

ON THE CONVOLUTION PRODUCT OF THE DISTRIBUTIONAL KERNEL $K_{\alpha,\beta,\gamma,\nu}$

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We introduce a distributional kernel $K_{\alpha,\beta,\gamma,\nu}$ which is related to the operator \oplus^k iterated k times and defined by $\oplus^k = [(\sum_{r=1}^p \partial^2 / \partial x_r^2)^4 - (\sum_{j=p+1}^{p+q} \partial^2 / \partial x_j^2)^4]^k$, where $p + q = n$ is the dimension of the space \mathbb{R}^n of the n -dimensional Euclidean space, $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$, k is a nonnegative integer, and α, β, γ , and ν are complex parameters. It is found that the existence of the convolution $K_{\alpha,\beta,\gamma,\nu} * K_{\alpha',\beta',\gamma',\nu'}$ is depending on the conditions of p and q .

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1. Introduction. The operator \oplus^k can be factorized in the form

$$\begin{aligned} \oplus^k &= \left[\left(\sum_{r=1}^p \frac{\partial^2}{\partial x_r^2} \right)^2 - \left(\sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2} \right)^2 \right]^{2k} \\ &\times \left[\sum_{r=1}^p \frac{\partial^2}{\partial x_r^2} + i \sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2} \right]^k \left[\sum_{r=1}^p \frac{\partial^2}{\partial x_r^2} - i \sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2} \right]^k, \end{aligned} \quad (1.1)$$

where $p + q = n$ is the dimension of the space \mathbb{R}^n , $i = \sqrt{-1}$, and k is a nonnegative integer. The operator $(\sum_{r=1}^p \partial^2 / \partial x_r^2)^2 - (\sum_{j=p+1}^{p+q} \partial^2 / \partial x_j^2)^2$ is first introduced by Kananthai [2] and named the Diamond operator denoted by

$$\diamond = \left(\sum_{r=1}^p \frac{\partial^2}{\partial x_r^2} \right)^2 - \left(\sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2} \right)^2. \quad (1.2)$$

We denote the operators L_1 and L_2 by

$$\begin{aligned} L_1 &= \sum_{r=1}^p \frac{\partial^2}{\partial x_r^2} + i \sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2}, \\ L_2 &= \sum_{r=1}^p \frac{\partial^2}{\partial x_r^2} - i \sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2}. \end{aligned} \quad (1.3)$$

Thus (1.1) can be written by

$$\oplus^k = \diamond^k L_1^k L_2^k. \tag{1.4}$$

Now consider the convolution $R_\alpha^H(u) * R_\beta^e(v) * S_y(w) * T_\nu(z)$ where $R_\alpha^H(u)$, $R_\beta^e(v)$, $S_y(w)$, and $T_\nu(z)$ are defined by (2.2), (2.4), (2.6), and (2.7), respectively.

We defined the distributional kernel $K_{\alpha,\beta,\gamma,\nu}$ by

$$K_{\alpha,\beta,\gamma,\nu} = R_\alpha^H(u) * R_\beta^e(v) * S_\gamma(w) * T_\nu(z). \tag{1.5}$$

Since the functions $R_\alpha^H(u)$, $R_\beta^e(v)$, $S_\gamma(w)$, and $T_\nu(z)$ are all tempered distributions and the supports of $R_\alpha^H(u)$ and $R_\beta^e(v)$ are compact (see [2, pages 30-31] and [1, pages 152-153]), then the convolution on the right-hand side of (1.5) exists and also is a tempered distributions. Thus $K_{\alpha,\beta,\gamma,\nu}$ is well defined and also is a tempered distribution.

For $\alpha = \beta = \gamma = \nu = 2k$, we obtain $(-1)^k K_{2k,2k,2k,2k}$ as an elementary solution of the operator \oplus^k , see [3]. That is $\oplus^k (-1)^k K_{2k,2k,2k,2k}(x) = \delta$ where δ is the Dirac-delta distribution and \oplus^k is defined by (1.4).

2. Preliminaries

DEFINITION 2.1. Let $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$ and write

$$x = x_1^2 + x_2^2 + \dots + x_p^2 - x_{p+1}^2 - x_{p+2}^2 - \dots - x_{p+q}^2, \quad p + q = n. \tag{2.1}$$

Denote by $\Gamma_+ = \{x \in \mathbb{R}^n : x_1 > 0 \text{ and } u > 0\}$ the interior of forward cone and $\bar{\Gamma}_+$ denote its closure. For any complex number α , we define the function

$$R_\alpha^H(x) = \begin{cases} \frac{u^{(\alpha-n)/2}}{K_n(\alpha)}, & \text{if } x \in \Gamma_+, \\ 0, & \text{if } x \notin \Gamma_+, \end{cases} \tag{2.2}$$

where the constant $K_n(\alpha)$ is given by the formula

$$K_n(\alpha) = \frac{\pi^{(n-1)/2} \Gamma((2 + \alpha - n)/2) \Gamma((1 - \alpha)/2) \Gamma(\alpha)}{\Gamma((2 + \alpha - p)/2) \Gamma((p - \alpha)/2)}, \tag{2.3}$$

the function R_α^H is first introduced by Nozaki [4, page 72] and is called the ultra-hyperbolic kernel of Marcel Riesz. Hence $R_\alpha^H(x)$ is an ordinary function if $\text{Re}(\alpha) \geq n$ and is a distribution of α if $\text{Re}(\alpha) < n$. Let $\text{supp } R_\alpha^H(u) \subset \bar{\Gamma}_+$ where $\text{supp } R_\alpha^H(u)$ denotes the support of $R_\alpha^H(u)$.

DEFINITION 2.2. Let $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$ and write $v = x_1^2 + x_2^2 + \dots + x_n^2$.

For any complex number β , define the function

$$R_\beta^e(v) = \frac{v^{(\beta-n)/2}}{W_n(\beta)}, \tag{2.4}$$

where $W_n(\beta) = \pi^{n/2} 2^\beta \Gamma(\beta) / \Gamma((n-\beta)/2)$, the function $R_\beta^e(v)$ is called the elliptic kernel of Marcel Riesz and is ordinary function if $\text{Re}(\beta) \geq n$ and is a distribution of β if $\text{Re}(\beta) < n$.

DEFINITION 2.3. Let $x = (x_1, x_2, \dots, x_n)$ be a point of the space \mathbb{R}^n . Write

$$\begin{aligned} w &= x_1^2 + x_2^2 + \dots + x_p^2 - i(x_{p+1}^2 + x_{p+2}^2 + \dots + x_{p+q}), \\ z &= x_1^2 + x_2^2 + \dots + x_p^2 + i(x_{p+1}^2 + x_{p+2}^2 + \dots + x_{p+q}), \quad p+q = n, \quad i = \sqrt{-1}. \end{aligned} \tag{2.5}$$

For any complex numbers y and v , define

$$S_y(\omega) = \frac{\omega^{(y-n)/2}}{W_n(y)}, \tag{2.6}$$

$$T_v(z) = \frac{z^{(v-n)/2}}{W_n(v)}, \tag{2.7}$$

where

$$W_n(y) = \frac{\pi^{n/2} 2^y \Gamma(y/2)}{\Gamma((n-y)/2)}, \quad W_n(v) = \frac{\pi^{n/2} 2^v \Gamma(v/2)}{\Gamma((n-v)/2)}. \tag{2.8}$$

LEMMA 2.4 (the convolution product of $R_\beta^e(v)$). *The convolution $R_\beta^e * R_{\beta'}^e = R_{\beta+\beta'}^e$ where R_β^e and $R_{\beta'}$ are given by (2.2).*

PROOF. See [5, page 20]. □

LEMMA 2.5 (the convolution product of $R_\alpha^H(x)$). *The convolution product is given by*

(i)

$$R_\alpha^H * R_{\alpha'}^H = \frac{\cos(\alpha(\pi/2)) \cos(\alpha'(\pi/2))}{\cos((\alpha+\beta)/2)\pi} \cdot R_{\alpha+\alpha'}^H, \tag{2.9}$$

where R_α^H and $R_{\alpha'}^H$ are defined by (2.1) with p even,

(ii) $R_\alpha^H * R_\alpha^H = R_{\alpha+\alpha'}^H + A_{\alpha,\alpha'}$ for p odd, where

$$A_{\alpha,\alpha'} = \frac{2\pi i}{4} \frac{C((-\alpha - \alpha')/2)}{C(-\alpha/2)C(-\alpha'/2)} [H_{\alpha+\alpha'}^+ - H_{\alpha+\alpha'}^-],$$

$$C(r) = \Gamma(r)\Gamma(1-r),$$

$$H_r^\pm = H_r(x \pm i0, n) = e^{\mp r(\pi/2)i} e^{\pm q(\pi/2)i} a\left(\frac{r}{2}\right) (u \pm i0)^{(r-n)/2}, \tag{2.10}$$

$$a\left(\frac{r}{2}\right) = \Gamma\left(\frac{n-r}{2}\right) \left(2^r \pi^{n/2} \Gamma\left(\frac{r}{2}\right)\right)^{-1},$$

$$(u \pm i0, n)^\lambda = \lim_{\epsilon \rightarrow 0} (u \pm i\epsilon, n)^\lambda,$$

$u = u(x)$ is defined by (2.1) and $|x| = (x_1^2 + x_2^2 + \dots + x_n^2)^{1/2}$ in particular $R_\alpha^H * R_{-2k}^H = R_{\alpha-2k}^H$ and $R_\alpha^H * R_{2k}^H = R_{\alpha+2k}^H$.

The proof of this lemma is given by Téllez [6, pages 121-123].

LEMMA 2.6 (the convolutions product of $S_y(w)$ and $T_v(z)$). *The convolutions product is given by*

- (i) $S_y * S_{y'} = (i)^{q/2} S_{y+y'}$,
- (ii) $T_v * T_{v'} = (-i)^{q/2} T_{v+v'}$ where S_y and T_v are defined by (2.6) and (2.7), respectively.

PROOF. (i) Now

$$\langle S_y(w), \varphi(x) \rangle = \frac{1}{W_n(y)} \int_{\mathbb{R}^n} \omega^{(y-n)/2} \varphi(x) dx, \tag{2.11}$$

where $\varphi \in \mathcal{D}$ the space of infinitely differentiable function with compact supports. We have $\omega = x_1^2 + x_2^2 + \dots + x_p^2 - i(x_{p+1}^2 + x_{p+2}^2 + \dots + x_{p+q}^2)$, $p + q = n$. By changing the variables $x_1 = y_1, x_2 = y_2, \dots, x_p = y_p, x_{p+1} = y_{p+1}/\sqrt{-i}, x_{p+2} = y_{p+2}/\sqrt{-i}, \dots$, and $x_{p+q} = y_{p+q}/\sqrt{-i}$. Thus we obtain $\omega = y_1^2 + y_2^2 + \dots + y_p^2 + y_{p+1}^2 + y_{p+2}^2 + \dots + y_{p+q}^2$. Let $r^2 = y_1^2 + y_2^2 + \dots + y_{p+q}^2$, $p + q = n$. Thus (2.11) can be written in the form

$$\begin{aligned} \langle S_y(w), \varphi(x) \rangle &= \frac{1}{W_n(y)} \int_{\mathbb{R}^n} r^{y-n} \varphi \frac{\partial(x_1, x_2, \dots, x_n)}{\partial(y_1, y_2, \dots, y_n)} dy_1 dy_2 \dots dy_n \\ &= \frac{1}{(-i)^{q/2}} \frac{1}{W_n(y)} \int_{\mathbb{R}^n} r^{y-n} \varphi dy \\ &= \frac{(i)^{q/2}}{W_n(y)} \langle r^{y-n}, \varphi \rangle. \end{aligned} \tag{2.12}$$

Thus $S_y(w) = ((i)^{q/2}/w_n(y))r^{y-n} = (i)^{q/2}R_y^e(w)$ by (2.4).

Consider the convolution $S_y * S_{y'}$. We have

$$\begin{aligned}
 S_y * S_{y'} &= (i)^{q/2} R_y^e(w) * (i)^{q/2} R_{y'}^e(w) \\
 &= (i)^q R_{y+y'}^e(w) \text{ by Lemma 2.4 and [1, pages 157-159]} \\
 &= (i)^{q/2} (i)^{q/2} R_{y+y'}^e(w) \\
 &= (i)^{q/2} S_{y+y'}.
 \end{aligned}
 \tag{2.13}$$

Similarly, for (ii) we also have

$$T_v * T_{v'} = (-i)^{q/2} T_{v+v'}.
 \tag{2.14}$$

□

3. Main results

THEOREM 3.1. *Let $K_{\alpha,\beta,\gamma,\nu}$ be the distributional kernel defined by (1.5). Then we obtain the following:*

- (i) $K_{\alpha,\beta,\gamma,\nu} * K_{\alpha',\beta',\gamma',\nu'} = K_{\alpha+\alpha',\beta+\beta',\gamma+\gamma',\nu+\nu'}$ for $\alpha, \beta, \gamma, \nu, \alpha', \beta', \gamma',$ and ν' positive even numbers with $\alpha = \beta = \gamma = \nu, \alpha' = \beta' = \gamma' = \nu'$;
- (ii) $K_{\alpha,\beta,\gamma,\nu} * K_{\alpha',\beta',\gamma',\nu'} = B_{\alpha,\alpha'} \cdot K_{\alpha+\alpha',\beta+\beta',\gamma+\gamma',\nu+\nu'}$ for p even, $\alpha, \beta, \gamma, \nu, \alpha', \beta', \gamma',$ and ν' any complex numbers, and $B_{\alpha,\alpha'} = \cos(\alpha\pi/2) \cos(\alpha'\pi/2) / \cos((\alpha + \alpha')/2)\pi$;
- (iii) $K_{\alpha,\beta,\gamma,\nu} * K_{\alpha',\beta',\gamma',\nu'} = K_{\alpha+\alpha',\beta+\beta',\gamma+\gamma',\nu+\nu'} + A_{\alpha,\alpha'} * R_{\beta+\beta'}^e * S_{\gamma+\gamma'} * T_{\nu+\nu'}$ for p odd, $\alpha, \beta, \gamma, \nu, \alpha', \beta', \gamma',$ and ν' any complex numbers $R_{\beta}^e, S_{\gamma},$ and T_{ν} defined by (2.4), (2.6), and (2.7), respectively. And

$$\begin{aligned}
 A_{\alpha,\alpha'} &= \frac{C((-\alpha - \alpha')/2)}{C(-\alpha/2)} C\left(-\frac{\alpha'}{2}\right) \cdot \frac{2\pi i}{4} [H_{\alpha+\alpha'}^+ - H_{\alpha+\alpha'}^-], \\
 C(r) &= \Gamma(r)\Gamma(1-r), \\
 H_r^{\pm} &= H_r(u \pm i0, n) = e^{\mp r(\pi/2)i} e^{\pm q(\pi/2)i} a\left(\frac{r}{2}\right) (u \pm i0)^{r-n/2}, \\
 a\left(\frac{r}{2}\right) &= \Gamma\left(\frac{n-r}{2}\right) \left[2^r \pi^{n/2} \Gamma\left(\frac{r}{2}\right)\right]^{-1}, \\
 (u \pm i0)^\lambda &= \lim_{\epsilon \rightarrow 0} (u + i \in |\mathcal{x}|^2)^\lambda,
 \end{aligned}
 \tag{3.1}$$

where $u = u(\mathcal{x})$ is defined by (2.1) and

$$|\mathcal{x}| = (x_1^2 + x_2^2 + \dots + x_n^2)^{1/2}.
 \tag{3.2}$$

PROOF. The proof of (i) follows from [3, Theorem 3.1, page 66]. The proof of (ii) and (iii) is obtained by Lemmas 2.4, 2.5, and 2.6. □

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