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K-THEORY FOR CUNTZ-KRIEGER ALGEBRAS ARISING FROM REAL QUADRATIC MAPS

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We compute the K-groups for the Cuntz-Krieger algebras $\mathcal{O}_{A_{\mathcal{K}(f_\mu)}}$, where $A_{\mathcal{K}(f_\mu)}$ is the Markov transition matrix arising from the kneading sequence $\mathcal{K}(f_\mu)$ of the one-parameter family of real quadratic maps f_μ.

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Consider the one-parameter family of real quadratic maps $f_\mu : [0,1] \to [0,1]$ defined by $f_\mu(x) = \mu x(1-x)$, with $\mu \in [0,4]$. Using Milnor-Thurston kneading theory [14], Guckenheimer [5] has classified, up to topological conjugacy, a certain class of maps, which includes the quadratic family. The idea of kneading theory is to encode information about the orbits of a map in terms of infinite sequences of symbols and to exploit the natural order of the interval to establish topological properties of the map. In what follows, I denotes the unit interval $[0,1]$ and c the unique turning point of f_μ. For $x \in I$, let

$$\varepsilon_n(x) = \begin{cases} -1, & \text{if } f_\mu^n(x) > c, \\ 0, & \text{if } f_\mu^n(x) = c, \\ +1, & \text{if } f_\mu^n(x) < c. \end{cases} \tag{1}$$

The sequence $\varepsilon(x) = (\varepsilon_n(x))_{n=0}^{\infty}$ is called the itinerary of x. The itinerary of $f_\mu(c)$ is called the *kneading sequence* of f_μ and will be denoted by $\mathcal{K}(f_\mu)$. Observe that $\varepsilon_n(f_\mu(x)) = \varepsilon_{n+1}(x)$, that is, $\varepsilon(f_\mu(x)) = \sigma\varepsilon(x)$, where σ is the shift map. Let $\sum = \{-1,0,+1\}$ be the alphabet set. The sequences on $\sum^{\mathbb{N}}$ are ordered lexicographically. However, this ordering is not reflected by the mapping $x \to \varepsilon(x)$ because the map f_μ reverses orientation on $[c,1]$. To take this into account, for a sequence $\varepsilon = (\varepsilon_n)_{n=0}^{\infty}$ of the symbols -1, 0, and $+1$, another sequence $\theta = (\theta_n)_{n=0}^{\infty}$ is defined by $\theta_n = \prod_{i=0}^{n} \varepsilon_i$. If $\varepsilon = \varepsilon(x)$ is the itinerary of a point $x \in I$, then $\theta = \theta(x)$ is called the *invariant coordinate* of x. The fundamental observation of Milnor and Thurston [14] is the monotonicity of the invariant coordinates:

$$x < y \Longrightarrow \theta(x) \le \theta(y). \tag{2}$$

We now consider only those kneading sequences that are periodic, that is,

$$\begin{aligned}\mathcal{K}(f_\mu) &= \varepsilon_0(f_\mu(c))\cdots\varepsilon_{n-1}(f_\mu(c))\varepsilon_0(f_\mu(c))\cdots\varepsilon_{n-1}(f_\mu(c))\cdots \\ &= \big(\varepsilon_0(f_\mu(c))\cdots\varepsilon_{n-1}(f_\mu(c))\big)^\infty \equiv \big(\varepsilon_1(c)\cdots\varepsilon_n(c)\big)^\infty\end{aligned} \tag{3}$$

for some $n \in \mathbb{N}$. The sequences $\sigma^i(\mathcal{K}(f_\mu)) = \varepsilon_{i+1}(c)\varepsilon_{i+2}(c)\cdots$, $i = 0,1,2,\ldots$, will then determine a Markov partition of I into $n-1$ line intervals $\{I_1, I_2, \ldots, I_{n-1}\}$ [15], whose definitions will be given in the proof of Theorem 1. Thus, we will have a Markov transition matrix $A_{\mathcal{K}(f_\mu)}$ defined by

$$A_{\mathcal{K}(f_\mu)} := (a_{ij}) \quad \text{with } a_{ij} = \begin{cases} 1, & \text{if } f_\mu(\operatorname{int} I_i) \supseteq \operatorname{int} I_j, \\ 0, & \text{otherwise.}\end{cases} \tag{4}$$

It is easy to see that the matrix $A_{\mathcal{K}(f_\mu)}$ is not a permutation matrix and no row or column of $A_{\mathcal{K}(f_\mu)}$ is zero. Thus, for each one of these matrices and following the work of Cuntz and Krieger [3], one can construct the Cuntz-Krieger algebra $\mathcal{O}_{A_{\mathcal{K}(f_\mu)}}$. In [2], Cuntz proved that

$$K_0(\mathcal{O}_A) \cong \mathbb{Z}^r/(1-A^T)\mathbb{Z}^r, \qquad K_1(\mathcal{O}_A) \cong \ker(I - A^t : \mathbb{Z}^r \longrightarrow \mathbb{Z}^r), \tag{5}$$

for an $r \times r$ matrix A that satisfies a certain condition (I) (see [3]), which is readily verified by the matrices $A_{\mathcal{K}(f_\mu)}$. In [1], Bowen and Franks introduced the group $BF(A) := \mathbb{Z}^r/(1-A)\mathbb{Z}^r$ as an invariant for flow equivalence of topological Markov subshifts determined by A.

We can now state and prove the following theorem.

Theorem 1. *Let $\mathcal{K}(f_\mu) = (\varepsilon_1(c)\varepsilon_2(c)\cdots\varepsilon_n(c))^\infty$ for some $n \in \mathbb{N}\setminus\{1\}$. Thus,*

$$\begin{aligned} K_0\big(\mathcal{O}_{A_{\mathcal{K}(f_\mu)}}\big) &\cong \mathbb{Z}_a \quad \text{with } a = \left|1 + \sum_{l=1}^{n-1}\prod_{i=1}^{l}\varepsilon_i(c)\right|, \\ K_1\big(\mathcal{O}_{A_{\mathcal{K}(f_\mu)}}\big) &\cong \begin{cases}\{0\}, & \text{if } a \neq 0, \\ \mathbb{Z}, & \text{if } a = 0.\end{cases}\end{aligned} \tag{6}$$

Proof. Set $z_i = \varepsilon_i(c)\varepsilon_{i+1}(c)\cdots$ for $i = 1,2,\ldots$. Let $z_i' = f_\mu^i(c)$ be the point on the unit interval $[0,1]$ represented by the sequence z_i for $i = 1,2,\ldots$. We have $\sigma(z_i) = z_{i+1}$ for $i = 1,\ldots,n-1$ and $\sigma(z_n) = z_1$. Denote by ω the $n\times n$ matrix representing the shift map σ. Let C_0 be the vector space spanned by the formal basis $\{z_1',\ldots,z_n'\}$. Now, let ρ be the permutation of the set $\{1,\ldots,n\}$, which allows us to order the points $z_1',\ldots,z_n'$ on the unit interval $[0,1]$, that is,

$$0 < z'_{\rho(1)} < z'_{\rho(2)} < \cdots < z'_{\rho(n)} < 1. \tag{7}$$

Set $x_i := z'_{\rho(i)}$ with $i = 1,\ldots,n$ and let π denote the permutation matrix which takes the formal basis $\{z'_1,\ldots,z'_n\}$ to the formal basis $\{x_1,\ldots,x_n\}$. We will denote by C_1 the $(n-1)$-dimensional vector space spanned by the formal basis $\{x_{i+1} - x_i : i = 1,\ldots,n-1\}$. Set

$$I_i := [x_i, x_{i+1}] \quad \text{for } i = 1,\ldots,n-1. \tag{8}$$

Thus, we can define the Markov transition matrix $A_{\mathcal{K}(f_\mu)}$ as above. Let φ denote the incidence matrix that takes the formal basis $\{x_1,\ldots,x_n\}$ of C_0 to the formal basis $\{x_2 - x_1,\ldots,x_n - x_{n-1}\}$ of C_1. Put $\eta := \varphi\pi$. As in [7, 8], we obtain an endomorphism α of C_1, that makes the following diagram commutative:

$$\begin{array}{ccc} C_0 & \xrightarrow{\ \eta\ } & C_1 \\ \Big\downarrow{\scriptstyle \omega} & & \Big\downarrow{\scriptstyle \alpha} \\ C_0 & \xrightarrow[\ \eta\ ]{} & C_1. \end{array} \tag{9}$$

We have $\alpha = \eta\omega\eta^T(\eta\eta^T)^{-1}$. Remark that if we neglect the negative signs on the matrix α, then we will obtain precisely the Markov transition matrix $A_{\mathcal{K}(f_\mu)}$. In fact, consider the $(n-1)\times(n-1)$ matrix

$$\beta := \begin{bmatrix} 1_{n_L} & 0 \\ 0 & -1_{n_R} \end{bmatrix}, \tag{10}$$

where 1_{n_L} and 1_{n_R} are the identity matrices of ranks n_L and n_R, respectively, with n_L (n_R) being the number of intervals I_i of the Markov partition placed on the left- (right-) hand side of the turning point of f_μ. Therefore, we have

$$A_{\mathcal{K}(f_\mu)} = \beta\alpha. \tag{11}$$

Now, consider the following matrix:

$$\gamma_{\mathcal{K}(f_\mu)} := (\gamma_{ij}) \quad \text{with} \begin{cases} \gamma_{ii} = \varepsilon_i(c), & i = 1,\ldots,n, \\ \gamma_{in} = -\varepsilon_i(c), & i = 1,\ldots,n, \\ \gamma_{ij} = 0, & \text{otherwise.} \end{cases} \tag{12}$$

The matrix $\gamma_{\mathcal{K}(f_\mu)}$ makes the diagram

$$\begin{array}{ccc} C_0 & \xrightarrow{\ \eta\ } & C_1 \\ \Big\downarrow{\scriptstyle \gamma_{\mathcal{K}(f_\mu)}} & & \Big\downarrow{\scriptstyle \beta} \\ C_0 & \xrightarrow[\ \eta\ ]{} & C_1 \end{array} \tag{13}$$

commutative. Finally, set $\theta_{\mathcal{K}(f_\mu)} := \theta_{\mathcal{K}(f_\mu)}\omega$. Then, the diagram

$$\begin{array}{ccc} C_0 & \xrightarrow{\ \eta\ } & C_1 \\ {\scriptstyle \theta_{\mathcal{K}(f_\mu)}}\downarrow & & \downarrow{\scriptstyle A_{\mathcal{K}(f_\mu)}} \\ C_0 & \xrightarrow[\ \eta\ ]{} & C_1 \end{array} \tag{14}$$

is also commutative. Now, notice that the transpose of η has the following factorization:

$$\eta^T = YiX, \tag{15}$$

where Y is an invertible (over $\mathbb{Z}$) $n \times n$ integer matrix given by

$$Y := \begin{pmatrix} 1 & 0 & \cdots & & & 0 \\ 0 & 1 & 0 & \cdots & & 0 \\ \vdots & 0 & \ddots & \ddots & & \vdots \\ & \vdots & & & 0 & \\ 0 & 0 & \cdots & 0 & 1 & 0 \\ -1 & -1 & \cdots & & -1 & 1 \end{pmatrix}, \tag{16}$$

i is the inclusion $C_1 \hookrightarrow C_0$ given by

$$i := \begin{pmatrix} 1 & 0 & & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & & 0 \\ & & & 1 \\ 0 & \cdots & & 0 \end{pmatrix}, \tag{17}$$

and X is an invertible (over $\mathbb{Z}$) $(n-1) \times (n-1)$ integer matrix obtained from the $(n-1) \times n$ matrix η^T by removing the nth row of η^T. Thus, from the commutative diagram

$$\begin{array}{ccc} C_1 & \xrightarrow{\ \eta^T\ } & C_0 \\ {\scriptstyle A^T_{\mathcal{K}(f_\mu)}}\downarrow & & \downarrow{\scriptstyle \theta^T_{\mathcal{K}(f_\mu)}} \\ C_1 & \xrightarrow[\ \eta^T\ ]{} & C_0, \end{array} \tag{18}$$

we will have the following commutative diagram with short exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C_1 & \xrightarrow{i} & C_0 & \xrightarrow{p} & C_0/C_1 & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle A'} & & \downarrow{\scriptstyle \theta'} & & \downarrow{\scriptstyle 0} & & \\ 0 & \longrightarrow & C_1 & \xrightarrow[i]{} & C_0 & \xrightarrow[p]{} & C_0/C_1 & \longrightarrow & 0, \end{array} \tag{19}$$

where the map p is represented by the $1\times n$ matrix $[\,0 \cdots 0\ 1\,]$ and

$$A' = XA^T_{\mathcal{K}(f_\mu)}X^{-1}, \qquad \theta' = Y^{-1}\theta^T_{\mathcal{K}(f_\mu)}Y, \tag{20}$$

that is, A' is similar to $A^T_{\mathcal{K}(f_\mu)}$ over $\mathbb{Z}$ and θ' is similar to $\theta^T_{\mathcal{K}(f_\mu)}$ over $\mathbb{Z}$. Hence, for example, by [10] we obtain, respectively,

$$\begin{gathered} \mathbb{Z}^{n-1}/(1-A')\mathbb{Z}^{n-1} \cong \mathbb{Z}^{n-1}/(1-A_{\mathcal{K}(f_\mu)})\mathbb{Z}^{n-1}, \\ \mathbb{Z}^{n}/(1-\theta')\mathbb{Z}^{n} \cong \mathbb{Z}^{n}/(1-\theta_{\mathcal{K}(f_\mu)})\mathbb{Z}^{n}. \end{gathered} \tag{21}$$

Now, from the last diagram we have, for example, by [9],

$$\theta' = \begin{bmatrix} A' & * \\ 0 & 0 \end{bmatrix}. \tag{22}$$

Therefore,

$$\begin{gathered} \mathbb{Z}^{n-1}/(1-A')\mathbb{Z}^{n-1} \cong \mathbb{Z}^{n}/(1-\theta')\mathbb{Z}^{n}, \\ \mathbb{Z}^{n-1}/(1-A_{\mathcal{K}(f_\mu)})\mathbb{Z}^{n-1} \cong \mathbb{Z}^{n}/(1-\theta_{\mathcal{K}(f_\mu)})\mathbb{Z}^{n}. \end{gathered} \tag{23}$$

Next, we will compute $\mathbb{Z}^n/(1-\theta_{\mathcal{K}(f_\mu)})\mathbb{Z}^n$. From the previous discussions and notations, the $n\times n$ matrix $\theta_{\mathcal{K}(f_\mu)}$ is explicitly given by

$$\theta_{\mathcal{K}(f_\mu)} := \begin{pmatrix} -\varepsilon_1(c) & \varepsilon_1(c) & 0 & \cdots & 0 \\ \vdots & 0 & \ddots & \ddots & \vdots \\ & \vdots & \ddots & & 0 \\ -\varepsilon_{n-1}(c) & & & & \varepsilon_{n-1}(c) \\ 0 & 0 & \cdots & & 0 \end{pmatrix}. \tag{24}$$

Notice that the matrix $\theta_{\mathcal{K}(f_\mu)}$ completely describes the dynamics of f_μ. Finally, using row and column elementary operations over $\mathbb{Z}$, we can find invertible (over $\mathbb{Z}$) matrices U_1 and U_2 with integer entries such that

$$1-\theta_{\mathcal{K}(f_\mu)} = U_1 \begin{pmatrix} 1+\sum_{l=1}^{n-1}\prod_{i=1}^{l}\varepsilon_i(c) & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix} U_2. \tag{25}$$

Thus, we obtain

$$K_0\left(\mathcal{O}_{A_{\mathcal{K}(f_\mu)}}\right) \cong \mathbb{Z}^{n-1}/\left(1-A^T_{\mathcal{K}(f_\mu)}\right)\mathbb{Z}^{n-1} \cong \mathbb{Z}_a, \tag{26}$$

where

$$a = \left|1+\sum_{l=1}^{n-1}\prod_{i=1}^{l}\varepsilon_i(c)\right|, \quad n \in \mathbb{N}\backslash\{1\}. \tag{27}$$

□

Example 2. Set

$$\mathcal{K}(f_\mu) = (RLLRRC)^\infty, \tag{28}$$

where $R=-1$, $L=+1$, and $C=0$. Thus, we can construct the 5×5 Markov transition matrix $A_{\mathcal{K}(f_\mu)}$ and the matrices $\theta_{\mathcal{K}(f_\mu)}$, ω, φ, and π:

$$A_{\mathcal{K}(f_\mu)} = \begin{pmatrix} 0&1&1&0&0\\ 0&0&0&1&1\\ 0&0&0&0&1\\ 0&0&1&1&0\\ 1&1&0&0&0 \end{pmatrix}, \qquad \theta_{\mathcal{K}(f_\mu)} = \begin{pmatrix} 1&-1&0&0&0&0\\ -1&0&1&0&0&0\\ -1&0&0&1&0&0\\ 1&0&0&0&-1&0\\ 1&0&0&0&0&-1\\ 0&0&0&0&0&0 \end{pmatrix},$$

$$\omega = \begin{pmatrix} 0&1&0&0&0&0\\ 0&0&1&0&0&0\\ 0&0&0&1&0&0\\ 0&0&0&0&1&0\\ 0&0&0&0&0&1\\ 1&0&0&0&0&0 \end{pmatrix}, \qquad \varphi = \begin{pmatrix} -1&1&&&&\\ &-1&1&&&\\ &&-1&1&&\\ &&&-1&1&\\ &&&&-1&1 \end{pmatrix}, \tag{29}$$

$$\pi = \begin{pmatrix} 0&1&0&0&0&0\\ 0&0&1&0&0&0\\ 0&0&0&0&0&1\\ 0&0&0&1&0&0\\ 0&0&0&0&1&0\\ 1&0&0&0&0&0 \end{pmatrix}.$$

We have

$$K_0\left(\mathbb{O}_{A_{\mathcal{K}(f_\mu)}}\right) \cong \mathbb{Z}_2, \qquad K_1\left(\mathbb{O}_{A_{\mathcal{K}(f_\mu)}}\right) \cong \{0\}. \tag{30}$$

Remark 3. In the statement of Theorem 1 the case $a = 0$ may occur. This happens when we have a star product factorizable kneading sequence [4]. In this case the correspondent Markov transition matrix is reducible.

Remark 4. In [6], Katayama et al. have constructed a class of C^*-algebras from the β-expansions of real numbers. In fact, considering a semiconjugacy from the real quadratic map to the tent map [14], we can also obtain Theorem 1 using [6] and the λ-expansions of real numbers introduced in [4].

Remark 5. In [13] (see also [12]) and [11], the BF-groups are explicitly calculated with respect to another kind of maps on the interval.

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