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GAUSSIAN QUADRATURE RULES AND A-STABILITY OF GALERKIN SCHEMES FOR ODE

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The A-stability properties of continuous and discontinuous Galerkin methods for solving ordinary differential equations (ODEs) are established using properties of Legendre polynomials and Gaussian quadrature rules. The influence on the A-stability of the numerical integration using Gaussian quadrature rules involving a parameter is analyzed.

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1. Introduction. In this paper, the A-stability of various (continuous and discontinuous) Galerkin schemes for the solution of an initial value problem in ordinary differential equation (ODE) is analyzed. Even if A-stability of a method for solving ODE is an old well-studied subject, the contribution of this paper is in the presentation of a new proof of A-stability in Section 4.1 which points out the link between A-stability, Legendre polynomials, and Gaussian quadrature rules.

The stability results are obtained using a variety of Gaussian quadrature formulas of integrals defining the Galerkin finite element methods for problems of the form

$$\begin{aligned} \dot{y}(t) &= f(y(t),t), \quad 0 \le t \le T, \\ y(0) &= y_0. \end{aligned} \tag{1.1}$$

Continuous and discontinuous Galerkin methods have played an important role in the recently developed approach to global error estimation and control for numerical approximations of ODEs [8, 15, 16, 24]. In particular, the stability analysis is an important issue and has motivated our work.

2. Polynomial approximations and Galerkin methods. Galerkin methods for (1.1) are based on a variational formulation and use a (continuous or discontinuous) piecewise polynomial approximation of the solution of the ODE. They can be briefly described in the following way [3, 5].

The interval $[0,T]$ is partitioned into N intervals $I_n = [t_{n-1}, t_n]$ $(n = 1,\dots,N)$ by specifying the sequence $\{t_n\}_{n=0}^{N}$, $0 = t_0 < t_1 < \cdots < t_{N-1} < t_N = T$, of real numbers. If $P^K(I_n)$ denotes the set of real polynomials of degree K on I_n, we consider the following variational problem.

Problem 2.1. Let $U_0 = y_0$, and for $n = 1,\dots,N$, find $u_n(\cdot) \in P^K(I_n)$ and $U_n \in \mathbb{R}$ such that $\langle Du_n - f(u_n), v_n\rangle_n + [u_n(t_{n-1}) - U_{n-1}]v_n(t_{n-1}) + [U_n - u_n(t_n)]v_n(t_n) = 0$ for all $v_n(\cdot) \in P^{K+1-L}(I_n)$, and $u_n(\cdot)$ is subject to L additional collocation conditions.

In Problem 2.1, $f(u)$ stands for $f(u(t),t)$ and

$$\langle f,g\rangle_n = \int_{t_{n-1}}^{t_n} f(\tau)g(\tau)d\tau. \tag{2.1}$$

We say that we have an *approximate problem* if we use exact integration in Problem 2.1 and we have a *discretized problem* if we use a quadrature rule to deal with the integrals.

Continuous Galerkin methods for ODEs have been introduced in [19, 20] and discontinuous Galerkin methods in [25]. Discontinuous Galerkin methods were first analyzed for linear nonstiff ODEs in [4], and later for nonlinear nonstiff systems in [2, 6]. A general framework and analysis of Galerkin methods for ODEs have been developed in [3, 5]. In particular, existence, uniqueness, and convergence results have been obtained for approximate and discretized problems under appropriate assumptions on $f(\cdot,\cdot)$.

Similarly, Galerkin methods for parabolic problems have been analyzed, for example, in [1, 14, 23, 26], and later used for finding adaptive finite element methods for parabolic problems in [9, 10, 11, 12, 13, 17].

3. Gaussian quadrature rules.

Let γ be a parameter in $[-1,1]$ and $\pi_M(\cdot)$ be the Legendre polynomial of degree M on $[-1,1]$ such that $\pi_M(1) = 1 = (-1)^M\pi_M(-1)$. The quadrature rules we consider are based on the following result.

Lemma 3.1. *For $\gamma \in [-1,1]$, the M roots of the polynomial $\pi_M(\cdot) - \gamma\pi_{M-1}(\cdot)$, denoted $\tau_m(\gamma)$ ($m = 1,\dots,M$), are all real and distinct. Moreover,*

(i) $-1 \le \tau_1(\gamma) < \cdots < \tau_m(\gamma) < \cdots < \tau_M(\gamma) \le 1$;

(ii) $\tau_m(\cdot) \in C^\infty([-1,1];[-1,1])$ *for* $m = 1,\dots,M$;

(iii) $(d\tau_m/d\gamma)(\gamma) > 0$ *for any* $\gamma \in (-1,1)$;

(iv) $\tau_1(-1) = -1$ *and* $\tau_M(1) = 1$.

Proof. This result is obtained from the interlacing properties of the roots of Legendre polynomials and from the implicit function theorem. □

If we define the weights

$$\omega_m(\gamma) = \int_{-1}^{1} \prod_{\substack{k=1\\k\neq m}}^{M} \frac{\tau - \tau_k(\gamma)}{\tau_m(\gamma) - \tau_k(\gamma)} d\tau, \tag{3.1}$$

for $m = 1,\dots,M$, the quadrature rule

$$\int_{-1}^{1} \psi(\tau)d\tau \cong \sum_{m=1}^{M} \omega_m(\gamma)\psi(\tau_m(\gamma)), \tag{3.2}$$

which depends on a parameter $\gamma \in [-1,1]$, is exact for polynomials of degree $M-1$. It follows that the formula is also exact for polynomials of degree $2M-2$ and for polynomials of degree $2M-1$ if $\gamma = 0$. Gauss-Legendre quadrature rules correspond to $\gamma = 0$, Gauss-Radau quadrature rules correspond to $\gamma = \pm 1$, and for $0 < |\gamma| < 1$, we obtain intermediate quadrature rules.

We will use the following notation for the interval $[-1,1]$:

$$\begin{aligned} \langle f,g\rangle &= \int_{-1}^{1} f(\tau)g(\tau)d\tau, \\ \langle f,g\rangle^d &= \sum_{m=1}^{M} \omega_m(\gamma)f(\tau_m(\gamma))g(\tau_m(\gamma)). \end{aligned} \tag{3.3}$$

We remark the following identities for Legendre polynomials:

$$\langle \pi_i,\pi_j\rangle^d = \langle \pi_i,\pi_j\rangle = \begin{cases} 0 & \text{if } i<j, \\ \dfrac{2}{2i+1} & \text{if } i=j, \end{cases} \tag{3.4}$$

for $j = 0,\dots,M-1$,

$$\langle \pi_i,\pi_M\rangle^d = \begin{cases} \langle \pi_i,\pi_M\rangle = 0 & \text{if } i<M-1, \\ \dfrac{2\gamma}{2M-1} & \text{if } i=M-1, \\ \dfrac{2\gamma^2}{2M-1} & \text{if } i=M, \end{cases} \tag{3.5}$$

since $\pi_M(\tau_m(\gamma)) = \gamma\pi_{M-1}(\tau_m(\gamma))$ and $\langle \pi_i,\pi_M\rangle^d = \gamma\langle \pi_i,\pi_{M-1}\rangle^d$. Also, if $i+j \le 2M-1$, then

$$\begin{aligned} \langle D\pi_i,\pi_j\rangle^d &= \langle D\pi_i,\pi_j\rangle \\ &= \begin{cases} 0 & \text{if } i\le j, \\ \pi_i\pi_j\big|_{-1}^{1} - \langle \pi_i, D\pi_j\rangle = 1-(-1)^{i+j} & \text{if } i>j. \end{cases} \end{aligned} \tag{3.6}$$

4. A-stability analysis. We will analyze the A-stability properties of the methods corresponding to approximate and discretized problems with respect to the parameter γ for the following cases:

(1) $L = 0$: the completely discontinuous method introduced in [3];

(2) $L = 2$ with $u_n(t_{n-1}) = U_{n-1}$ and $u_n(t_n) = U_n$: the continuous method presented in [19, 20];

(3) $L = 1$ with $u_n(t_n) = U_n$: the discontinuous method described in [25], and also a special case of the discontinuous α-method of [4] ($\alpha_n = 1$ for all n);

(4) $L = 1$ with $u_n(t_n) = U_{n-1}$: a special case of the discontinuous α-method of [4] ($\alpha_n = 0$ for all n).

For the A-stability analysis, we consider the following form of (1.1):

$$\begin{aligned} \dot{y}(t) &= \lambda y(t), \quad 0 \le t \le T, \\ y(0) &= y_0, \end{aligned} \tag{4.1}$$

and we solve Problem 2.1 to obtain

$$U_n = R\left(\frac{\lambda h_n}{2}\right) U_{n-1}, \tag{4.2}$$

where $R(z) = P(z)/Q(z)$ is a rational function and $h_n = t_n - t_{n-1}$.

Definition 4.1. The *region of stability* of a method is the set

$$S = \{z \in \mathbb{C} \mid |R(z)| \le 1\}. \tag{4.3}$$

Let $\operatorname{Re}(z)$ be the real part of z; a method is said to be

(i) *A-stable* if $|R(z)| < 1$ whenever $\operatorname{Re}(z) < 0$;

(ii) *stiff A-stable* if it is A-stable and $\lim_{\operatorname{Re}(z)\to-\infty} |R(z)| = 0$.

To obtain (4.2) from Problem 2.1, let

$$\pi_{ni}(t) = \pi_i\left(\frac{2t - (t_{n-1} + t_n)}{t_n - t_{n-1}}\right) \tag{4.4}$$

be the polynomial of degree i defined on $I_n = [t_{n-1}, t_n]$ normalized such that $\pi_{ni}(t_n) = 1 = (-1)^i \pi_{ni}(t_{n-1})$. Then $\{\pi_{ni}\}_{i=0}^{K}$ and $\{\pi_{nj}\}_{j=0}^{K+1-L}$ form a basis for $P^K(I_n)$ and $P^{K+1-L}(I_n)$. Hence, the polynomial $u_n(\cdot)$ that we look for can be written as $u_n(t) = \sum_{i=0}^{K} a_{ni}\pi_{ni}(t)$. Then Problem 2.1 becomes the following problem.

Problem 4.2. Let $U_0 = y_0$, and for $n = 1, \ldots, N$, find $a_{n0}, \ldots, a_{nK}, U_n \in \mathbb{R}$, such that $\sum_{i=0}^{K} [(\lambda h_n/2)\langle \pi_i, \pi_j\rangle - \langle D\pi_i, \pi_j\rangle + 1 - (-1)^{i+j}] a_{ni} = U_n - (-1)^j U_{n-1}$ for $j = 0, \ldots, K+1-L$, and $u_n(\cdot)$ is subject to L additional conditions.

In the sequel of this paper, we will use the notation $R_{L,K}(z;\gamma)$ for the amplifying factor $R(\lambda h_n/2)$, and $\lambda h_n/2$ is replaced by z.

4.1. The case $L = 0$. For the interval I_n, we use the quadrature formula (3.2) with $M = K+1$. Consequently, $\langle D\pi_i, \pi_j\rangle^d$ is always exact, and $\langle \pi_i, \pi_j\rangle^d$ is exact for $i + j \le 2K$ or for $i + j \le 2K+1$ if $\gamma = 0$. Hence, for the stability analysis, the approximate problem is equivalent to the discretized problem for $\gamma = 0$.

In this case, we have the following system:

$$\sum_{i=0}^{j}[1-(-1)^{i+j}]a_{ni}+\frac{2z}{2j+1}a_{nj}=U_n-(-1)^{j}U_{n-1} \tag{4.5a}$$

for $j=0,\dots,K$, and

$$\sum_{i=1}^{K}[1-(-1)^{i+K+1}]a_{ni}+\frac{2\gamma z}{2K+1}a_{nK}=U_n-(-1)^{K+1}U_{n-1}, \tag{4.5b}$$

which is equivalent to

$$U_n=U_{n-1}+2za_{n0} \tag{4.6a}$$

and

$$\begin{bmatrix} 1-z & z & & & & & & 0 \\ -z & 3 & z & & & & & \\ & \ddots & \ddots & \ddots & & & & \\ & & -z & 2i+1 & z & & & \\ & & & \ddots & \ddots & \ddots & & \\ & 0 & & & -z & 2K-1 & & z \\ & & & & & -z & & [(2K+1)+\gamma z] \end{bmatrix} \begin{bmatrix} a_{n0} \\ \frac{a_{n1}}{3} \\ \vdots \\ \frac{a_{ni}}{(2i+1)} \\ \vdots \\ \frac{a_{nK}}{(2K+1)} \end{bmatrix} = \begin{bmatrix} U_{n-1} \\ 0 \\ \vdots \\ 0 \\ \vdots \\ 0 \end{bmatrix}. \tag{4.6b}$$

The computation of $R_{0,K}(z;\gamma)$ is based on the following two lemmas.

Lemma 4.3. *Let $A_{k,k-1}=1$, $A_{k,k-2}=0$, and for $k\le\ell$*

$$A_{k\ell}=\begin{vmatrix} 2k+1 & z & & & \\ -z & 2k+3 & z & 0 & \\ & \ddots & \ddots & \ddots & \\ & 0 & -z & 2\ell-1 & z \\ & & & -z & 2\ell+1 \end{vmatrix}. \tag{4.7}$$

Then, for $k\le\ell$

$$\begin{aligned} A_{k\ell}&=(2k+1)A_{k+1,\ell}+z^2A_{k+2,\ell},\\ A_{k\ell}&=(2\ell+1)A_{k,\ell-1}+z^2A_{k,\ell-2}. \end{aligned} \tag{4.8}$$

Lemma 4.4. *For $\ell \geq k$, $A_{k\ell}$ is a polynomial in z^2. More precisely,*

(i) *if $\ell - k = 2n+1$, then $A_{k\ell} = z^{2n+2} + p_n(z^2)$,*

(ii) *if $\ell - k = 2n$, then $A_{k\ell} = (n+1)(2k+2n+1)z^{2n} + p_{n-1}(z^2)$,*

for $n = 0,1,2,\dots$, and where $p_j(z)$ is a polynomial of degree j in z $(p_{-1}(z) = 0)$.

As a direct consequence of Cramer's rule, we have

$$a_{n0} = U_{n-1} \frac{A_{1K} + \gamma z A_{1K-1}}{(A_{0K} + \gamma z A_{0K-1}) - z(A_{1K} + \gamma z A_{1K-1})} \tag{4.9}$$

and from the properties of $A_{k\ell}$, we obtain

$$R_{0K}(z;\gamma) = \frac{(A_{0K} + \gamma z A_{0K-1}) + z(A_{1K} + \gamma z A_{1K-1})}{(A_{0K} + \gamma z A_{0K-1}) - z(A_{1K} + \gamma z A_{1K-1})} \tag{4.10a}$$

or

$$R_{0K}(z;\gamma) = \frac{(A_{0K} + z A_{1K}) + \gamma z(A_{0K-1} + z A_{1K-1})}{(A_{0K} - z A_{1K}) + \gamma z(A_{0K-1} - z A_{1K-1})}. \tag{4.10b}$$

Let

$$Q_K(z) = A_{0K-1} + z A_{1K-1} \tag{4.11}$$

for $K = 0,1,2,\dots$. From Lemma 4.4,

$$Q_K(-z) = A_{0K-1} - z A_{1K-1} \tag{4.12}$$

and we have the following results.

Lemma 4.5. *The polynomials $Q_k(z)$ can be generated recursively by $Q_0(z) = 1$, $Q_1(z) = 1+z$, and for $K \geq 1$*

$$Q_{K+1}(z) = (2K+1)Q_K(z) + z^2 Q_{K-1}(z). \tag{4.13}$$

Proof. The proof is a direct consequence of Lemma 4.3. □

Lemma 4.6. *For $K \geq 0$, $Q_K(z) = \sum_{i=0}^{K}((2K-i)!/2^{K-i}(K-i)!i!)z^i$.*

Proof. Since $Q_K(z)$ is a polynomial of degree K, let $Q_K(z) = \sum_{i=0}^{K} a_{Ki} z^i$. Then $a_{00} = a_{10} = a_{11} = 1$, and for $K \geq 2$, we have

$$a_{Ki} = (2K-1)a_{K-1,i} + a_{K-2,i-2}, \tag{4.14}$$

for $i = 0,\dots,K$, considering $a_{Kj} = 0$ if $j < 0$ or $j > K$. Then the result follows by induction. □

Theorem 4.7. *Let $P_{K+1}(z;\gamma) = Q_{K+1}(z) + \gamma z Q_K(z)$. Then*

$$P_{K+1}(z;\gamma) = \sum_{i=0}^{K+1} \frac{(2(K+1)-i)!}{2^{K+1-i}(K+1-i)!i!}\left[1+\gamma\frac{i}{2(K+1)-i}\right]z^i,$$
$$R_{0K}(z;\gamma) = \frac{P_{K+1}(z;\gamma)}{P_{K+1}(-z;-\gamma)}. \tag{4.15}$$

Moreover, the following limits exist:

$$\lim_{|z|\to-\infty} \left|R_{0K}(z;\gamma)\right| = \frac{1+\gamma}{1-\gamma} \quad \text{for } \gamma \in [-1,1),$$
$$\lim_{|z|\to+\infty} \left|R_{0K}(z;1)\right| = +\infty \quad \text{for } \gamma = 1. \tag{4.16}$$

From the properties of $A_{k\ell}$ of Lemma 4.3, we obtain the following expression.

Lemma 4.8. *For any complex number z,*

$$z\left(A_{1K}+\gamma z A_{1K-1}\right)\left(\overline{A_{0K}+\gamma z A_{0K-1}}\right)$$
$$= \gamma|z|^{2K+2} + \sum_{i=0}^{K}(2i+1)z_i|z|^{2i}\left|A_{i+1,K}+\gamma z A_{i+1,K-1}\right|^2, \tag{4.17}$$

where $z_i = z$ for i even and $z_i = \bar{z}$ for i odd.

Lemma 4.9. *Let X and Y be two complex numbers, then*

$$\frac{|X+Y|}{|X-Y|} < 1 \quad \text{iff } \operatorname{Re}(Y\bar{X}) < 0. \tag{4.18}$$

Theorem 4.10. *Let $L = 0$ and $K \geq 0$.*

(i) *The method corresponding to the approximate problem (4.5) is A-stable but not stiff A-stable.*

(ii) *Let $\gamma \in [-1,1]$ and $M = K+1$ in (3.2). Then the method corresponding to the discretized problem (4.5) is A-stable for $\gamma = [-1,0]$ and stiff A-stable for $\gamma = -1$.*

Proof. We recall that the result for the approximate problem corresponds to the result for the discretized problem for $\gamma = 0$. For the A-stability, using Lemma 4.9, we observe that $|R_{0K}(z;\gamma)| < 1$ is equivalent to

$$\operatorname{Re}\left\{z\left(A_{1K}+\gamma z A_{1K-1}\right)\left(\overline{A_{0K}+\gamma z A_{0K-1}}\right)\right\} < 0. \tag{4.19}$$

From (4.10) and the expression in Lemma 4.8, (4.19) is satisfied for $\operatorname{Re}(z) < 0$ if $\gamma \leq 0$. From the limits of Theorem 4.7, it follows that for any $\gamma > 0$, there exists z such that $\operatorname{Re}(z) < 0$ and $|R_{0K}(z;\gamma)| > 1$. Then the result on A-stability follows. The stiff A-stability for $\gamma = -1$ follows also from Theorem 4.7. □

Example 4.11. (i) $K = 0$,

$$R_{00}(z;\gamma) = \frac{1+(1+\gamma)z}{1-(1-\gamma)z}; \tag{4.20}$$

(ii) $K = 1$,

$$R_{01}(z;\gamma) = \frac{3+(3+\gamma)z+(1+\gamma)z^2}{3-(3-\gamma)z+(1-\gamma)z^2}; \tag{4.21}$$

(iii) $K = 2$,

$$R_{02}(z;\gamma) = \frac{15+(15+3\gamma)z+(6+3\gamma)z^2+(1+\gamma)z^3}{15-(15-3\gamma)z+(6-3\gamma)z^2-(1-\gamma)z^3}; \tag{4.22}$$

(iv) $K = 3$,

$$R_{03}(z;\gamma) = \frac{105+(105+15\gamma)z+(45+15\gamma)z^2+(10+6\gamma)z^3+(1+\gamma)z^4}{105-(105-15\gamma)z+(45-15\gamma)z^2-(10-6\gamma)z^3+(1-\gamma)z^4}. \tag{4.23}$$

Remark 4.12. Since $R_{0K}(z;\gamma) = 1/R_{0K}(-z;-\gamma)$, the stability region for $-1 \le \gamma < 0$ is the exterior of the mirror image in the imaginary axis of the corresponding region for $-\gamma$. For $\gamma = 0$, the stability region is the left half-plane. Thus, it is sufficient to describe the bounded regions for $0 < \gamma \le 1$.

Remark 4.13. When $\gamma = -1, 0$, and 1, the ratios of Example 4.11 correspond to the subdiagonal, diagonal, and superdiagonal element of the Padé table for e^{2z}, respectively. Other values of γ give intermediate rational approximations of the exponential. These rational approximations of e^{2z} have already been analyzed; they appear in [7, 18, 27, 28].

Remark 4.14. The proof of the A-stability presented here seems to be new in the sense that it uses only elementary properties of Legendre polynomials and Gaussian quadrature rules. However, we do not obtain the stability region in the case $\gamma \in (0,1]$, we obtain only the fact that it is not A-stable. One way to obtain the stability region is to use the order stars approach [21, 22].

4.2. The case $L = 2$: $u_n(t_{n-1}) = U_{n-1}$ and $u_n(t_n) = U_n$. In this case, we use the quadrature formula (3.2) with $M = K$. Hence, $\langle D\pi_i, \pi_j\rangle^d$ is always exact, and $\langle \pi_i, \pi_j\rangle^d$ is exact for $i+j \le 2K-2$ or for $i+j \le 2K-1$ if $\gamma = 0$.

Then the system is

$$\sum_{i=0}^{j} [1-(-1)^{i+j}]a_{ni} + \frac{2z}{2j+1}a_{nj} = U_n - (-1)^j U_{n-1} \tag{4.24a}$$

for $j = 0,\dots,K-2$,

$$\sum_{i=0}^{K-1}[1-(-1)^{i+K-1}]a_{ni}+\frac{2z}{2K-1}a_{nK-1}+\frac{2\gamma z}{2K-1}a_{nK}=U_n-(-1)^{K-1}U_{n-1}, \tag{4.24b}$$

$$\sum_{i=0}^{K}(-1)^i a_{ni}=U_{n-1}, \tag{4.24c}$$

$$\sum_{i=0}^{K}a_{ni}=U_n, \tag{4.24d}$$

which is equivalent to

$$U_n=U_{n-1}+2za_{n0}, \tag{4.25a}$$

$$\sum_{i=0}^{K}(-1)^{i+1}a_{ni}=-U_{n-1}, \tag{4.25b}$$

$$-2za_{nj}+(2j+1)\sum_{i=j+1}^{K}[1-(-1)^{i+j}]a_{ni}=0, \tag{4.25c}$$

for $j = 0,2,\dots,K-2$, and

$$-za_{nK-1}+[(2K-1)-\gamma z]a_{nK}=0. \tag{4.25d}$$

Solving (4.25) for a_{n0} using Cramer's rule, we obtain

$$\begin{aligned} a_{n0}&=U_{n-1}\frac{A_{1K-1}-\gamma zA_{1K-2}}{(A_{0K-1}-\gamma zA_{0K-2})-z(A_{1K-1}-\gamma zA_{1K-2})},\\ R_{2K}(z;\gamma)&=\frac{(A_{0K-1}-\gamma zA_{0K-2})+z(A_{1K-1}-\gamma zA_{1K-2})}{(A_{0K-1}-\gamma zA_{0K-2})-z(A_{1K-1}-\gamma zA_{1K-2})}, \end{aligned} \tag{4.26}$$

and the next result follows.

Theorem 4.15. *For any $\gamma\in[-1,1]$,*

$$R_{2,K+1}(z;\gamma)=R_{0K}(z;-\gamma). \tag{4.27}$$

Theorem 4.16. *Let $L=2$, $u_n(t_n)=U_n$, $u_n(t_{n-1})=U_{n-1}$, and $K\ge 1$.*

(1) *The method corresponding to the approximate problem (4.24a) is A-stable but not stiff A-stable.*

(2) *Let $\gamma\in[-1,1]$ and $M=K$ in (3.2). Then the method corresponding to the discretized problem (4.24a) is A-stable for $\gamma\in[0,1]$ and stiff A-stable for $\gamma=1$.*

4.3. The case $L=1$: $u_n(t_n)=U_n$. In this case, using the quadrature rule (3.2) with $M=K+1$, any term of the forms $\langle\pi_i,\pi_j\rangle^d$ or $\langle D\pi_i,\pi_j\rangle^d$ is integrated exactly for $i,j\le K$. Hence, the approximate problem and the discretized problems have the same A-stability property.

The system is

$$\sum_{i=0}^{j}[1-(-1)^{i+j}]a_{ni}+\frac{2z}{2j+1}a_{nj}=U_n-(-1)^jU_{n-1} \tag{4.28a}$$

for $j=0,\dots,K$, and

$$\sum_{i=0}^{K}a_{ni}=U_n. \tag{4.28b}$$

But, this system is equivalent to

$$U_n=U_{n-1}+2za_{n0}, \tag{4.29a}$$

$$(1-z)a_{n0}+\frac{z}{3}a_{n1}=U_{n-1}, \tag{4.29b}$$

$$-\frac{z}{2j-1}a_{nj-1}+a_{nj}+\frac{z}{2j+1}a_{nj+1}=0, \tag{4.29c}$$

for $j=1,\dots,K-1$, and

$$-\frac{z}{2K-1}a_{nK-1}+\left(1-\frac{z}{2K+1}\right)a_{nK}=0. \tag{4.29d}$$

Hence,

$$\begin{aligned}a_{n0}&=U_{n-1}\frac{A_{1K}-zA_{1K-1}}{(A_{0K}-zA_{0K-1})-z(A_{1k}-zA_{1K-1})},\\ R^r_{1K}(z;\gamma)&=\frac{(A_{0K}-zA_{0K-1})+z(A_{1K}-zA_{1K-1})}{(A_{0K}-zA_{0K-1})-z(A_{1K}-zA_{1K-1})},\end{aligned} \tag{4.30}$$

and we have the next result.

Theorem 4.17. *For any $\gamma\in[-1,1]$,*

$$R^r_{1K}(z;\gamma)=R_{0K}(z;-1), \tag{4.31}$$

and the methods corresponding to the approximate and the discretized problems (4.28) are stiff A-stable.

4.4. The case $L=1$: $u_n(t_{n-1})=U_{n-1}$. In this case, we have

$$\sum_{i=0}^{j}[1-(-1)^{i+j}]a_{ni}+\frac{2z}{2j+1}a_{nj}=U_n-(-1)^jU_{n-1} \tag{4.32a}$$

for $j=0,\dots,K$, and

$$\sum_{i=0}^{K}(-1)^ia_{ni}=U_{n-1}. \tag{4.32b}$$

This system is equivalent to

$$U_n = U_{n-1} + 2za_{n0}, \tag{4.33a}$$

$$(1-z)a_{n0} + \frac{z}{3}a_{n1} = U_{n-1}, \tag{4.33b}$$

$$-\frac{z}{2j-1}a_{nj-1} + a_{nj} + \frac{z}{2j+1}a_{nj+1} = 0, \tag{4.33c}$$

for $j = 1,\dots,K-1$, and

$$-\frac{z}{2K-1}a_{nK-1} + \left(1+\frac{z}{2K+1}\right)a_{nK} = 0. \tag{4.33d}$$

Then

$$\begin{aligned} a_{n0} &= U_{n-1}\frac{A_{1K} + zA_{1K-1}}{(A_{0K}+zA_{0K-1}) - z(A_{1k}+zA_{1K-1})},\\ R^{\ell}_{1K}(z;\gamma) &= \frac{(A_{0K}+zA_{0K-1}) + z(A_{1K}+zA_{1K-1})}{(A_{0K}+zA_{0K-1}) - z(A_{1K}+zA_{1K-1})}, \end{aligned} \tag{4.34}$$

and we obtain the last result.

Theorem 4.18. *For any $\gamma \in [-1,1]$,*

$$R^{\ell}_{1K}(z;\gamma) = R_{0K}(z;1), \tag{4.35}$$

and the methods corresponding to the approximate and the discretized problems (4.32) are not A-stable.

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