisfy the following two conditions:

- (i) $\Omega(x)$ is continuous on S^{n-1} and satisfies the Lip_γ condition on S^{n-1} ($0 \leq \gamma \leq 1$), that is,

$$|\Omega(x') - \Omega(y')| \leq M|x' - y'|^\gamma, \quad x', y' \in S^{n-1}; \quad (1.1)$$

- (ii) $\int_{S^{n-1}} \Omega(x') d\sigma = 0$.

Let m be a positive integer and A be a function on \mathbb{R}^n . The multilinear Marcinkiewicz integral operator is defined by

$$\mu_\Omega^A(f)(x) = \left[\int_0^\infty |F_t^A(f)(x)|^2 \frac{dt}{t^3} \right]^{1/2}, \quad (1.2)$$

where

$$F_t^A(f)(x) = \int_{|x-y| \leq t} \frac{\Omega(x-y)}{|x-y|^{n-1}} \frac{R_{m+1}(A; x, y)}{|x-y|^m} f(y) dy, \quad (1.3)$$

$$R_{m+1}(A; x, y) = A(x) - \sum_{|\alpha| \leq m} \frac{1}{\alpha!} D^\alpha A(y) (x-y)^\beta.$$

We denote that $F_t(f)(x) = \int_{|x-y| \leq t} \Omega(x-y)/|x-y|^{n-1} f(y) dy$. We also denote that

$$\mu_\Omega(f)(x) = \left(\int_0^\infty |F_t(f)(x)|^2 \frac{dt}{t^3} \right)^{1/2}, \quad (1.4)$$

which is the Marcinkiewicz integral operator (see [5, 6, 12]).

Note that when $m = 0$, μ_Ω^A is just the commutator of Marcinkiewicz operator (see [5, 12]). It is well known that multilinear operators are of great interest in harmonic analysis and have been widely studied by many authors (see [1, 2, 3, 4, 5]). The main purpose of this paper is to consider the continuity of the multilinear Marcinkiewicz operators on certain Hardy and Herz-Hardy spaces. We first introduce some definitions (see [7, 8, 9, 10, 11]).

DEFINITION 1.1. Let A be a function on \mathbb{R}^n , m a positive integer, and $0 < p \leq 1$. A bounded measurable function a on \mathbb{R}^n is said to be a $(p, D^m A)$ -atom if

- (i) $\text{supp } a \subset B = B(x_0, r)$,
- (ii) $\|a\|_{L^\infty} \leq |B|^{-1/p}$,
- (iii) $\int a(y) dy = \int a(y) D^\alpha A(y) dy = 0$, $|\alpha| = m$.

A temperate distribution f is said to belong to $H_{D^m A}^p(\mathbb{R}^n)$, if, in the Schwartz distributional sense, it can be written as

$$f(x) = \sum_{j=0}^{\infty} \lambda_j a_j(x), \quad (1.5)$$

where a_j 's are $(p, D^m A)$ -atoms, $\lambda_j \in \mathbb{C}$, and $\sum_{j=0}^{\infty} |\lambda_j|^p < \infty$. Moreover, $\|f\|_{H_{D^m A}^p(\mathbb{R}^n)} \sim (\sum_{j=0}^{\infty} |\lambda_j|^p)^{1/p}$.

Let $B_k = \{x \in \mathbb{R}^n : |x| \leq 2^k\}$, $C_k = B_k \setminus B_{k-1}$, $k \in \mathbb{Z}$, and $m_k(\lambda, f) = |\{x \in C_k : |f(x)| > \lambda\}|$; for $k \in \mathbb{N}$, let $\tilde{m}_k(\lambda, f) = m_k(\lambda, f)$ and $\tilde{m}_0(\lambda, f) = |\{x \in B_0 : |f(x)| > \lambda\}|$.

DEFINITION 1.2. Let $0 < p, q < \infty$, and $\alpha \in \mathbb{R}$.

(1) The homogeneous Herz space is defined by

$$\dot{K}_q^{\alpha, p} = \left\{ f \in L_{\text{loc}}^q(\mathbb{R}^n \setminus \{0\}) : \|f\|_{\dot{K}_q^{\alpha, p}(\mathbb{R}^n)} < \infty \right\}, \quad (1.6)$$

where

$$\|f\|_{\dot{K}_q^{\alpha, p}(\mathbb{R}^n)} = \left[\sum_{k=-\infty}^{\infty} 2^{k\alpha p} \|f \chi_k\|_{L^q}^p \right]^{1/p}. \quad (1.7)$$

(2) The nonhomogeneous Herz space is defined by

$$K_q^{\alpha, p}(\mathbb{R}^n) = \left\{ f \in L_{\text{loc}}^q(\mathbb{R}^n) : \|f\|_{K_q^{\alpha, p}(\mathbb{R}^n)} < \infty \right\}, \quad (1.8)$$

where

$$\|f\|_{K_q^{\alpha, p}(\mathbb{R}^n)} = \left[\sum_{k=1}^{\infty} 2^{k\alpha p} \|f \chi_k\|_{L^q}^p + \|f \chi_{B_0}\|_{L^q}^p \right]^{1/p}, \quad (1.9)$$

where

$$\|f\|_{WK_q^{\alpha,p}(\mathbb{R}^n)} = \sup_{\lambda>0} \lambda \left[\sum_{k=0}^{\infty} 2^{k\alpha p} \tilde{m}_k(\lambda, f)^{p/q} \right]^{1/p}. \tag{1.10}$$

DEFINITION 1.3. Let m be a positive integer and A a function on \mathbb{R}^n , $\alpha \in \mathbb{R}$, and $1 < q \leq \infty$. A function $a(x)$ on \mathbb{R}^n is called a central $(\alpha, q, D^m A)$ -atom (or a central $(\alpha, q, D^m A)$ -atom of restrict type), if

- (1) $\text{supp } a \subset B(0, r)$ for some $r > 0$ (or for some $r \geq 1$),
- (2) $\|a\|_{L^q} \leq |B(0, r)|^{-\alpha/n}$,
- (3) $\int a(x) dx = \int a(x) D^\beta A(x) dx = 0$, $|\beta| = m$.

A temperate distribution f is said to belong to $HK_{q,D^m A}^{\alpha,p}(\mathbb{R}^n)$ (or $HK_{q,D^m A}^{\alpha,p}(\mathbb{R}^n)$) if it can be written as $f = \sum_{j=-\infty}^{\infty} \lambda_j a_j$ (or $f = \sum_{j=0}^{\infty} \lambda_j a_j$) in the $S'(\mathbb{R}^n)$ sense, where a_j is a central $(\alpha, q, D^m A)$ -atom (or a central $(\alpha, q, D^m A)$ -atom of restrict type) supported on $B(0, 2^j)$ and $\sum_{j=-\infty}^{\infty} |\lambda_j|^p < \infty$ (or $\sum_{j=0}^{\infty} |\lambda_j|^p < \infty$). Moreover, $\|f\|_{HK_{q,D^m A}^{\alpha,p}}$ (or $\|f\|_{HK_{q,D^m A}^{\alpha,p}}$) $\sim (\sum_j |\lambda_j|^p)^{1/p}$.

2. Theorems and proofs. We begin with some preliminary lemmas.

LEMMA 2.1 (see [2]). *Let A be a function on \mathbb{R}^n and $D^\alpha A \in L^q(\mathbb{R}^n)$ for $|\alpha| = m$ and some $q > n$. Then,*

$$|R_m(A; x, y)| \leq C|x-y|^m \sum_{|\alpha|=m} \left(\frac{1}{|\tilde{Q}(x, y)|} \int_{\tilde{Q}(x, y)} |D^\alpha A(z)|^q dz \right)^{1/q}, \tag{2.1}$$

where \tilde{Q} is the cube centered at x and having side length $5\sqrt{n}|x-y|$.

LEMMA 2.2. *Let $1 < p < \infty$ and $D^\alpha A \in L^r(\mathbb{R}^n)$, $|\alpha| = m$, $1 < r \leq \infty$, and $1/q = 1/p + 1/r$. Then, μ_Ω^A is bound from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$, that is,*

$$\|\mu_\Omega^A(f)\|_{L^q} \leq C \sum_{|\alpha|=m} \|D^\alpha A\|_{L^r} \|f\|_{L^p}. \tag{2.2}$$

PROOF. By Minkowski inequality and the condition of Ω , we have

$$\begin{aligned} \mu_\Omega^A(f)(x) &\leq \int_{\mathbb{R}^n} \frac{|f(y)| |R_{m+1}(A; x, y)|}{|x-y|^m} \left(\int_{|x-y|}^{\infty} \frac{dt}{t^3} \right)^{1/2} dy \\ &\leq C \int_{\mathbb{R}^n} \frac{|R_{m+1}(A; x, y)|}{|x-y|^{m+n}} |f(y)| dy. \end{aligned} \tag{2.3}$$

Thus, the lemma follows from [3, 4]. □

THEOREM 2.3. *Let $1 \geq p > n/(n+y)$, and let $D^\beta A \in \text{BMO}(\mathbb{R}^n)$ for $|\beta| = m$. Then, μ_Ω^A is bounded from $H_{D^m A}^p(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$.*

PROOF. It suffices to show that there exists a constant $c > 0$ such that, for every $(p, D^m A)$ -atom a ,

$$\|\mu_\Omega^A(a)\|_{L^p} \leq C. \quad (2.4)$$

Let a be a $(p, D^m A)$ -atom supported on a ball $B = B(x_0, r)$. We write

$$\begin{aligned} \int_{\mathbb{R}^n} [\mu_\Omega^A(a)(x)]^p dx &= \int_{|x-x_0| \leq 2r} [\mu_\Omega^A(a)(x)]^p dx \\ &\quad + \int_{|x-x_0| > 2r} [\mu_\Omega^A(a)(x)]^p dx \\ &\equiv I + II. \end{aligned} \quad (2.5)$$

For I , taking $q > 1$ and by Hölder's inequality and the L^q -boundedness of μ_Ω^A (see Lemma 2.2), we see that

$$\begin{aligned} I &\leq C \|\mu_\Omega^A(a)\|_{L^q}^p \cdot |B(x_0, 2r)|^{1-p/q} \\ &\leq C \|a\|_{L^q}^p |B|^{1-p/q} \\ &\leq C. \end{aligned} \quad (2.6)$$

To obtain the estimate of II , we need to estimate $\mu_\Omega^A(a)(x)$ for $x \in (2B)^c$. Let $\tilde{B} = 5\sqrt{n}B$, and let $\tilde{A}(x) = A(x) - \sum_{|\alpha|=m} (1/\alpha!) (D^\alpha A)_{\tilde{B}} \cdot x^\alpha$. Then, $R_m(A; x, y) = R_m(\tilde{A}; x, y)$. By the vanishing moment of a , we write

$$\begin{aligned} F_t^A(a)(x) &= \int_{|x-y| \leq t} \left[\frac{\Omega(x-y)}{|x-y|^{m+n-1}} - \frac{\Omega(x-x_0)}{|x-x_0|^{m+n-1}} \right] R_m(\tilde{A}; x, y) a(y) dy \\ &\quad + \int_{|x-y| \leq t} \frac{\Omega(x-x_0)}{|x-x_0|^{m+n-1}} [R_m(\tilde{A}; x, y) - R_m(\tilde{A}; x, x_0)] a(y) dy \\ &\quad - \sum_{|\alpha|=m} \frac{1}{\alpha!} \int_{|x-y| \leq t} \frac{\Omega(x-y)(x-y)^\alpha}{|x-y|^{m+n-1}} (D^\alpha A(y) - (D^\alpha A)_B) a(y) dy, \end{aligned} \quad (2.7)$$

thus,

$$\begin{aligned}
\mu_{\Omega}^A(a)(x) &\leq \left[\int_0^{\infty} \left(\int_{|x-y|\leq t} \left| \frac{\Omega(x-y)}{|x-y|^{m+n-1}} - \frac{\Omega(x-x_0)}{|x-x_0|^{m+n-1}} \right| \right. \right. \\
&\quad \left. \left. \times |R_m(\tilde{A}; x, y)| |a(y)| dy \right)^2 \frac{dt}{t^3} \right]^{1/2} \\
&\quad + \left[\int_0^{\infty} \left(\int_{|x-y|\leq t} \frac{|\Omega(x-x_0)|}{|x-x_0|^{m+n-1}} \right. \right. \\
&\quad \left. \left. \times |R_m(\tilde{A}; x, y) - R_m(\tilde{A}; x, x_0)| |a(y)| dy \right)^2 \frac{dt}{t^3} \right]^{1/2} \\
&\quad + \left[\int_0^{\infty} \left| \sum_{|\alpha|=m} \frac{1}{\alpha!} \int_{|x-y|\leq t} \frac{\Omega(x-y)(x-y)^{\alpha}}{|x-y|^{m+n-1}} \right. \right. \\
&\quad \left. \left. \times (D^{\alpha}A(y) - (D^{\alpha}A)_B) a(y) dy \right|^2 \frac{dt}{t^3} \right]^{1/2} \\
&\equiv II_1 + II_2 + II_3.
\end{aligned} \tag{2.8}$$

By [Lemma 2.1](#), for $y \in B$ and $x \in 2^{k+1}B \setminus 2^k B$, we know

$$|R_m(\tilde{A}; x, y)| \leq C|x-y|^m \sum_{|\alpha|=m} |D^{\alpha}A(x) - (D^{\alpha}A)_{2^k B}|. \tag{2.9}$$

By the condition of Ω and Minkowski's inequality, and noting that $|x-y| \sim |x-x_0|$ for $y \in B$ and $x \in \mathbb{R}^n \setminus B$, we obtain

$$\left| \frac{\Omega(x-y)}{|x-y|^{m+n-1}} - \frac{\Omega(x-x_0)}{|x-x_0|^{m+n-1}} \right| \leq C \left(\frac{r}{|x-x_0|^{m+n}} + \frac{r^y}{|x-x_0|^{m+n+y-1}} \right). \tag{2.10}$$

Thus,

$$\begin{aligned}
II_1 &\leq C \int_B |R_m(\tilde{A}; x, y)| |a(y)| \left(\int_{|x-y|}^{\infty} \frac{dt}{t^3} \right)^{1/2} \\
&\quad \times \left(\frac{r}{|x-x_0|^{m+n}} + \frac{r^y}{|x-x_0|^{m+n+y-1}} \right) dy \\
&\leq C \left(\frac{r}{|x-x_0|^{n+1}} + \frac{r^y}{|x-x_0|^{n+y}} \right) |B|^{1-1/p} \sum_{|\alpha|=m} |D^{\alpha}A(x) - (D^{\alpha}A)_{2^k B}|.
\end{aligned} \tag{2.11}$$

On the other hand, by the following formula (see [2]):

$$R_m(\tilde{A}; x, y) - R_m(\tilde{A}; x, x_0) = \sum_{|\beta| < m} \frac{1}{\beta!} R_{m-|\beta|}(D^\beta \tilde{A}; y, x_0) (x - x_0)^\beta \quad (2.12)$$

and [Lemma 2.1](#), we get

$$\begin{aligned} & |R_m(\tilde{A}; x, y) - R_m(\tilde{A}; x, x_0)| \\ & \leq C \sum_{|\beta| < m} \sum_{|\alpha| = m} |x_0 - y|^{m-|\beta|} |x - x_0|^{|\beta|} \|D^\alpha A\|_{\text{BMO}}, \end{aligned} \quad (2.13)$$

so that

$$\begin{aligned} II_2 & \leq C \int_B |x - x_0|^{-(n+m)} \sum_{|\beta| < m} |R_{m-|\beta|}(D^\beta \tilde{A}; y, x_0)| |x - x_0|^{|\beta|} |a(y)| dy \\ & \leq C \int_B |x - x_0|^{-(n+m)} \sum_{|\beta| < m} |x_0 - y|^{m-|\beta|} |x - x_0|^{|\beta|} \\ & \quad \times \sum_{|\alpha| = m} \|D^\alpha A\|_{\text{BMO}} |a(y)| dy \\ & \leq C \sum_{|\alpha| = m} \|D^\alpha A\|_{\text{BMO}} \int_B \frac{|x_0 - y|}{|x - x_0|^{n+1}} |a(y)| dy \\ & \leq C \sum_{|\alpha| = m} \|D^\alpha A\|_{\text{BMO}} |x - x_0|^{-n-1} |B|^{1/n-1/p+1}. \end{aligned} \quad (2.14)$$

For II_3 , and by the vanishing moment of a , we write,

$$\begin{aligned} & \int \frac{\Omega(x-y)(x-y)^\alpha}{|x-y|^m} (D^\alpha A(y) - (D^\alpha A)_B) a(y) dy \\ & = \int \left[\frac{\Omega(x-y)(x-y)^\alpha}{|x-y|^m} - \frac{\Omega(x-x_0)(x-x_0)^\alpha}{|x-x_0|^{m+n-1}} \right] [D^\alpha A(y) - (D^\alpha A)_B] a(y) dy. \end{aligned} \quad (2.15)$$

Similar to the estimate of II_1 , we obtain

$$\begin{aligned} II_3 & \leq C \sum_{|\alpha| = m} \left(\frac{r}{|x - x_0|^{n+1}} + \frac{r^y}{|x - x_0|^{n+y}} \right) \\ & \quad \times \int_B |x_0 - y| |D^\alpha A(y) - (D^\alpha A)_B| |a(y)| dy \\ & \leq C \sum_{|\alpha| = m} \|D^\alpha A\|_{\text{BMO}} |B|^{1-1/p} \left(\frac{r}{|x - x_0|^{n+1}} + \frac{r^y}{|x - x_0|^{n+y}} \right). \end{aligned} \quad (2.16)$$

Recalling that $p > n/(n+y)$, therefore,

$$\begin{aligned}
II &\leq \sum_{k=1}^{\infty} \int_{2^{k+1}B \setminus 2^k B} [\mu_{\Omega}^A(a)(x)]^p dx \\
&\leq C \sum_{k=1}^{\infty} \int_{2^{k+1}B \setminus 2^k B} \left(\frac{r}{|x-x_0|^{n+1}} + \frac{r^y}{|x-x_0|^{n+y}} \right)^p |B|^{p-1} \\
&\quad \times \left(\sum_{|\alpha|=m} |D^{\alpha}A(x) - (D^{\alpha}A)_{2^{k+1}B}| \right)^p dx \\
&\quad + C \left(\sum_{|\alpha|=m} \|D^{\alpha}A\|_{\text{BMO}} \right)^p \sum_{k=1}^{\infty} \int_{2^{k+1}B \setminus 2^k B} |x-x_0|^{-p(n+1)} |B|^{p(1+1/n-1/p)} dx \\
&\leq C \left(\sum_{|\alpha|=m} \|D^{\alpha}A\|_{\text{BMO}} \right)^p \sum_{k=1}^{\infty} (2^{k(n-p-pn)} + 2^{k(n-pn-py)}) \\
&\leq C \left(\sum_{|\alpha|=m} \|D^{\alpha}A\|_{\text{BMO}} \right)^p,
\end{aligned} \tag{2.17}$$

which, together with the estimate for I , yields the desired result. This finishes the proof of [Theorem 2.3](#). \square

THEOREM 2.4. *Let $0 < p < \infty$, $1 < q < \infty$, $n(1-1/q) \leq \alpha < n(1-1/q) + y$, and $D^{\beta}A \in \text{BMO}(\mathbb{R}^n)$ for $|\beta| = m$. Then, μ_{Ω}^A is bounded from $H\dot{K}_{q,D^m A}^{\alpha,p}(\mathbb{R}^n)$ to $\dot{K}_q^{\alpha,p}(\mathbb{R}^n)$.*

PROOF. Let $f \in H\dot{K}_{q,D^m A}^{\alpha,p}(\mathbb{R}^n)$ and $f(x) = \sum_{j=-\infty}^{\infty} \lambda_j a_j(x)$ be the atomic decomposition for f as in [Definition 1.3](#). We write

$$\begin{aligned}
\|\mu_{\Omega}^A(f)\|_{\dot{K}_q^{\alpha,p}(\mathbb{R}^n)} &\leq C \left[\sum_{k=-\infty}^{\infty} 2^{k\alpha p} \left(\sum_{j=-\infty}^{k-3} |\lambda_j| \|\mu_{\Omega}^A(a_j)\chi_k\|_{L^q} \right)^p \right]^{1/p} \\
&\quad + C \left[\sum_{k=-\infty}^{\infty} 2^{k\alpha p} \left(\sum_{j=k-2}^{\infty} |\lambda_j| \|\mu_{\Omega}^A(a_j)\chi_k\|_{L^q} \right)^p \right]^{1/p} \tag{2.18} \\
&\equiv I + II.
\end{aligned}$$

For II , and by the boundedness of μ_{Ω}^A on $L^q(\mathbb{R}^n)$ (see [Lemma 2.2](#)), we have

$$II \leq C \left[\sum_{k=-\infty}^{\infty} 2^{k\alpha p} \left(\sum_{j=k-2}^{\infty} |\lambda_j| \|a_j\|_{L^q} \right)^p \right]^{1/p}$$

$$\begin{aligned}
&\leq C \left[\sum_{k=-\infty}^{\infty} 2^{k\alpha p} \left(\sum_{j=k-2}^{\infty} |\lambda_j| 2^{-j\alpha} \right)^p \right]^{1/p} \\
&\leq C \begin{cases} \left[\sum_{k=-\infty}^{\infty} 2^{k\alpha p} \sum_{j=k-2}^{\infty} |\lambda_j|^p 2^{-j\alpha p} \right]^{1/p}, & 0 < p \leq 1 \\ \left[\sum_{k=-\infty}^{\infty} 2^{k\alpha p} \left(\sum_{j=k-2}^{\infty} |\lambda_j|^p 2^{-j\alpha p/2} \right) \left(\sum_{j=k-2}^{\infty} 2^{-j\alpha p'/2} \right)^{p/p'} \right]^{1/p}, & p > 1 \end{cases} \\
&\leq C \begin{cases} \left[\sum_{j=-\infty}^{\infty} |\lambda_j|^p \left(\sum_{k=-\infty}^{j+2} 2^{(k-j)\alpha p} \right) |\lambda_j|^p 2^{-j\alpha p} \right]^{1/p}, & 0 < p \leq 1 \\ \left[\sum_{j=-\infty}^{\infty} |\lambda_j|^p \left(\sum_{k=-\infty}^{j+2} 2^{(k-j)\alpha p/2} \right) \right]^{1/p}, & p > 1 \end{cases} \\
&\leq C \left(\sum_{j=-\infty}^{\infty} |\lambda_j|^p \right)^{1/p} \leq C \|f\|_{HK_{q,Dm_A}^{\alpha,p}(\mathbb{R}^n)}.
\end{aligned} \tag{2.19}$$

For I , and similar to the proof of [Theorem 2.3](#), we have, for $x \in C_k$, $j \leq k-3$,

$$\begin{aligned}
\mu_{\Omega}^A(a_j)(x) &\leq C \left(|x|^{-n-m-1} |B_j|^{1/n} + |x|^{-n-m-\gamma} |B_j|^{y/n} \right) \\
&\quad \times \left(\int_{B_j} |a_j(y)| |R_m(\tilde{A}; x, y)| dy \right) \\
&\quad + C \sum_{|\beta|=m} \|D^{\beta}A\|_{\text{BMO}} |x|^{-n-1} |B_j|^{1/n} \int_{B_j} |a(y)| dy \\
&\quad + C \left(|x|^{-n-1} |B_j|^{1+1/n} + |x|^{-n-\gamma} |B_j|^{1+y/n} \right) \\
&\quad \times \sum_{|\beta|=m} \int_{B_j} |D^{\beta}A(y) - (D^{\beta}A)_{B_j}| |a(y)| dy \\
&\leq C \left(2^{-k(n+1)} 2^{j(1+n(1-1/q)-\alpha)} + 2^{-k(n+\gamma)} 2^{j(y+n(1-1/q)-\alpha)} \right) \\
&\quad \times \left(\sum_{|\beta|=m} |D^{\beta}A(x) - (D^{\beta}A)_{B_k}| \right) \\
&\quad + C \sum_{|\alpha|=m} \|D^{\alpha}A\|_{\text{BMO}}(k-j) \\
&\quad \times \left(2^{-k(n+1)} 2^{j(1+n(1-1/q)-\alpha)} + 2^{-k(n+\gamma)} 2^{j(y+n(1-1/q)-\alpha)} \right)
\end{aligned} \tag{2.20}$$

thus,

$$\begin{aligned}
I &\leq C \left[\sum_{k=-\infty}^{\infty} 2^{k\alpha p} \left(\sum_{j=-\infty}^{k-3} |\lambda_j| (2^{-k(n+1)+j(1+n(1-1/q)-\alpha)} + 2^{-k(n+y)+j(\gamma+n(1-1/q)-\alpha)}) \right. \right. \\
&\quad \left. \left. \times \sum_{|\alpha|=m} \left(\int_{B_k} |D^\alpha A(x) - (D^\alpha A)_{B_k}|^q dx \right)^{1/q} \right)^p \right]^{1/p} \\
&\quad + C \left[\sum_{k=-\infty}^{\infty} 2^{k\alpha p} \left(\sum_{j=-\infty}^{k-3} |\lambda_j| (k-j) (2^{-k(n+1)+j(1+n(1-1/q)-\alpha)} \right. \right. \\
&\quad \left. \left. + 2^{-k(n+y)+j(\gamma+n(1-1/q)-\alpha)}) 2^{kn/q} \sum_{|\alpha|=m} \|D^\alpha A\|_{\text{BMO}} \right)^p \right]^{1/p} \\
&\equiv I_1 + I_2.
\end{aligned} \tag{2.21}$$

To estimate I_1 and I_2 , we consider two cases.

CASE 1 ($0 < p \leq 1$). We have

$$\begin{aligned}
I_1 &\leq C \left[\sum_{k=-\infty}^{\infty} 2^{k\alpha p} \sum_{j=-\infty}^{k-3} |\lambda_j|^p (2^{[-k(n+1)+j(1+n(1-1/q)-\alpha)]p} \right. \\
&\quad \left. + 2^{[-k(n+y)+j(\gamma+n(1-1/q)-\alpha)]p}) 2^{knp/q} \left(\sum_{|\beta|=m} \|D^\beta A\|_{\text{BMO}} \right)^p \right]^{1/p} \\
&= C \sum_{|\beta|=m} \|D^\beta A\|_{\text{BMO}} \left[\sum_{j=-\infty}^{\infty} |\lambda_j|^p \sum_{k=j+3}^{\infty} (2^{(j-k)(1+n(1-1/q)-\alpha)p} \right. \\
&\quad \left. + 2^{(j-k)(\gamma+n(1-1/q)-\alpha)p}) \right]^{1/p} \\
&\leq C \sum_{|\beta|=m} \|D^\beta A\|_{\text{BMO}} \left(\sum_{j=-\infty}^{\infty} |\lambda_j|^p \right)^{1/p} \\
&\leq C \|f\|_{\dot{H}_{q,D^m A}^{\alpha,p}(\mathbb{R}^n)}.
\end{aligned} \tag{2.22}$$

Similarly,

$$I_2 \leq C \|f\|_{\dot{H}_{q,D^m A}^{\alpha,p}(\mathbb{R}^n)}. \tag{2.23}$$

CASE 2 ($p > 1$). By Hölder's inequality, we deduce that

$$\begin{aligned}
 I_1 &\leq C \sum_{|\beta|=m} \|D^\beta A\|_{\text{BMO}} \left[\sum_{j=-\infty}^{\infty} \left(\sum_{j=-\infty}^{k-3} |\lambda_j|^p 2^{(j-k)p(\gamma+n(1-1/q)-\alpha)/2} \right) \right. \\
 &\quad \left. \times \left(\sum_{j=-\infty}^{k-3} 2^{(j-k)p'(\gamma+n(1-1/q)-\alpha)/2} \right)^{p/p'} \right]^{1/p} \\
 &\leq C \left(\sum_{j=-\infty}^{\infty} |\lambda_j|^p \right)^{1/p} \leq C \|f\|_{\dot{H}_{q,D^m A}^{\alpha,p}(\mathbb{R}^n)},
 \end{aligned} \tag{2.24}$$

$$I_2 \leq C \|f\|_{\dot{H}_{q,D^m A}^{\alpha,p}(\mathbb{R}^n)}.$$

This finishes the proof of Theorem 2. \square

REMARK 2.5. Theorem 2.4 also holds for nonhomogeneous Herz-type space.

REFERENCES

- [1] J. Cohen and J. A. Gosselin, *On multilinear singular integrals on \mathbf{R}^n* , *Studia Math.* **72** (1982), no. 3, 199-223.
- [2] ———, *A BMO estimate for multilinear singular integrals*, *Illinois J. Math.* **30** (1986), no. 3, 445-464.
- [3] Y. Ding, *A note on multilinear fractional integrals with rough kernel*, *Adv. Math. (China)* **30** (2001), no. 3, 238-246.
- [4] Y. Ding and S. Lu, *Weighted boundedness for a class of rough multilinear operators*, *Acta Math. Sin. (Engl. Ser.)* **17** (2001), no. 3, 517-526.
- [5] Y. Ding, S. Lu, and Q. Xue, *On Marcinkiewicz integral with homogeneous kernels*, *J. Math. Anal. Appl.* **245** (2000), no. 2, 471-488.
- [6] Y. Ding and Q. Xue, *Commutator of Marcinkiewicz integral on Hardy space*, preprint.
- [7] G. Hu, S. Lu, and D. C. Yang, *The applications of weak Herz spaces*, *Adv. in Math. (China)* **26** (1997), no. 5, 417-428.
- [8] ———, *The weak Herz spaces*, *Beijing Shifan Daxue Xuebao* **33** (1997), no. 1, 27-34.
- [9] S. Lu and D. C. Yang, *The decomposition of weighted Herz space on \mathbf{R}^n and its applications*, *Sci. China Ser. A* **38** (1995), no. 2, 147-158.
- [10] ———, *The weighted Herz-type Hardy space and its applications*, *Sci. China Ser. A* **38** (1995), no. 6, 662-673.
- [11] ———, *The continuity of commutators on Herz-type spaces*, *Michigan Math. J.* **44** (1997), no. 2, 255-281.
- [12] A. Torchinsky and S. L. Wang, *A note on the Marcinkiewicz integral*, *Colloq. Math.* **60/61** (1990), no. 1, 235-243.

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