IJMMS 2003:39, 2507–2518
PII. S0161171203201071
http://ijmms.hindawi.com

NETTED MATRICES

PANTELIMON STĂNICĂ

Received 11 January 2002

We prove that powers of 4-netted matrices (the entries satisfy a four-term recurrence $\delta a_{i,j} = \alpha a_{i-1,j} + \beta a_{i-1,j-1} + \gamma a_{i,j-1}$) preserve the property of nettedness: the entries of the eth power satisfy $\delta_e a^{(e)}_{i,j} = \alpha_e a^{(e)}_{i-1,j} + \beta_e a^{(e)}_{i-1,j-1} + \gamma_e a^{(e)}_{i,j-1}$, where the coefficients are all instances of the same sequence $x_{e+1} = (\beta+\delta)x_e - (\beta\delta+\alpha\gamma)x_{e-1}$. Also, we find a matrix $Q_n(a,b)$ and a vector v such that $Q_n(a,b)^e \cdot v$ acts as a shifting on the general second-order recurrence sequence with parameters a, b. The shifting action of $Q_n(a,b)$ generalizes the known property $\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}^e \cdot (1,0)^t = (F_{e-1}, F_e)^t$. Finally, we prove some results about congruences satisfied by the matrix $Q_n(a,b)$.

2000 Mathematics Subject Classification: 05A10, 11B39, 11B65, 11C20, 15A36.

1. Introduction. In [4], Peele and Stănică studied $n \times n$ matrices with the (i,j) entry the binomial coefficients $\binom{i-1}{j-1}$ (matrix L_n) and $\binom{i-1}{n-j}$ (matrix R_n), respectively, and derived many interesting results on powers of these matrices. The matrix L_n was easily subdued, but curiously enough, closed forms for entries of powers of R_n, say R_n^e, were not found. However, recurrences among various entries of R_n^e were proved and precise results on congruences modulo any prime p were found. To accomplish that, the authors of [4] proved that the entries $a^{(e)}_{i,j}$ of the eth power of R_n satisfy

$$F_{e-1} a^{(e)}_{i,j} = F_e a^{(e)}_{i-1,j} + F_{e+1} a^{(e)}_{i-1,j-1} - F_e a^{(e)}_{i,j-1}, \tag{1.1}$$

where F_e is the Fibonacci sequence, $F_{e+1} = F_e + F_{e-1}$, $F_0 = 0$, and $F_1 = 1$. As we will see in our first result, this is not a singular phenomenon. The goal of this note is two-fold: we prove the results of [4] for a class of matrices, containing R_n, where the entries satisfy any four-term recurrence (we call these 4-*netted* matrices), and we find a possible generalization of the Q-matrix, namely a matrix $Q_n(a,b)$, with the property that any power multiplied by a fixed vector gives an n-tuple of consecutive terms of the general Pell or Fibonacci sequence. It generalizes the known property $\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}^e \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} F_{e-1} \\ F_e \end{pmatrix}$. We also find the generating function for the entries of powers of these matrices. As applications, we find some interesting identities for general Fibonacci (or Pell) numbers. In Section 5, we provide a few results on the order of these matrices modulo a prime.

2. Matrices with entries satisfying a general four-term recurrence. Define a tableau with elements $a_{i,j}$, $i \ge 0$, $j \ge 0$, which satisfy (for $i \ge 1$, $j \ge 1$)

$$\delta a_{i,j} = \alpha a_{i-1,j} + \beta a_{i-1,j-1} + \gamma a_{i,j-1} \tag{2.1}$$

with the boundary conditions

$$\begin{aligned} \beta a_{i,0} + \gamma a_{i+1,0} &= 0 \quad \forall 1 \le i \le n-1, \\ \delta a_{i+1,n+1} - \alpha a_{i,n+1} &= 0 \quad \forall 1 \le i \le n-1. \end{aligned} \tag{2.2}$$

We remark that if the 0th and $(n+1)$th columns are made up of zeros, then conditions (2.2) are fulfilled.

In our main result of this section we prove that (2.1) is preserved for higher powers of the $n \times n$ matrix $(a_{i,j})_{i,j=1,\dots,n}$. Precisely, we prove the following theorem.

Theorem 2.1. *The entries of the eth power of the matrix*

$$R = (a_{i,j})_{i,j=1,\dots,n} \tag{2.3}$$

satisfy the recurrence

$$\delta_e a_{i,j}^{(e)} = \alpha_e a_{i-1,j}^{(e)} + \beta_e a_{i-1,j-1}^{(e)} + \gamma_e a_{i,j-1}^{(e)}, \quad i,j \le n, \tag{2.4}$$

where the sequences α_e, β_e, γ_e, and δ_e are all instances of the sequence x_e satisfying

$$x_{e+1} = (\beta + \delta)x_e - (\beta\delta + \alpha\gamma)x_{e-1} \tag{2.5}$$

with initial conditions $(\delta_1 = \delta;\ \delta_2 = \delta^2 - \alpha\gamma)$, $(\alpha_1 = \alpha;\ \alpha_2 = \alpha(\delta + \beta))$, $(\beta_1 = \beta;\ \beta_2 = \beta^2 - \alpha\gamma)$, *and* $(\gamma_1 = \gamma;\ \gamma_2 = \gamma(\beta + \delta))$.

Proof. We prove by induction on e that there exists a relation among the entries of any 2×2 cell, namely

$$\delta_e a_{i,j}^{(e)} = \alpha_e a_{i-1,j}^{(e)} + \beta_e a_{i-1,j-1}^{(e)} + \gamma_e a_{i,j-1}^{(e)}. \tag{2.6}$$

The above relation is certainly true for $e = 1$. We evaluate, for $i \ge 2$,

$$\begin{aligned} \alpha\delta_{e-1} a_{i-1,j}^{(e)} &= \sum_{s=1}^{n} \alpha\delta_{e-1} a_{i-1,s} a_{s,j}^{(e-1)} \\ &= \sum_{s=1}^{n} \alpha a_{i-1,s} \left(\alpha_{e-1} a_{s-1,j}^{(e-1)} + \beta_{e-1} a_{s-1,j-1}^{(e-1)} + \gamma_{e-1} a_{s,j-1}^{(e-1)}\right) \\ &= \sum_{s=1}^{n} (\delta a_{i,s} - \beta a_{i-1,s-1} - \gamma a_{i,s-1}) \left(\alpha_{e-1} a_{s-1,j}^{(e-1)} + \beta_{e-1} a_{s-1,j-1}^{(e-1)}\right) \\ &\quad + \sum_{s=1}^{n} \alpha\gamma_{e-1} a_{i-1,s} a_{s,j-1}^{(e-1)} \end{aligned}$$

$$
\begin{aligned}
&= \sum_{s=1}^{n} \delta a_{i,s}\left(\alpha_{e-1}a_{s-1,j}^{(e-1)} + \beta_{e-1}a_{s-1,j-1}^{(e-1)}\right)\\
&\quad - \gamma\alpha_{e-1}a_{i,j}^{(e)} - \beta\beta_{e-1}a_{i-1,j-1}^{(e)} - \gamma\beta_{e-1}a_{i,j-1}^{(e)} - \beta\alpha_{e-1}a_{i-1,j}^{(e)}\\
&\quad + \alpha\gamma_{e-1}a_{i-1,j-1}^{(e)} - \gamma\alpha_{e-1}\left(a_{i,0}a_{0,j}^{(e-1)} - a_{i,n}a_{n,j}^{(e-1)}\right)\\
&\quad - \beta\beta_{e-1}\left(a_{i-1,0}a_{0,j-1}^{(e-1)} - a_{i-1,n}a_{n,j-1}^{(e-1)}\right)\\
&\quad - \gamma\beta_{e-1}\left(a_{i,0}a_{0,j-1}^{(e-1)} - a_{i,n}a_{n,j-1}^{(e-1)}\right)\\
&\quad - \beta\alpha_{e-1}\left(a_{i-1,0}a_{0,j}^{(e-1)} - a_{i-1,n}a_{n,j}^{(e-1)}\right).
\end{aligned}
\tag{2.7}
$$

Using the boundary conditions (2.2), we obtain, for $i \geq 2$,

$$
\begin{aligned}
\alpha\delta_{e-1}a_{i-1,j}^{(e)} &= (\alpha\gamma_{e-1} - \beta\beta_{e-1})a_{i-1,j-1}^{(e)} - \gamma\alpha_{e-1}a_{i,j}^{(e)} - \gamma\beta_{e-1}a_{i,j-1}^{(e)}\\
&\quad - \beta\alpha_{e-1}a_{i-1,j}^{(e)} + \sum_{s=1}^{n}\delta a_{i,s}\left(\delta_{e-1}a_{s,j}^{(e-1)} - \gamma_{e-1}a_{s,j-1}^{(e-1)}\right)\\
&\quad + \left(\alpha_{e-1}a_{n,j}^{(e-1)} + \beta_{e-1}a_{n,j-1}^{(e-1)}\right)(\beta a_{i-1,n} + \gamma a_{i,n})\\
&\quad - \left(\alpha_{e-1}a_{0,j}^{(e-1)} + \beta_{e-1}a_{0,j-1}^{(e-1)}\right)(\beta a_{i-1,0} + \gamma a_{i,0})\\
&= (\alpha\gamma_{e-1} - \beta\beta_{e-1})a_{i-1,j-1}^{(e)} - \gamma\alpha_{e-1}a_{i,j}^{(e)} - \gamma\beta_{e-1}a_{i,j-1}^{(e)}\\
&\quad - \beta\alpha_{e-1}a_{i-1,j}^{(e)} + \delta\delta_{e-1}a_{i,j}^{(e)} - \delta\gamma_{e-1}a_{i,j-1}^{(e)}\\
&\quad + \left(\alpha_{e-1}a_{n,j}^{(e-1)} + \beta_{e-1}a_{n,j-1}^{(e-1)}\right)(\delta a_{i,n+1} - \alpha a_{i-1,n+1})\\
&= (\alpha\gamma_{e-1} - \beta\beta_{e-1})a_{i-1,j-1}^{(e)} + (\delta\delta_{e-1} - \gamma\alpha_{e-1})a_{i,j}^{(e)}\\
&\quad - (\gamma\beta_{e-1} + \delta\gamma_{e-1})a_{i,j-1}^{(e)} - \beta\alpha_{e-1}a_{i-1,j}^{(e)}.
\end{aligned}
\tag{2.8}
$$

Thus,

$$
\begin{aligned}
(\delta\delta_{e-1} - \gamma\alpha_{e-1})a_{i,j}^{(e)} &= (\alpha\delta_{e-1} + \beta\alpha_{e-1})a_{i-1,j}^{(e)}\\
&\quad + (\beta\beta_{e-1} - \alpha\gamma_{e-1})a_{i-1,j-1}^{(e)}\\
&\quad + (\gamma\beta_{e-1} + \delta\gamma_{e-1})a_{i,j-1}^{(e)}.
\end{aligned}
\tag{2.9}
$$

Therefore, we obtain the system of sequences

$$\delta_e = \delta\delta_{e-1} - \gamma\alpha_{e-1}, \tag{2.10}$$

$$\alpha_e = \alpha\delta_{e-1} + \beta\alpha_{e-1}, \tag{2.11}$$

$$
\begin{aligned}
\beta_e &= \beta\beta_{e-1} - \alpha\gamma_{e-1},\\
\gamma_e &= \gamma\beta_{e-1} + \delta\gamma_{e-1}.
\end{aligned}
\tag{2.12}
$$

From (2.10) we get $\alpha_{e-1} = (\delta/\gamma)\delta_{e-1} - (1/\gamma)\delta_e$, which when replaced in (2.11) gives the recurrence

$$\delta_{e+1} = (\beta+\delta)\delta_e - (\beta\delta+\gamma\alpha)\delta_{e-1}. \tag{2.13}$$

Similarly,

$$\begin{aligned} \alpha_{e+1} &= (\beta+\delta)\alpha_e - (\beta\delta+\gamma\alpha)\alpha_{e-1}, \\ \beta_{e+1} &= (\beta+\delta)\beta_e - (\beta\delta+\gamma\alpha)\beta_{e-1}, \\ \gamma_{e+1} &= (\beta+\delta)\gamma_e - (\beta\delta+\gamma\alpha)\gamma_{e-1}. \end{aligned} \tag{2.14}$$

The initial conditions are $(\delta_1 = \delta;\ \delta_2 = \delta^2 - \alpha\gamma)$, $(\alpha_1 = \alpha;\ \alpha_2 = \alpha(\delta+\beta))$, $(\beta_1 = \beta;\ \beta_2 = \beta^2 - \alpha\gamma)$, and $(\gamma_1 = \gamma;\ \gamma_2 = \gamma(\beta+\delta))$. □

Example 2.2. As examples of tableaux satisfying our conditions, we have

$$\begin{aligned} a^1_{i,j} &= \binom{i-1}{j-1} \quad (\delta = 1,\ \alpha = 1,\ \beta = 1,\ \gamma = 0), \\ a^2_{i,j} &= \binom{i-1}{n-j} \quad (\delta = 0,\ \alpha = 1,\ \beta = 1,\ \gamma = -1), \\ a^3_{i,j} &= \binom{n-i}{n-j} \quad (\delta = 1,\ \alpha = 0,\ \beta = -1,\ \gamma = 1). \end{aligned} \tag{2.15}$$

Other examples are given by the alternating matrices $(-1)^{i+j}a^k_{i,j}$ (or $(-1)^{i-1}a^k_{i,j}$ or $(-1)^{j-1}a^k_{i,j}$, etc.), $k = 1,2,3$. In the next section, we present more examples.

3. Higher-order Fibonacci matrices.

In this section, we uncover a very interesting side of our previous results. A matrix of the form $M = \begin{pmatrix} 0 & 1 \\ 1 & m \end{pmatrix}$ is called a *Fibonacci* or *Q-matrix*. It is known that if the sequence $U_{e+1} = mU_e + U_{e-1}$, $U_0 = 0$, $U_1 = 1$, then $M^e = \begin{pmatrix} U_{e-1} & U_e \\ U_e & U_{e+1} \end{pmatrix}$ and $M^e \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} U_{e-1} \\ U_e \end{pmatrix}$. Next, we find a matrix $Q_n(a,b)$ such that $Q_n(a,b)^e \cdot v$ is a vector of n consecutive terms of the sequence U_n for any power e and any vector v of alternating consecutive terms in the sequence $U_{n+1} = aU_n + bU_{n-1}$. Let I_n be the identity matrix of dimension n and let M^t be the *transpose* of a matrix M.

Let $a_{i,j} = a^{(1)}_{i,j} = a^{i+j-n-1}b^{n-j}\binom{i-1}{n-j}$ and $Q_n(a,b) = (a_{i,j})_{i,j}$. We use our previous results to show the following theorem.

Theorem 3.1. *Let* $w = ((-1)^n U_{n-1}, (-1)^{n-1}U_{n-2}, \ldots, -U_0)^t$. *Then*

$$Q_n(a,b)^{e+1} \cdot w = \left(U_{(n-1)e}, U_{(n-1)e+1}, \ldots, U_{(n-1)(e+1)}\right)^t, \tag{3.1}$$

$$U_{e-1}a^{(e)}_{i,j} + U_e a^{(e)}_{i,j-1} = U_e a^{(e)}_{i-1,j} + U_{e+1}a^{(e)}_{i-1,j-1}, \tag{3.2}$$

where $a^{(e)}_{i,j}$ *are the entries of* $Q_n(a,b)^e$ *and* U_e *is the sequence satisfying* $U_{e+1} = aU_e + bU_{e-1}$, $U_0 = 0$, $U_1 = 1$. *Moreover,* $Q_n(a,b)$ *is unique with the properties* $a_{1,j} = 0$, $j < n$, $a_{i,n} = a^{i-1}$, *and* $a_{i,j} = aa_{i-1,j} + ba_{i-1,j+1}$.

Proof. First, the ith entry of $Q_n(a,b)^{e+1}\cdot w$ is

$$\begin{aligned}\sum_{j=1}^{n}(-1)^{n+1-j}a_{i,j}^{(e+1)}U_{n-j} &= \sum_{j=1}^{n}(-1)^{n+1-j}\sum_{k=1}^{n}a_{i,k}a_{k,j}^{(e)}U_{n-j}\\ &= \sum_{k=1}^{n}a_{i,k}\sum_{j=1}^{n}(-1)^{n+1-j}a_{k,j}^{(e)}U_{n-j}\\ &= \sum_{j=1}^{n}a_{i,k}U_{(n-1)(e-1)+k-1}.\end{aligned}\tag{3.3}$$

The initial condition and the step of induction (on e) will both follow if we can prove that the matrix Q_n acts as an index-translation on our sequence U_l, namely

$$\sum_{k=1}^{n}a_{i,k}U_{t+k}=U_{t+n+i-1},\quad t\geq -1.\tag{3.4}$$

Let $W_i=\sum_{k=1}^{n}a_{i,k}U_{t+k}$ (t is assumed fixed). First,

$$W_1=\sum_{k=1}^{n}a_{1,k}U_{t+k}=a_{1,n}U_{t+n}=U_{t+n}.\tag{3.5}$$

Then,

$$W_2=\sum_{k=1}^{n}a_{2,k}U_{t+k}=a_{2,n-1}U_{t+n-1}+a_{2,n}U_{t+n}=bU_{t+n-1}+aU_{t+n}=U_{t+n+1}.\tag{3.6}$$

Now, for $1\leq i\leq n-1$,

$$\begin{aligned}W_{i+1} &= \sum_{k=1}^{n}a_{i+1,k}U_{t+k}=\sum_{k=1}^{n}(aa_{i,k}+ba_{i,k+1})U_{t+k}\\ &= aW_i+\sum_{k=1}^{n}a_{i,k+1}bU_{t+k}\\ &= aW_i+\sum_{k=1}^{n-1}a_{i,k+1}(U_{t+k+2}-aU_{t+k+1})\\ &= aW_i+\sum_{u=2}^{n}a_{i,u}U_{t+u+1}-a\sum_{u=2}^{n}a_{i,u}U_{t+u}\quad\text{since } u=k+1\\ &= bV_i-ba_{i,1}U_{t+2}+aa_{i,1}U_{t+1}=V_i\quad\text{with } a_{i,1}=0 \text{ if } i\leq n-1,\end{aligned}\tag{3.7}$$

where

$$\begin{aligned} V_i &= \sum_{u=1}^{n} a_{i,u} U_{t+u+1} = \sum_{u=1}^{n} (a a_{i-1,u} + b a_{i-1,u+1}) U_{t+u+1} \\ &= a V_{i-1} + b \sum_{u=1}^{n-1} a_{i-1,u+1} U_{t+u+1} \quad \text{since } a_{i+1,n+1} = 0 \\ &= a V_{i-1} + b \sum_{u=2}^{n} a_{i-1,s} U_{t+s} \quad \text{since } u+1 = s \\ &= a V_{i-1} + b W_{i-1} - b a_{i-1,1} U_{t+1} \\ &= a V_{i-1} + b W_{i-1} \quad \text{with } a_{i-1,1} = 0 \text{ if } i \le n-1. \end{aligned} \tag{3.8}$$

Using $V_i = W_{i+1}$ in the previous recurrence, we get $W_{i+1} = aW_i + bW_{i-1}$. Therefore, $W_i = U_{t+n+i-1}$ since $W_1 = U_{t+n}$, $W_2 = U_{t+n+1}$.

Using Theorem 2.1, with $\delta = 0$, $\alpha = b$, $\beta = a$, and $\gamma = -1$, we get the recurrence between the entries of the higher power of Q_n, namely $U_{e-1} a_{i,j}^{(e)} + U_e a_{i,j-1}^{(e)} = U_e a_{i-1,j}^{(e)} + U_{e+1} a_{i-1,j-1}^{(e)}$, with the appropriate initial conditions. The fact that Q_n is the unique matrix with the given properties follows easily observing that such a matrix could be defined inductively as follows. Let $Q_1 = 1$. Assume that $Q_{n-1} = (a_{i,j})_{i,j=1,2,\dots,n-1}$ and construct Q_n by bordering Q_{n-1} with the first column and the last row (left and bottom). The first column is $(0,0,\dots,0,1)^t$ and the last row is given by $a_{n,n} = a^{n-1}$ and $a_{n,j} = a a_{n-1,j} + b a_{n-1,j+1}$. □

Definition 3.2. We call such a matrix $Q_n(a,b)$ a *generalized Fibonacci* or *Q-matrix* of dimension n and parameters a, b.

Example 3.3. We give here the first few powers of $Q_3(a,b)$:

$$\begin{gathered} Q_3(a,b) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & b & a \\ b^2 & 2ab & a^2 \end{pmatrix}, \\ Q_3(a,b)^2 = \begin{pmatrix} b^2 & 2ab & a^2 \\ ab^2 & b(2a^2+b) & a(a^2+b) \\ a^2b^2 & 2ab(a^2+b) & (a^2+b)^2 \end{pmatrix}, \\ Q_3(a,b)^3 = \begin{pmatrix} a^2b^2 & 2ab(a^2+b) & (a^2+b)^2 \\ ab^2(a^2+b) & b(2a^4+4a^2b+b^2) & a^5+3a^3b+2ab^2 \\ b^2(a^2+b)^2 & 2ab(a^4+3a^2b+2b^2) & (a^3+2ab)^2 \end{pmatrix}. \end{gathered} \tag{3.9}$$

4. Some generating functions and an inverse. Although we cannot find simple closed forms for *all* entries of $Q_n(a,b)^e$, we prove the following theorem.

Theorem 4.1. *The generating function for $a_{i,j}^{(e)}$ is*

$$B_n^{(e)}(x,y)=\frac{(U_{e-1}+U_e y)(bU_{e-1}+yU_e)^{n-1}}{U_{e-1}+U_e y-x(U_e+U_{e+1}y)}. \tag{4.1}$$

Proof. Multiplying the recurrence (3.2) by $x^{i-1}y^{j-1}$ and summing for $i,j\geq 2$, we get

$$\begin{aligned} &U_{e-1}\sum_{i,j\geq 2}a_{i,j}^{(e)}x^{i-1}y^{j-1}+U_e y\sum_{i,j\geq 2}a_{i,j-1}^{(e)}x^{i-1}y^{j-2}\\ &\quad=U_e x\sum_{i,j\geq 2}a_{i-1,j}^{(e)}x^{i-2}y^{j-1}+U_{e+1}xy\sum_{i,j\geq 2}a_{i-1,j-1}^{(e)}x^{i-2}y^{j-2}. \end{aligned} \tag{4.2}$$

Thus,

$$\begin{aligned} &U_{e-1}\left(B_n^{(e)}(x,y)-\sum_{i\geq 1}a_{i,1}^{(e)}x^{i-1}-\sum_{j\geq 1}a_{1,j}^{(e)}y^{j-1}+a_{1,1}^{(e)}\right)\\ &\quad+U_e y\left(B_n^{(e)}(x,y)-\sum_{j\geq 1}a_{1,j}^{(e)}y^{j-1}\right)\\ &\quad=U_e x\left(B_n^{(e)}(x,y)-\sum_{i\geq 1}a_{i,1}^{(e)}x^{i-1}\right)+U_{e+1}xyB_n^{(e)}(x,y). \end{aligned} \tag{4.3}$$

Solving for $B_n^{(e)}(x,y)$, we get

$$\begin{aligned} &B_n^{(e)}(x,y)(U_{e-1}+U_e y-x(U_e+U_{e+1}y))\\ &\quad=(U_{e-1}-U_e x)\sum_{i\geq 1}a_{i,1}^{(e)}x^{i-1}\\ &\qquad+(U_{e-1}+U_e y)\sum_{j\geq 1}a_{1,j}^{(e)}y^{j-1}-U_{e-1}a_{1,1}^{(e)}. \end{aligned} \tag{4.4}$$

We need to find $a_{i,1}^{(e)}$ and $a_{1,j}^{(e)}$. As in [4], we prove that

$$\begin{aligned} a_{1,j}^{(e)}&=b^{n-j}U_{e-1}^{n-j}U_e^{j-1}\binom{n-1}{j-1},\\ a_{i,1}^{(e)}&=b^{n-1}U_{e-1}^{n-i}U_e^{i-1}. \end{aligned} \tag{4.5}$$

There is no difficulty to show the relations for $e = 1, 2$. Assume that $e \geq 3$. First we deal with the elements in the first row:

$$\begin{aligned}
a_{1,j}^{(e+1)} &= \sum_{s=1}^{n} a_{1,s}^{(e)} a_{s,j} \\
&= \sum_{s=1}^{n} b^{n-s} U_{e-1}^{n-s} U_e^{s-1} a^{s+j-n-1} b^{n-j} \binom{n-1}{s-1} \binom{s-1}{n-j} \\
&= \sum_{s=1}^{n} U_{e-1}^{n-s} U_e^{s-1} a^{s+j-n-1} b^{2n-j-s} \binom{n-1}{j-1} \binom{j-1}{n-s} \\
&= b^{n-j} a^{j-1} U_e^{n-1} \binom{n-1}{j-1} \sum_{s=1}^{n} \left(\frac{bU_{e-1}}{aU_e}\right)^{n-s} \binom{j-1}{n-s} \\
&= b^{n-j} a^{j-1} U_e^{n-1} \binom{n-1}{j-1} \left(1 + \frac{bU_{e-1}}{aU_e}\right)^{j-1} \\
&= b^{n-j} U_e^{n-j} U_{e+1}^{j-1} \binom{n-1}{j-1}.
\end{aligned} \tag{4.6}$$

Now we find the elements in the first column:

$$\begin{aligned}
a_{i,1}^{(e+1)} &= \sum_{s=1}^{n} a_{i,s} a_{s,1}^{(e)} \\
&= \sum_{s=1}^{n} a^{i+s-n-1} b^{n-s} \binom{i-1}{n-s} b^{n-1} U_{e-1}^{n-s} U_e^{s-1} \\
&= a^{i-1} b^{n-1} U_e^{n-1} \sum_{s=1}^{n} \left(\frac{bU_{e-1}}{aU_e}\right)^{n-s} \binom{i-1}{n-s} \\
&= a^{i-1} b^{n-1} U_e^{n-1} \left(1 + \frac{bU_{e-1}}{aU_e}\right)^{i-1} = b^{n-1} U_e^{n-i} U_{e+1}^{i-1}.
\end{aligned} \tag{4.7}$$

Using (4.5), we get

$$\begin{aligned}
\sum_{j\geq 1} a_{1,j}^{(e)} y^{j-1} &= \sum_{j\geq 1} U_{e-1}^{n-j} U_e^{j-1} b^{n-j} y^{j-1} \binom{n-1}{j-1} \\
&= b^{n-1} \sum_{s\geq 0} U_{e-1}^{(n-1)-s} \left(\frac{yU_e}{b}\right)^{s} \binom{n-1}{s} \\
&= \left(bU_{e-1} + yU_e\right)^{n-1}.
\end{aligned} \tag{4.8}$$

Using (4.4), $(U_{e-1} - U_e x) \sum_{i\geq 1} U_{e-1}^{n-i} U_e^{i-1} b^{n-1} x^{i-1} = b^{n-1} U_{e-1}^{n}$, and $U_{e-1} a_{1,1}^{(e)} = b^{n-1} U_{e-1}^{n}$, we deduce the result. □

The inverse of $Q_n(a,b)$ is not difficult to find. We have the following theorem.

Theorem 4.2. *The inverse of $Q_n(a,b)$ is*

$$Q_n(a,b)^{-1} = \left((-1)^{n+i+j+1}a^{n+1-i-j}b^{i-n}\binom{n-i}{j-1}\right)_{i,j}. \tag{4.9}$$

Proof. The proof is straightforward. □

In general, finding *simple* closed forms for the entries of powers of $Q_n(a,b)$ seems to be a very difficult matter. We can derive (after some work) simple formulas for the entries of the second row and column of $Q_n(a,b)^e$.

Proposition 4.3. *The entries of the second row and column of $Q_n(a,b)$ are given by*

$$\begin{aligned} a_{2,j}^{(e)} &= b^{n-j}U_{e-1}^{n-j-1}U_e^j\binom{n-2}{j-1} + b^{n-j}U_{e-1}^{n-j}U_e^{j-2}U_{e+1}\binom{n-2}{j-2}, \\ a_{i,2}^{(e)} &= (n-i)b^{n-2}U_{e-1}^{n-i-1}U_e^i + (i-1)b^{n-2}U_{e-1}^{n-i}U_e^{i-2}U_{e+1}. \end{aligned} \tag{4.10}$$

Remark 4.4. Since $b^{n-j}a_{j,n}^{(e-1)} = a_{j,1}^{(e)}$ and $a_{n,j}^{(e-1)} = a_{1,j}^{(e)}$, we get closed forms for the last row and column of $Q_n(a,b)^e$ as well.

By taking some particular cases of our previous results we get some very interesting binomial sums. For instance, we have the following corollary.

Corollary 4.5. *The following identities are true:*

$$\begin{gathered} \sum_{j=1}^{n}(-1)^{n+1-j}a^{i+j-n-1}b^{n-j}\binom{i-1}{n-j}U_{n-j} = U_{i-1}, \\ \sum_{j=1}^{n}\sum_{k=1}^{n}(-1)^{n+1-j}a^{i+j+2k-2n-2}b^{2n-j-k}\binom{i-1}{n-k}\binom{k-1}{n-j}U_{n-j} = U_{n+i-2}, \\ \sum_{j=1}^{n}U_{l-1}^{n-j}U_l^{j-1}U_{(n-1)p+j-1}b^{n-j}\binom{n-1}{j-1} = U_{(n-1)(l+p)}, \quad \textit{for any } l,p, \\ \sum_{j=1}^{n}U_{(n-1)p+j-1}U_{l-1}^{n-j-1}U_l^{j-2}b^{n-j}\left[U_l^2\binom{n-1}{j-1} + (-1)^l\binom{n-2}{j-2}\right] \\ = U_{(n-1)(l+p)+1}, \quad \textit{for any } l,p. \end{gathered} \tag{4.11}$$

Proof. Using Theorem 3.1, with $e = 1,2$, we obtain the first two identities. Now, with the help of Theorem 3.1 and the trivial identity $Q_n(a,b)^{l+p} = Q_n(a,b)^lQ_n(a,b)^p$, we get

$$\begin{aligned} &(Q_n(a,b)^lQ_n(a,b)^p)\cdot v \\ &\quad = Q_n(a,b)^l\cdot\left(U_{(n-1)p},U_{(n-1)p+1},\dots,U_{(n-1)(p+1)}\right)^t \\ &\quad = \left(U_{(n-1)(l+p)},U_{(n-1)(l+p)+1},\dots,U_{(n-1)(l+p+1)}\right). \end{aligned} \tag{4.12}$$

Since $a_{1,j}^{(l)} = U_{l-1}^{n-j}U_l^{j-1}b^{n-j}\binom{n-1}{j-1}$, we obtain the third identity.

Using

$$a_{2,j}^{(l)} = b^{n-j}U_{l-1}^{n-j-1}U_l^j\binom{n-2}{j-1} + b^{n-j}U_{l-1}^{n-j}U_l^{j-2}U_{l+1}\binom{n-2}{j-2}, \tag{4.13}$$

Cassini's identity (see [2, page 292]) (usually given for the Fibonacci numbers, but certainly true for the sequence U_l, as well, as the reader can check easily), and $U_{l-1}U_{l+1} - U_l^2 = (-1)^l$, we get the fourth identity. □

Corollary 4.6. *In general,*

$$\sum_{j=1}^{n} U_{(n-1)p+j-1}a_{i,j}^{(l)} = U_{(n-1)(l+p)+i-1} \tag{4.14}$$

for any i, l, and p.

5. Order of $Q_n(a,b)$ modulo p. Let $a,b \in \mathbb{Z}$ and $p \in \mathbb{Z}$ prime. Using the recurrence among the entries of $Q_n(a,b)$ and reasoning as in [4], we prove the following theorem.

Theorem 5.1. *If e is the least positive integer (entry point) such that $U_e \equiv 0 \pmod p$, then*

$$\begin{aligned} Q_{2k}(a,b)^e &\equiv (-1)^{(k+1)e}U_{e-1}I_{2k} \pmod p, \\ Q_{2k+1}(a,b)^e &\equiv (-1)^{ke}I_{2k+1} \pmod p. \end{aligned} \tag{5.1}$$

Moreover, $Q_n(a,b)^{4e} \equiv I_n \pmod p$. Furthermore, considering the parity of e,

$$\begin{aligned} Q_n(a,b)^{2e} &\equiv I_n \pmod p \quad \text{if } e \text{ is even}, \\ Q_n(a,b)^{2e} &\equiv r^{n-1}I_n \pmod p \quad \text{if } e \equiv 3 \pmod 4, \\ Q_n(a,b)^{2e} &\equiv (-r)^{n-1}I_n \pmod p \quad \text{if } e \equiv 1 \pmod 4, \end{aligned} \tag{5.2}$$

where $r \equiv (U_{(e+1)/2}/U_{(e-1)/2}) \pmod p$, so $r^2 \equiv -1 \pmod p$.

Proof. Using (3.2), if $U_e \equiv 0 \pmod p$, then

$$U_{e-1}a_{i,j}^{(e)} \equiv U_{e+1}a_{i-1,j-1}^{(e)}. \tag{5.3}$$

Since p divides neither U_{e-1} nor U_{e+1} (otherwise it would divide $U_1 = 1$), we get

$$\begin{aligned} a_{i,j}^{(e)} &\equiv 0 \pmod p \quad \text{if } i \neq j, \\ a_{i,i}^{(e)} &\equiv a_{i-1,i-1}^{(e)} \equiv \cdots \equiv a_{1,1}^{(e)} \equiv U_{e-1}^{n-1} \pmod p. \end{aligned} \tag{5.4}$$

Therefore

$$Q_n(a,b)^e \equiv U_{e-1}^{n-1}I_n \pmod p. \tag{5.5}$$

Using Cassini's identity $U_{l-1}U_{l+1} - U_l^2 = (-1)^l$, for $l = e$, we get, if $n = 2k$,

$$\begin{aligned} U_{e-1}^{n-1} = U_{e-1}^{2k-1} &\equiv (U_{e-1}^2)^k U_{e-1}^{-1} \equiv (-1)^{ke} U_{e-1}^{-1} \\ &\equiv (-1)^{(k+1)e} U_{e-1} \pmod p \end{aligned} \tag{5.6}$$

since $U_{e-1}^2 \equiv U_{e+1}^2 \equiv (-1)^e \pmod p$. If $n = 2k+1$, then

$$U_{e-1}^{n-1} = U_{e-1}^{2k} \equiv (U_{e-1}^2)^k \equiv (-1)^{ke} \pmod p. \tag{5.7}$$

The previous two congruences, replaced in $Q_n(a,b)^e \equiv U_{e-1}^{n-1} I_n \pmod p$, prove the first claim.

By [3, Lemma 3.4],

$$\begin{aligned} &U_{e-1} \equiv (-1)^{(e-2)/2} \quad \text{if } e \text{ is even}, \\ &U_{e-1} \equiv r(-1)^{(e-3)/2}, \quad r^2 \equiv -1 \pmod p, \quad \text{if } e \text{ is odd}. \end{aligned} \tag{5.8}$$

The residue r in the previous relation is just $r \equiv (U_{(e+1)/2}/U_{(e-1)/2}) \pmod p$. Thus, if e is even, then $U_{e-1}^2 \equiv 1 \pmod p$, so $Q_n(a,b)^{2e} \equiv I_n \pmod p$ for any n. The remaining cases are similar. □

Similarly, we can prove the following theorem.

Theorem 5.2. (1) *If $p|U_{p-1}$, then $Q_n(a,b)^{p-1} \equiv I_n \pmod p$.*
(2) *If $p|U_{p+1}$, then*

$$Q_{2k+1}(a,b)^{p+1} \equiv I_{2k+1} \pmod p, \qquad Q_{2k}(a,b)^{p+1} \equiv -I_{2k} \pmod p. \tag{5.9}$$

A consequence of [1, Theorem 1] is the following lemma.

Lemma 5.3. *For a prime p which divides $f(x) = x^2 - ax - 1$ for some integer x, the sequence $\{U_e\}_e$, satisfying the recurrence $U_e = aU_{e-1} + U_{e-2}$, has a period $p-1 \pmod p$ provided p is not a divisor of $D = a^2 + 4$.*

Our final result is the following theorem.

Theorem 5.4. *Let p be a prime divisor of $x^2 - ax - 1$ for some integer x and $\gcd(p, a^2+4) = 1$. Then, $Q_n(a,1)^{p-1} \equiv I_n \pmod p$.*

Proof. The proof is straightforward, using Lemma 5.3 and Theorem 3.1 or Theorem 5.1. □

6. Further comments. We observed that netted matrices defined using three-term or four-term recurrences with constant coefficients (we call these 3- or 4-*netted* matrices) preserve a *four-term* recurrence among the entries of their powers. We ask the question: *what is the order of the recurrence for higher powers of a 5-netted, and so forth, matrices?*

We might attempt to prove that a k-netted matrix will preserve a k-term recurrence. However, that is not true, and it can be seen from our work since a two-term recurrence is not preserved (see Theorem 2.1). Our guess is that a k^2-netted matrix preserves a k^2-term recurrence among the entries of its higher powers. Our guess is based on work already done and on many computer hours running examples. However, it is too early to promote it to a conjecture.

Acknowledgment. The author is on leave from the Institute of Mathematics of Romanian Academy, Bucharest, Romania.

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Pantelimon Stănică: Department of Mathematics, Auburn University Montgomery, Montgomery, AL 36124-4023, USA

E-mail address: pstanica@mail.aum.edu