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SOME CHARACTERIZATIONS OF SPECIALLY MULTIPLICATIVE FUNCTIONS

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A multiplicative function f is said to be specially multiplicative if there is a completely multiplicative function f_A such that $f(m)f(n)=\sum_{d|(m,n)} f(mn/d^2)f_A(d)$ for all m and n. For example, the divisor functions and Ramanujan's τ-function are specially multiplicative functions. Some characterizations of specially multiplicative functions are given in the literature. In this paper, we provide some further characterizations of specially multiplicative functions.

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1. Introduction. An arithmetical function f is said to be multiplicative if $f(1)=1$ and

$$f(m)f(n)=f(mn) \tag{1.1}$$

whenever $(m,n)=1$. If (1.1) holds for all m and n, then f is said to be completely multiplicative. A multiplicative function is known if the values $f(p^n)$ are known for all prime numbers p and positive integers n. A completely multiplicative function is known if the values $f(p)$ are known for all prime numbers p.

A multiplicative function f is said to be specially multiplicative if there is a completely multiplicative function f_A such that

$$f(m)f(n)=\sum_{d|(m,n)} f\left(\frac{mn}{d^2}\right)f_A(d) \tag{1.2}$$

for all m and n, or equivalently

$$f(mn)=\sum_{d|(m,n)} f\left(\frac{m}{d}\right)f\left(\frac{n}{d}\right)\mu(d)f_A(d) \tag{1.3}$$

for all m and n, where μ is the Möbius function. If $f_A=\delta$, where $\delta(1)=1$ and $\delta(n)=0$ for $n>1$, then (1.2) reduces to (1.1). Therefore, the class of completely multiplicative functions is a subclass of the class of specially multiplicative functions.

The study of specially multiplicative functions was initiated in [7], and arose in an effort to understand the identity

$$\sigma_\alpha(mn)=\sum_{d\mid(m,n)}\sigma_\alpha\left(\frac{m}{d}\right)\sigma_\alpha\left(\frac{n}{d}\right)\mu(d)d^\alpha, \tag{1.4}$$

where $\sigma_\alpha(n)$ denotes the sum of the αth powers of the positive divisors of n. Vaidyanathaswamy used the term "quadratic function," while the term "specially multiplicative function" was coined by Lehmer [3]. For more background information, reference is made to the books by McCarthy [4] and Sivaramakrishnan [6].

The Dirichlet convolution of two arithmetical functions f and g is defined as

$$(f*g)(n)=\sum_{d\mid n}f(d)g\left(\frac{n}{d}\right). \tag{1.5}$$

The function δ serves as the identity under the Dirichlet convolution. An arithmetical function f possesses a Dirichlet inverse f^{-1} if and only if $f(1)\neq 0$.

We next review some basic characterizations of specially multiplicative functions, see [4, 6].

Proposition 1.1. *The following statements are equivalent.*

(1) *The function f is a specially multiplicative function.*

(2) *The function f is the Dirichlet convolution of two completely multiplicative functions a and b. (In this case $f_A=ab$, the usual product of a and b.)*

(3) *The function f is a multiplicative function, and for each prime number p,*

$$f^{-1}(p^n)=0,\quad n\geq 3. \tag{1.6}$$

(In this case $f_A(p)=f^{-1}(p^2)$ for all prime numbers p.)

(4) *The function f is a multiplicative function, and for each prime number p, there exists a complex number $g(p)$ such that*

$$f(p^{n+1})=f(p)f(p^n)-g(p)f(p^{n-1}),\quad n\geq 1. \tag{1.7}$$

(In this case $f_A(p)=g(p)$ for all prime numbers p.)

(5) *The function f is a multiplicative function, and for each prime number p, there exists a complex number $g(p)$ such that*

$$f(p^n)=\sum_{k=0}^{\lfloor n/2\rfloor}(-1)^k\binom{n-k}{k}[f(p)]^{n-2k}[g(p)]^k,\quad n\geq 0. \tag{1.8}$$

(In this case $f_A(p)=g(p)$ for all prime numbers p.)

Remark 1.2. Completely multiplicative functions a and b in part 2 need not be unique. The usual product ab, however, is unique. For example, let a, b, c, and d be completely multiplicative functions such that $a(p) = 1$ and $b(p) = 2$ for all prime numbers p, and $c(2) = 2$, $c(p) = 1$, $d(2) = 1$, and $d(p) = 2$ for all prime numbers $p \neq 2$. Then $a * b = c * d$, but $a, b \neq c$ and $a, b \neq d$. However, $ab = cd$.

The purpose of this paper is to provide some further characterizations of specially multiplicative functions. As applications, we obtain formulas for the usual products $\sigma_\alpha\phi_\beta$, $\sigma_\alpha\sigma_\beta$, and $\sigma_\alpha\tau$, where ϕ_β is a generalized Euler totient function and τ is Ramanujan's τ-function. The function ϕ_β is given by $\phi_\beta = N^\beta * \mu$, where $N^\beta(n) = n^\beta$ for all n. In particular, we denote $N^1 = N$, $N^0 = \zeta$, and $\phi_1 = \phi$, where ϕ is the Euler totient function. Ramanujan's τ-function is a specially multiplicative function with $\tau_A = N^{11}$.

In the characterizations, we need the concepts of the unitary convolution and the kth convolute. The unitary convolution of two arithmetical functions f and g is defined as

$$(f \oplus g)(n) = \sum_{d \| n} f(d) g\left(\frac{n}{d}\right), \tag{1.9}$$

where $d \| n$ means that $d | n$, $(d, n/d) = 1$. The kth convolute of an arithmetical function f is defined as $\Omega_k(f)(n) = f(n^{1/k})$ if n is a kth power, and $\Omega_k(f)(n) = 0$ otherwise.

2. Characterizations

Theorem 2.1. *If f is a specially multiplicative function and g is a completely multiplicative function, then*

$$h * f(g * \mu) = fg, \tag{2.1}$$

where h is the specially multiplicative function such that

$$h(p) = f(p), \qquad h_A(p) = g(p) f_A(p) \tag{2.2}$$

for all prime numbers p. Conversely, if $f(1) = 1$ and there exist completely multiplicative functions a, b, g, and k such that

$$a * b * f(g * \mu) = fg, \tag{2.3}$$

where

$$a(p) + b(p) = f(p), \qquad a(p)b(p) = g(p)k(p), \qquad (g * \mu)(n) \neq g(n) \tag{2.4}$$

for all prime numbers p and integers n (≥ 2), then f is a specially multiplicative function with $f_A = k$.

Proof. By multiplicativity, it suffices to show that (2.1) holds at prime powers, that is,

$$[f(g*\mu)](p^e) = (fg*h^{-1})(p^e) \tag{2.5}$$

for all prime powers p^e. If $e = 1$, then both sides of (2.5) are equal to $f(p)g(p) - f(p)$. Assume that $e \geq 2$. Then

$$\begin{aligned}(fg*h^{-1})(p^e) &= f(p^e)g(p^e) + f(p^{e-1})g(p^{e-1})h^{-1}(p) \\ &\quad + f(p^{e-2})g(p^{e-2})h^{-1}(p^2) \\ &= f(p^e)g(p^e) - f(p^{e-1})g(p^{e-1})f(p) \\ &\quad + f(p^{e-2})g(p^{e-2})g(p)f_A(p).\end{aligned} \tag{2.6}$$

By (1.7), we obtain

$$(fg*h^{-1})(p^e) = f(p^e)g(p^e) - f(p^e)g(p^{e-1}) = f(p^e)(g*\mu)(p^e). \tag{2.7}$$

Thus we have proved (2.5).

To prove the converse, we write (2.3) in the form

$$(f(g*\mu))(n) = (fg*a^{-1}*b^{-1})(n). \tag{2.8}$$

We write $n = p^{e+1}$ $(e \geq 1)$ and, after some simplifications, obtain

$$f(p^{e+1}) = f(p^e)f(p) - f(p^{e-1})k(p). \tag{2.9}$$

Therefore, by (1.7), it remains to prove that f is multiplicative. Denote $n = p_1^{e_1}\cdots p_r^{e_r}p_{r+1}\cdots p_{r+s}$, where $e_i > 1$ $(i = 1,2,\ldots,r)$. We proceed by induction on $e_1+\cdots+e_r+s$ to prove that

$$f(n) = f(p_1^{e_1})\cdots f(p_r^{e_r})f(p_{r+1})\cdots f(p_{r+s}). \tag{2.10}$$

If $e_1+\cdots+e_r+s = 1$, then (2.10) holds. Suppose that (2.10) holds when $e_1+\cdots+e_r+s < m$. Then for $e_1+\cdots+e_r+s = m$, we have after some manipulation

$$\begin{aligned}&f(n)(g*\mu)(n) \\ &\quad = (fg*a^{-1}*b^{-1})(n) \\ &\quad = f(n)g(n) + \sum_{\substack{d\mid n\\ d>1}} f\left(\frac{n}{d}\right)g\left(\frac{n}{d}\right)(a^{-1}*b^{-1})(d) \\ &\quad = f(n)g(n) - \prod_{p^e\parallel n} f(p^e)g(p^e) + \prod_{p^e\parallel n}(fg*a^{-1}*b^{-1})(p^e)\end{aligned}$$

$$
\begin{aligned}
&= f(n)g(n) - \prod_{p^e \| n} f(p^e)g(p^e) \\
&\quad + \prod_{i=1}^{r} \left[f\left(p_i^{e_i}\right) g\left(p_i^{e_i}\right) - f\left(p_i^{e_i-1}\right) f(p_i) g\left(p_i^{e_i-1}\right) + f\left(p_i^{e_i-2}\right) k(p_i) g\left(p_i^{e_i-1}\right) \right] \\
&\quad \times \prod_{i=1}^{s} \left(f(p_{r+i}) g(p_{r+i}) - f(p_{r+i}) \right).
\end{aligned}
\tag{2.11}
$$

Using (2.9), we obtain

$$
f(n)(g*\mu)(n) = f(n)g(n) - g(n) \prod_{p^e \| n} f(p^e) + (g*\mu)(n) \prod_{p^e \| n} f(p^e). \tag{2.12}
$$

This gives (2.10). □

Remark 2.2. The converse part of Theorem 2.1 can also be written as follows. If $f(1) = 1$ and there exist completely multiplicative functions g and k, and a specially multiplicative function h such that

$$
h * f(g*\mu) = fg, \tag{2.13}
$$

where

$$
h(p) = f(p), \qquad h_A(p) = g(p)k(p), \qquad (g*\mu)(n) \neq g(n) \tag{2.14}
$$

for all prime numbers p and integers n (≥ 2), then f is a specially multiplicative function with $f_A = k$.

Corollary 2.3. *If f is a specially multiplicative function, then*

$$
h * f\phi = fN, \tag{2.15}
$$

where h is the specially multiplicative function such that

$$
h(p) = f(p), \qquad h_A(p) = p f_A(p) \tag{2.16}
$$

for all prime numbers p. Conversely, if $f(1) = 1$ and if there exist completely multiplicative functions a, b, and k such that

$$
a * b * f\phi = fN, \tag{2.17}
$$

where

$$
a(p) + b(p) = f(p), \qquad a(p)b(p) = pk(p) \tag{2.18}
$$

for all prime numbers p, then f is a specially multiplicative function with $f_A = k$.

Corollary 2.4. *If f and g are completely multiplicative functions, then*

$$f * f(g * \mu) = fg. \tag{2.19}$$

Conversely, if $f(1) = 1$ and if there exists a completely multiplicative function g such that

$$f * f(g * \mu) = fg, \tag{2.20}$$

where

$$(g * \mu)(n) \neq g(n) \tag{2.21}$$

for all integers n (≥ 2), then f is a completely multiplicative function.

Corollary 2.5 (Sivaramakrishnan [5])**.** *If $f(1) = 1$, then f is a completely multiplicative function if and only if*

$$f * f\phi = fN. \tag{2.22}$$

Example 2.6. We have

$$\sigma_\alpha \phi_\beta = \sigma_\alpha N^\beta * h^{-1}, \tag{2.23}$$

where h is the specially multiplicative function such that

$$h(p) = \sigma_\alpha(p) = p^\alpha + 1, \qquad h_A(p) = p^\beta p^\alpha = p^{\alpha+\beta} \tag{2.24}$$

for all prime numbers p.

Theorem 2.7. *If f is a specially multiplicative function and g is a completely multiplicative function, then*

$$f(g * \mu) = fg * (\mu f \oplus \Omega_2(\mu^2 f_A g)). \tag{2.25}$$

Conversely, if $f(1) \neq 0$ and if there exist completely multiplicative functions g and k such that

$$f(g * \mu) = fg * (\mu f \oplus \Omega_2(\mu^2 k g)), \tag{2.26}$$

where

$$(g * \mu)(n) \neq g(n) \tag{2.27}$$

for all n, then f is a specially multiplicative function with $f_A = k$.

Proof. We observe that

$$\begin{aligned}(\mu f \oplus \Omega_2(\mu^2 f_A g))(p) &= -f(p),\\ (\mu f \oplus \Omega_2(\mu^2 f_A g))(p^2) &= f_A(p) g(p),\\ (\mu f \oplus \Omega_2(\mu^2 f_A g))(p^n) &= 0\end{aligned} \tag{2.28}$$

for all prime numbers p and integers n (≥ 3). Therefore $\mu f \oplus \Omega_2(\mu^2 f_A g) = h^{-1}$, where h is the specially multiplicative function in Theorem 2.1. Thus (2.25) follows from (2.1).

The converse follows from Theorem 2.1 since $\mu f \oplus \Omega_2(\mu^2 gk) = a^{-1} * b^{-1}$, where a and b are completely multiplicative functions as given in Theorem 2.1. □

Theorem 2.8. *If f is a specially multiplicative function and g is a completely multiplicative function, then*

$$f(g*\mu) = fg * (f^{-1} \oplus \Omega_2(\mu^2 f_A(g \oplus \mu))). \tag{2.29}$$

Conversely, if $f(1) = 1$ and there exist completely multiplicative functions c, d, and g such that

$$f(g*\mu) = fg * ((c*d)^{-1} \oplus \Omega_2(\mu^2 cd(g \oplus \mu))), \tag{2.30}$$

where

$$c(p) + d(p) = f(p), \qquad (g*\mu)(n) \neq g(n) \tag{2.31}$$

for all prime numbers p and integers n (≥ 2), then f is the specially multiplicative function given as $f = c*d$.

Proof. Proof of Theorem 2.8 is similar to that of Theorem 2.7. □

Example 2.9. We have

$$\begin{aligned} \sigma_\alpha \phi_\beta &= \sigma_\alpha N^\beta * (\mu\sigma_\alpha \oplus \Omega_2(\mu^2 N^{\alpha+\beta})), \\ \sigma_\alpha \phi_\beta &= \sigma_\alpha N^\beta * (\sigma_\alpha^{-1} \oplus \Omega_2(\mu^2 N^\alpha (N^\beta \oplus \mu))). \end{aligned} \tag{2.32}$$

Lemma 2.10. *Suppose that f is an arithmetical function such that $f(1) = 1$ and $f^{-1}(p^i) = 0$ for $3 \leq i < k$ ($k \geq 4$). Then*

$$f(p^k) = f(p)f(p^{k-1}) - f^{-1}(p^2)f(p^{k-2}) - f^{-1}(p^k). \tag{2.33}$$

Proof. Lemma 2.10 follows from the equation

$$\sum_{i=0}^{k} f^{-1}(p^i) f(p^{k-i}) = 0. \tag{2.34}$$

□

Theorem 2.11. *If f is a specially multiplicative function and g is a completely multiplicative function, then*

$$f(g*\zeta) = fg * f * \Omega_2(f_A g)^{-1}. \tag{2.35}$$

Conversely, if f is a multiplicative function such that

$$f(g*\zeta)=fg*f*\Omega_2(hg)^{-1}, \tag{2.36}$$

where g is a completely multiplicative function with $g(p)(g*\zeta)(p^e)\neq 0$ for all prime powers p^e and where h is a completely multiplicative function, then f is a specially multiplicative function with $f_A=h$.

Proof. Let $f=a*b$, where a and b are completely multiplicative functions. It is known [7] that

$$f(g*\zeta)=(a*b)(g*\zeta)=ag*a\zeta*bg*b\zeta*\Omega_2(abg\zeta)^{-1}. \tag{2.37}$$

Using elementary properties of arithmetical functions, we obtain

$$f(g*\zeta)=(a*b)g*(a*b)*\Omega_2(f_Ag)^{-1}=fg*f*\Omega_2(f_Ag)^{-1}. \tag{2.38}$$

This proves (2.35).

Assume that (2.36) holds. Then (2.36) at p^2 gives

$$h(p)=f(p)^2-f(p^2). \tag{2.39}$$

Since $f^{-1}(p^2)=f(p)^2-f(p^2)$ for all multiplicative functions, we obtain

$$h(p)=f^{-1}(p^2). \tag{2.40}$$

We next prove that

$$f^{-1}(p^i)=0 \quad \forall i\geq 3. \tag{2.41}$$

We proceed by induction on i. Calculating (2.36) at p^3 and using (2.40) gives

$$f(p^3)=f(p)f(p^2)-f(p)f^{-1}(p^2). \tag{2.42}$$

Since

$$f(p^3)-f(p^2)f(p)+f(p)f^{-1}(p^2)+f^{-1}(p^3)=0, \tag{2.43}$$

we see that $f^{-1}(p^3)=0$.

Suppose that $f^{-1}(p^i)=0$ for all $3\leq i<k$ ($k>3$). We write (2.36) as

$$f(g*\zeta)*f^{-1}=fg*\Omega_2(hg)^{-1}. \tag{2.44}$$

Suppose that k is even, say $k = 2e$ $(e > 1)$. At p^{2e}, the left-hand side of (2.44) becomes

$$\begin{aligned}
&\sum_{i=0}^{2e} f(p^i)(g*\zeta)(p^i)f^{-1}(p^{2e-i}) \\
&\qquad = f^{-1}(p^{2e}) + f(p^{2e-2})(g*\zeta)(p^{2e-2})f^{-1}(p^2) \\
&\qquad\quad + f(p^{2e-1})(g*\zeta)(p^{2e-1})f^{-1}(p) + f(p^{2e})(g*\zeta)(p^{2e}) \\
&\qquad = f^{-1}(p^{2e}) - f^{-1}(p^{2e})(g*\zeta)(p^{2e-2}) \\
&\qquad\quad - f(p)f(p^{2e-1})g(p^{2e-1}) + f(p^{2e})g(p^{2e-1}) + f(p^{2e})g(p^{2e}) \\
&\qquad = f^{-1}(p^{2e}) - f^{-1}(p^{2e})(g*\zeta)(p^{2e-1}) \\
&\qquad\quad - f^{-1}(p^2)f(p^{2e-2})g(p^{2e-1}) + f(p^{2e})g(p^{2e}),
\end{aligned} \tag{2.45}$$

where the last two equations are derived by Lemma 2.10. Further, at p^{2e}, the right-hand side of (2.44) becomes

$$\begin{aligned}
&\sum_{i=0}^{2e} f(p^{2e-i})g(p^{2e-i})\Omega_2(hg)^{-1}(p^i) \\
&\qquad = \sum_{i=0}^{e} f(p^{2(e-i)})g(p^{2(e-i)})\mu(p^i)h(p^i)g(p^i) \\
&\qquad = f(p^{2e})g(p^{2e}) - f(p^{2(e-1)})g(p^{2(e-1)})h(p)g(p).
\end{aligned} \tag{2.46}$$

Now, we see that $f^{-1}(p^{2e}) = 0$, that is, $f^{-1}(p^k) = 0$.

If k is odd, a similar argument applies. Thus (2.41) holds and therefore, by (1.6), f is a specially multiplicative function with $f_A = h$. □

Corollary 2.12. *If f is a specially multiplicative function, then*

$$f\sigma_0 = f*f*\Omega_2(f_A)^{-1}. \tag{2.47}$$

Conversely, if f is a multiplicative function such that

$$f\sigma_0 = f*f*\Omega_2(h)^{-1}, \tag{2.48}$$

where h is a completely multiplicative function, then f is a specially multiplicative function with $f_A = h$.

Corollary 2.13 (Apostol [1]). *If f and g are completely multiplicative functions, then*

$$f(g*\zeta) = fg*f. \tag{2.49}$$

Conversely, if f is a multiplicative function such that

$$f(g*\zeta) = fg*f, \tag{2.50}$$

where g is a completely multiplicative function with $g(p)(g*\zeta)(p^e) \neq 0$ for all prime powers p^e, then f is a completely multiplicative function.

Corollary 2.14 (Carlitz [2]). *Suppose that f is a multiplicative function. Then f is a completely multiplicative function if and only if*

$$f\sigma_0 = f * f. \tag{2.51}$$

Corollary 2.15. *There exist*

$$\begin{aligned} \tau\sigma_\alpha &= \tau N^\alpha * \tau * \Omega_2 (N^{\alpha+11})^{-1}, \\ \sigma_\alpha\sigma_\beta &= \sigma_\alpha N^\beta * \sigma_\alpha * \Omega_2 (N^{\alpha+\beta})^{-1}. \end{aligned} \tag{2.52}$$

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