

COEFFICIENTS OF SINGULARITIES OF THE BIHARMONIC PROBLEM OF NEUMANN TYPE: CASE OF THE CRACK

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This paper concerns the biharmonic problem of Neumann type in a sector V . We give a representation of the solution u of the problem in a form of a series $u = \sum_{\alpha \in E} c_{\alpha} r^{\alpha} \phi_{\alpha}$, and the functions ϕ_{α} are solutions of an auxiliary problem obtained by the separation of variables.

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1. Introduction. Let V be a sector of angle $\omega \leq 2\pi$ defined by

$$V = \{(r \cos \theta, r \sin \theta) \in \mathbb{R}^2; 0 < r < \rho, 0 < \theta < \omega\} \quad (1.1)$$

and Σ the circular boundary part defined by

$$\Sigma = \{(\rho \cos \theta, \rho \sin \theta) \in \mathbb{R}^2; 0 < \theta < \omega\}. \quad (1.2)$$

We are interested in the study of functions u , belonging to the Sobolev spaces $H^2(V)$, solutions of

$$\begin{aligned} \Delta^2 u &= 0 \quad \text{in } V, \\ Mu = Tu &= 0 \quad \text{for } \theta = 0, \omega, \end{aligned} \quad (1.3)$$

where

$$\begin{aligned} Mu &= \nu \Delta u + (1 - \nu)(\partial_1^2 u n_1^2 + 2\partial_{12} u n_1 n_2 + \partial_2^2 u n_2^2), \\ Tu &= -\frac{\partial \Delta u}{\partial n} + (1 - \nu) \frac{d}{ds} (\partial_1^2 u n_1 n_2 - \partial_{12} u (n_1^2 - n_2^2) - \partial_2^2 u n_1 n_2), \end{aligned} \quad (1.4)$$

ν is a real number called Poisson coefficient ($0 < \nu < 1/2$).

We show that these functions u are written under the form

$$u(r, \theta) = \sum_{\alpha \in E} c_{\alpha} r^{\alpha} \phi_{\alpha}(\theta), \quad (1.5)$$

E is the set of solutions of the equation in α

$$\sin^2(\alpha - 1)\omega = \left(\frac{1 - \nu}{3 + \nu}\right)^2 (\alpha - 1)^2 \sin^2 \omega, \quad \text{Re } \alpha > 1. \tag{1.6}$$

For the study of the solutions of (1.6), see, for example, Blum and Rannacher [1] and Grisvard [2].

We are going to calculate the coefficients c_α of development (1.5). These calculations have already been done by Tcha-Kondor [3] for the Dirichlet’s boundary conditions. He has established, thanks to the Green’s formula, a relation of biorthogonality between the functions ϕ_α and ϕ_β allowing him to calculate the coefficients c_β . We follow the same approach. This needs the writing in the domain V of an appropriate Green formula. Using this formula, we establish a relation of biorthogonality between the functions ϕ_α and ϕ_β , which is reduced under some conditions to the simple relation obtained by Tcha-Kondor, which enables us to calculate the coefficients c_β in the particular case of the crack ($\omega = 2\pi$).

2. Separation of variables. Replacing u by $r^\alpha \phi_\alpha(\theta)$ in problem (1.3) leads us to the boundary value problem

$$\phi_\alpha^{(4)}(\theta) + [\alpha^2 + (\alpha - 2)^2] \phi_\alpha''(\theta) + \alpha^2(\alpha - 2)^2 \phi_\alpha(\theta) = 0, \tag{2.1}$$

$$[\nu \alpha^2 + (1 - \nu)\alpha] \phi_\alpha + \phi_\alpha'' = 0, \quad \theta = 0, \theta = \omega, \tag{2.2}$$

$$[(2 - \nu)\alpha^2 - 3(1 - \nu)\alpha + 2(1 - \nu)] \phi_\alpha' + \phi_\alpha^{(3)} = 0, \quad \theta = 0, \theta = \omega. \tag{2.3}$$

The relation similar to orthogonality for the biharmonic operator is given by the following theorem.

THEOREM 2.1. *Let ϕ_α and ϕ_β be solutions of (2.1) with α and β solutions of (1.6). So, for $\alpha \neq \bar{\beta}$, we have the following relation:*

$$\begin{aligned} [\phi_\alpha, \phi_\beta] &= \int_0^\omega \left\{ \left[(\alpha^2 - 2\alpha) \phi_\alpha - \frac{\nu(\alpha + \bar{\beta}) + (3 - \nu) - 2\alpha}{\alpha - \bar{\beta}} \phi_\alpha'' \right] \bar{\phi}_\beta \right. \\ &\quad \left. + \left[(\bar{\beta}^2 - 2\bar{\beta}) \bar{\phi}_\beta + \frac{\nu(\alpha + \bar{\beta}) + (3 - \nu) - 2\bar{\beta}}{\alpha - \bar{\beta}} \bar{\phi}_\beta'' \right] \phi_\alpha \right\} d\theta \tag{2.4} \\ &= 0. \end{aligned}$$

PROOF. We use the following Green formula:

$$\int_V (\nu \Delta^2 u - u \Delta^2 \nu) dx = \int_\Gamma \left\{ \left(u \Gamma \nu + \frac{\partial u}{\partial n} M \nu \right) - \left(\nu \Gamma u + \frac{\partial \nu}{\partial n} M u \right) \right\} d\sigma, \tag{2.5}$$

(Γ is the boundary of V). For two functions u, v solutions of (1.3), we get the Green's formula in the following form:

$$\int_{\Sigma} \left\{ \left(uTv + \frac{\partial u}{\partial n} Mv \right) - \left(vTu + \frac{\partial v}{\partial n} Mu \right) \right\} d\sigma = 0. \tag{2.6}$$

On Σ , we have, for the function $u_{\alpha} = r^{\alpha}\phi_{\alpha}$,

$$\begin{aligned} \frac{\partial u_{\alpha}}{\partial n} &= \frac{\partial u_{\alpha}}{\partial r} = \alpha r^{\alpha-1} \phi_{\alpha}, \\ Mu_{\alpha} &= r^{\alpha-2} \{ [\alpha^2 - (1-\nu)\alpha] \phi_{\alpha} + \nu \phi_{\alpha}'' \}, \\ Tu_{\alpha} &= r^{\alpha-3} \{ -\alpha^2(\alpha-2)\phi_{\alpha} + [(\nu-2)\alpha + (3-\nu)] \phi_{\alpha}'' \}. \end{aligned} \tag{2.7}$$

The theorem results from the application of formula (2.6) to the biharmonic functions $u_{\alpha} = r^{\alpha}\phi_{\alpha}$ and $\bar{u}_{\beta} = r^{\beta}\bar{\phi}_{\beta}$, and by using relations (2.7). \square

REMARK 2.2. This relation between the functions ϕ_{α} and ϕ_{β} is similar to the relation of biorthogonality obtained when the functions ϕ_{α} and ϕ_{β} fulfill (2.1) with the Dirichlet's boundary conditions $\phi_{\alpha} = \phi'_{\alpha} = \phi_{\beta} = \phi'_{\beta} = 0$ for $\theta = 0$ and $\theta = \omega$. In this case, the relation is simplified because we have

$$\int_0^{\omega} \phi_{\alpha} \phi_{\beta}'' d\theta = \int_0^{\omega} \phi_{\alpha}'' \phi_{\beta} d\theta. \tag{2.8}$$

REMARK 2.3. By a double integration by parts, we get

$$\int_0^{\omega} \phi_{\alpha} \phi_{\beta}'' d\theta = \int_0^{\omega} \phi_{\alpha}'' \phi_{\beta} d\theta + [\phi_{\alpha}, \phi'_{\beta}]_0^{\omega} - [\phi'_{\alpha}, \phi_{\beta}]_0^{\omega}. \tag{2.9}$$

COROLLARY 2.4. Let ϕ_{α} and ϕ_{β} be solutions of (2.1) with α and β solutions of (1.6); in addition,

$$[\phi_{\alpha}, \phi'_{\beta}]_0^{\omega} - [\phi'_{\alpha}, \phi_{\beta}]_0^{\omega} = 0. \tag{2.10}$$

So, for $\alpha \neq \bar{\beta}$, we have the following relation:

$$[\phi_{\alpha}, \phi_{\beta}] = \int_0^{\omega} \left\{ [(\alpha^2 - 2\alpha)\phi_{\alpha} + \phi_{\alpha}'] \bar{\phi}_{\beta} + [(\bar{\beta}^2 - 2\bar{\beta})\bar{\phi}_{\beta} + \bar{\phi}_{\beta}'] \phi_{\alpha} \right\} d\theta = 0. \tag{2.11}$$

REMARK 2.5. For $u_{\alpha} = r^{\alpha}\phi_{\alpha}$, we have

$$\Delta u_{\alpha} - \frac{2}{r} \frac{\partial u_{\alpha}}{\partial r} = r^{\alpha-2} [(\alpha^2 - 2\alpha)\phi_{\alpha} + \phi_{\alpha}']. \tag{2.12}$$

Let P be the operator $P = \Delta - (2/r)(\partial/\partial r)$. From [Corollary 2.4](#) and [Remark 2.5](#), we deduce the following corollary.

COROLLARY 2.6. *Set $u_\alpha = r^\alpha \phi_\alpha(\theta)$ and $\bar{u}_\beta = r^{\bar{\beta}} \bar{\phi}_\beta$, where ϕ_α and ϕ_β are solutions of (2.1) with α and β solutions of (1.6); in addition,*

$$[\phi_\alpha, \phi'_\beta]_0^\omega - [\phi'_\alpha, \phi_\beta]_0^\omega = 0. \tag{2.13}$$

If $\alpha \neq \bar{\beta}$, we have the following relation:

$$\int_\Sigma (Pu_\alpha \bar{u}_\beta + u_\alpha P\bar{u}_\beta) d\sigma = 0. \tag{2.14}$$

Now, using [Corollary 2.6](#), we calculate the coefficients c_α of the development of the solution u of (1.3). The calculations will be done in the case of the crack ($\omega = 2\pi$), which is a very important case of singularity of domains. The explicit knowledge of the roots manifestly simplifies the calculations.

3. Case of a crack. The crack corresponds to $\omega = 2\pi$; if we replace this value in (1.6), we find that solutions α of (1.6) are the real values $k/2$. In this case, all the roots are of multiplicity 2.

We are going to represent u as follows:

$$u = \sum_{\alpha \in E} c_\alpha u_\alpha + \sum_{\alpha \in E} d_\alpha v_\alpha, \quad E = \left\{ \frac{k}{2}, k > 2 \right\}, \tag{3.1}$$

$$u_\alpha = r^\alpha \varphi_\alpha, \quad v_\alpha = r^\alpha \psi_\alpha,$$

φ_α are the even solutions in θ

$$\varphi_\alpha(\theta) = r^\alpha \left[\cos(\alpha - 2)\theta + \frac{4 - (1 - \nu)\alpha}{(1 - \nu)\alpha} \cos \alpha \theta \right], \tag{3.2}$$

and ψ_α the odd solutions in θ

$$\psi_\alpha(\theta) = r^\alpha \left[\sin(\alpha - 2)\theta - \frac{4 + (1 - \nu)(\alpha - 2)}{(1 - \nu)\alpha} \sin \alpha \theta \right]. \tag{3.3}$$

In this case ($\omega = 2\pi$), we have $\alpha = k/2$, then

$$\varphi'_\alpha(\omega) = \varphi'_\alpha(0) = 0, \quad \psi_\alpha(\omega) = \psi_\alpha(0) = 0; \tag{3.4}$$

hence, $[\varphi'_\alpha, \varphi_\beta]_0^\omega = [\varphi_\alpha, \varphi'_\beta]_0^\omega = 0$ and $[\psi'_\alpha, \psi_\beta]_0^\omega = [\psi_\alpha, \psi'_\beta]_0^\omega = 0$. But $[\varphi'_\alpha, \psi_\beta]_0^\omega = 0$ and $[\varphi_\alpha, \psi'_\beta]_0^\omega \neq 0$. From here comes the idea of decomposing the solution u of (1.3) to its even and odd parts with respect to k

$$\begin{aligned}
 u &= u_1 + u_2, \\
 u_i &= \sum_{\alpha \in E_i} (c_\alpha u_\alpha + d_\alpha v_\alpha), \quad i = 1, 2, \\
 E_1 &= \{n, n > 1\}, \quad E_2 = \left\{ \frac{2n+1}{2}, 2n > 1 \right\}.
 \end{aligned}
 \tag{3.5}$$

3.1. Calculation of c_β and d_β . We consider the integrals

$$\int_\Sigma (Pu_i u_\beta + u_i P u_\beta) d\sigma, \quad \int_\Sigma (Pu_i v_\beta + u_i P v_\beta) d\sigma,
 \tag{3.6}$$

$$\begin{aligned}
 \text{if } \alpha \in E_1, \text{ then } \varphi_\alpha(\omega) &= \varphi_\alpha(0), \quad \psi'_\alpha(\omega) = \psi'_\alpha(0), \\
 \text{if } \alpha \in E_2, \text{ then } \varphi_\alpha(\omega) &= -\varphi_\alpha(0), \quad \psi'_\alpha(\omega) = -\psi'_\alpha(0).
 \end{aligned}
 \tag{3.7}$$

Equations (3.4) and (3.7) allow us to apply Corollary 2.6 to functions u_α and u_β (resp., u_α, v_β and v_α, v_β); then, we obtain

$$\begin{aligned}
 \int_\Sigma (Pu_i u_\beta + u_i P u_\beta) d\sigma &= 2c_\beta \int_\Sigma u_\beta P u_\beta d\sigma + d_\beta \int_\Sigma (Pv_\beta u_\beta + v_\beta P u_\beta) d\sigma, \\
 \int_\Sigma (Pu_i v_\beta + u_i P v_\beta) d\sigma &= c_\beta \int_\Sigma (P u_\beta v_\beta + u_\beta P v_\beta) d\sigma + 2d_\beta \int_\Sigma v_\beta P v_\beta d\sigma.
 \end{aligned}
 \tag{3.8}$$

Direct calculation gives us

$$\begin{aligned}
 \int_\Sigma (P u_\beta v_\beta + u_\beta P v_\beta) d\sigma &= 0, \\
 \int_\Sigma u_\beta P u_\beta d\sigma &= \frac{2\rho^{2\beta-1}\omega}{(1-\nu)^2\beta} [\beta(1-\nu)(3+\nu) - 8], \\
 \int_\Sigma v_\beta P v_\beta d\sigma &= -\frac{2\rho^{2\beta-1}\omega}{(1-\nu)^2\beta} [(1-\nu)(3+\nu)(\beta-2) + 8].
 \end{aligned}
 \tag{3.9}$$

So, we have just established the following proposition.

PROPOSITION 3.1. *Let u be the solution of (1.3) written in the form*

$$u = u_1 + u_2,
 \tag{3.10}$$

where

$$u_i = \sum_{\alpha \in E_i} (c_\alpha u_\alpha + d_\alpha v_\alpha).
 \tag{3.11}$$

Suppose that the series that gives u_i is uniformly convergent; so, if $\beta \in E_i$, $i = 1, 2$, then

$$\begin{aligned}
 c_\beta &= \frac{(1-\nu)^2 \beta \rho^{1-2\beta}}{4\omega[(1-\nu)(3+\nu)\beta-8]} \int_\Sigma (Pu_i u_\beta + u_i Pu_\beta) d\sigma, \\
 d_\beta &= \frac{-(1-\nu)^2 \beta \rho^{1-2\beta}}{4\omega[(1-\nu)(3+\nu)(\beta-2)+8]} \int_\Sigma (Pu_i v_\beta + u_i Pv_\beta) d\sigma.
 \end{aligned}
 \tag{3.12}$$

REMARK 3.2. We have $\zeta \in \tilde{H}^{3/2}(\Sigma)$, the trace of u on Σ and $\chi \in H^{1/2}(\Sigma)$, the trace of Pu on Σ . If ζ belongs to the space $H^4(]0, 2\pi[)$ and χ to $H^2(]0, 2\pi[)$, then we have a uniform convergence of the series in \bar{V}_{ρ_0} for all $\rho_0 \leq \rho$, [3].

3.2. Independence of the coefficients. We are going to prove that the coefficients c_β (resp., d_β) of the development of the solution u of (1.3) are independent from ρ .

We have the following result.

THEOREM 3.3. *The coefficients c_β and d_β are independent from ρ .*

PROOF. In order to prove that c_β is independent from ρ , we are going to show that its derivative with respect to ρ is null, and by observing the expression of c_β (Proposition 3.1), we just have to prove that

$$\gamma_\beta = \rho^{1-2\beta} \int_\Sigma (Pu_i u_\beta + u_i Pu_\beta) d\sigma
 \tag{3.13}$$

has the null derivative with respect to ρ . □

By derivation in regard to r , we have

$$\begin{aligned}
 \gamma'_\beta &= \int_0^\omega \left\{ \frac{\partial \Delta u_i}{\partial r} r^{2-\beta} \varphi_\beta + \left[(2-\beta) \Delta u_i - 2 \frac{\partial^2 u_i}{\partial r^2} + (\beta^2 - 2) \frac{1}{r} \frac{\partial u_i}{\partial r} \right] r^{1-\beta} \varphi_\beta \right. \\
 &\quad \left. + \frac{\partial u_i}{\partial r} r^{-\beta} \varphi''_\beta - \beta u_i r^{-\beta-1} [(\beta^2 - 2\beta) \varphi_\beta + \varphi''_\beta] \right\} d\theta.
 \end{aligned}
 \tag{3.14}$$

On Σ , we have

$$\begin{aligned}
 \frac{\partial \Delta u_i}{\partial r} &= -T u_i + (1-\nu) \left(\frac{1}{r^3} \frac{\partial^2 u_i}{\partial \theta^2} - \frac{1}{r^2} \frac{\partial^3 u_i}{\partial r \partial \theta^2} \right), \\
 (2-\beta) \Delta u_i - 2 \frac{\partial^2 u_i}{\partial r^2} &= -\beta M u_i + [2 - (1-\nu)\beta] \left[\frac{1}{r} \frac{\partial u_i}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u_i}{\partial \theta^2} \right].
 \end{aligned}
 \tag{3.15}$$

Reinjecting these formulas in the expression of y'_β , we obtain

$$\begin{aligned}
 y'_\beta &= - \int_0^\omega (Mu_i \beta r^{1-\beta} \varphi_\beta + Tu_i r^{2-\beta} \varphi_\beta) d\theta \\
 &+ \int_0^\omega \left\{ ([\beta^2 - (1-\nu)\beta] \varphi_\beta + \varphi''_\beta) \frac{\partial u_i}{\partial r} - (1-\nu) \frac{\partial^3 u_i}{\partial r \partial \theta^2} \varphi_\beta \right\} r^{-\beta} d\theta \\
 &+ \int_0^\omega \left\{ [2 - (1-\nu)(\beta-1)] \frac{\partial^2 u_i}{\partial \theta^2} \varphi_\beta - \beta u_i [(\beta^2 - 2\beta) \varphi_\beta + \varphi''_\beta] \right\} r^{-1-\beta} d\theta.
 \end{aligned} \tag{3.16}$$

By a double integration by parts, we verify that

$$\int_0^\omega \frac{\partial^2 u_i}{\partial \theta^2} \varphi_\beta d\theta = \int_0^\omega u_i \varphi''_\beta d\theta, \tag{3.17}$$

$$\int_0^\omega \frac{\partial^3 u_i}{\partial r \partial \theta^2} \varphi_\beta d\theta = \int_0^\omega \frac{\partial u_i}{\partial r} \varphi''_\beta d\theta. \tag{3.18}$$

Reinjecting in the expression of y'_β and putting $\rho^{1-2\beta} \cdot \rho$ in factor, we obtain

$$\begin{aligned}
 y'_\beta &= -\rho^{1-2\beta} \int_0^\omega [Mu_i (\beta r^{\beta-1} \varphi_\beta) + Tu_i (r^\beta \varphi_\beta)] \rho d\theta \\
 &+ \rho^{1-2\beta} \int_0^\omega [([\beta^2 - (1-\nu)\beta] \varphi_\beta + \nu \varphi''_\beta) r^{\beta-2}] \frac{\partial u_i}{\partial r} \rho d\theta \\
 &+ \rho^{1-2\beta} \int_0^\omega \{ -\beta^2 (\beta-2) \varphi_\beta + [-(2-\nu)\beta + (3-\nu)] \varphi''_\beta \} r^{\beta-3} u_i \rho d\theta.
 \end{aligned} \tag{3.19}$$

By taking account of (2.7), whose expressions appear explicitly in y'_β , we obtain

$$y'_\beta = \rho^{1-2\beta} \left\{ - \int_\Sigma \left(Mu_i \frac{\partial u_\beta}{\partial n} + Tu_i u_\beta \right) d\sigma + \int_\Sigma \left(Mu_\beta \frac{\partial u_i}{\partial n} + Tu_\beta u_i \right) d\sigma \right\} = 0 \tag{3.20}$$

since we come back to the Green's formula (2.6) applied to u_i and u_β .

We follow the same analysis to prove the independence of d_β with respect to ρ .

3.3. Convergence of the series. We write c_α and d_α in the form

$$c_\alpha = I_i \rho^{-\alpha}, \quad d_\alpha = j_i \rho^{-a}, \tag{3.21}$$

where

$$\begin{aligned}
 I_i &= \rho \frac{(1-\nu)^2 \alpha}{4\omega[(1-\nu)(3+\nu)\alpha-8]} \int_{\Sigma} \{Pu_i \varphi_{\alpha} + u_i[(\alpha^2-2\alpha)\varphi_{\alpha} + \varphi''_{\alpha}]\rho^{-2}\} d\sigma, \\
 J_i &= -\rho \frac{(1-\nu)^2 \alpha}{4\omega[(1-\nu)(3+\nu)(\alpha-2)+8]} \int_{\Sigma} \{Pu_i \psi_{\alpha} + u_i[(\alpha^2-2\alpha)\psi_{\alpha} + \psi''_{\alpha}]\rho^{-2}\} d\sigma.
 \end{aligned}
 \tag{3.22}$$

The solution u of (1.3) is then written as follows:

$$u = u_1 + u_2, \tag{3.23}$$

$$u_i = \sum_{\alpha \in E_i} \left[\left(\frac{r}{\rho}\right)^{\alpha} I_i \varphi_{\alpha} + \left(\frac{r}{\rho}\right)^{\alpha} J_i \psi_{\alpha} \right]. \tag{3.24}$$

We have the following result.

THEOREM 3.4. *The series (3.24) converges as soon as $r < \rho$.*

PROOF. Set

$$\begin{aligned}
 N_{i,\alpha} &= \int_0^{\omega} \{Pu_i \varphi_{\alpha} + u_i[(\alpha^2-2\alpha)\varphi_{\alpha} + \varphi''_{\alpha}]\rho^{-2}\} d\theta \\
 &= \int_0^{\omega} Pu_i \varphi_{\alpha} d\theta + (\alpha^2-2\alpha)\rho^{-2} \int_0^{\omega} u_i \varphi_{\alpha} d\theta + \rho^{-2} \int_0^{\omega} u_i \varphi''_{\alpha} d\theta.
 \end{aligned}
 \tag{3.25}$$

We show that $N_{i,\alpha}$ is a product of $1/\alpha$ by limited term for α large.

According to (3.17), we have

$$\int_0^{\omega} u_i \varphi''_{\alpha} d\theta = \int_0^{\omega} u_i'' \varphi_{\alpha} d\theta. \tag{3.26}$$

Replacing φ_{α} by its expression and integrating by parts, we get

$$\int_0^{\omega} u_i'' \varphi_{\alpha} d\theta = \frac{1}{\alpha} \left\{ \frac{-\alpha}{\alpha-2} \int_0^{\omega} u_i''' \sin(\alpha-2)\theta d\theta - \frac{4-(1-\nu)\alpha}{(1-\nu)\alpha} \int_0^{\omega} u_i''' \sin \alpha \theta d\theta \right\}. \tag{3.27}$$

On the other hand, by a triple integration by parts, we have

$$\begin{aligned}
 (\alpha^2-2\alpha) \int_0^{\omega} u_i \varphi_{\alpha} d\theta &= \frac{1}{\alpha} \left\{ \frac{\alpha^2}{(\alpha-2)^2} \int_0^{\omega} u_i''' \sin(\alpha-2)\theta d\theta \right. \\
 &\quad \left. + \frac{\alpha-2}{\alpha} \frac{4-(1-\nu)\alpha}{(1-\nu)\alpha} \int_0^{\omega} u_i''' \sin \alpha \theta d\theta \right\}.
 \end{aligned}
 \tag{3.28}$$

Also, by an integration by parts, we get

$$\int_0^\omega \left(\Delta u_i - \frac{2}{r} \frac{\partial u_i}{\partial r} \right) \varphi_\alpha d\theta = -\frac{1}{\alpha-2} \int_0^\omega \left(\frac{\partial \Delta u_i}{\partial \theta} - \frac{2}{r} \frac{\partial^2 u_i}{\partial r \partial \theta} \right) \sin(\alpha-2)\theta d\theta - \frac{1}{\alpha} \frac{4-(1-\nu)\alpha}{(1-\nu)\alpha} \int_0^\omega \left(\frac{\partial \Delta u_i}{\partial \theta} - \frac{2}{r} \frac{\partial^2 u_i}{\partial r \partial \theta} \right) \sin \alpha \theta d\theta. \tag{3.29}$$

Then, we deduce the existence of a constant C_0 so as

$$|N_{i,\alpha}| \leq \frac{C_0}{\alpha}. \tag{3.30}$$

Using this last inequality and remarking that φ_α is limited, as well as the term

$$\frac{(1-\nu)^2 \alpha}{4\omega[(1-\nu)(\nu+3)\alpha-8]} \tag{3.31}$$

for large α , we deduce the existence of a constant C so as

$$\left| \sum_{\alpha \in E_i} c_\alpha r^\alpha \varphi_\alpha \right| \leq \sum_{\alpha \in E_i} \frac{C}{\alpha} \left(\frac{r}{\rho} \right)^\alpha, \tag{3.32}$$

which converges as soon as $r < \rho$.

In the same way, we prove the convergence of the series $\sum_{\alpha \in E_i} d_\alpha r^\alpha \psi_\alpha$. □

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