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# COEFFICIENTS OF SINGULARITIES OF THE BIHARMONIC PROBLEM OF NEUMANN TYPE: CASE OF THE CRACK

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This paper concerns the biharmonic problem of Neumann type in a sector $V$. We give a representation of the solution $u$ of the problem in a form of a series $u = \sum_{\alpha \in E} c_\alpha r^\alpha \phi_\alpha$, and the functions $\phi_\alpha$ are solutions of an auxiliary problem obtained by the separation of variables.

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**1. Introduction.** Let $V$ be a sector of angle $\omega \le 2\pi$ defined by

$$V = \{(r\cos\theta, r\sin\theta) \in \mathbb{R}^2;\ 0 < r < \rho,\ 0 < \theta < \omega\} \tag{1.1}$$

and $\Sigma$ the circular boundary part defined by

$$\Sigma = \{(\rho\cos\theta, \rho\sin\theta) \in \mathbb{R}^2;\ 0 < \theta < \omega\}. \tag{1.2}$$

We are interested in the study of functions $u$, belonging to the Sobolev spaces $H^2(V)$, solutions of

$$\begin{gathered} \Delta^2 u = 0 \quad \text{in } V, \\ Mu = Tu = 0 \quad \text{for } \theta = 0, \omega, \end{gathered} \tag{1.3}$$

where

$$\begin{aligned} Mu &= \nu \triangle u + (1-\nu)\left(\partial_1^2 u n_1^2 + 2\partial_{12} u n_1 n_2 + \partial_2^2 u n_2^2\right), \\ Tu &= -\frac{\partial \triangle u}{\partial n} + (1-\nu)\frac{d}{ds}\left(\partial_1^2 u n_1 n_2 - \partial_{12} u\left(n_1^2 - n_2^2\right) - \partial_2^2 u n_1 n_2\right), \end{aligned} \tag{1.4}$$

$\nu$ is a real number called Poisson coefficient $(0 < \nu < 1/2)$.

We show that these functions $u$ are written under the form

$$u(r,\theta) = \sum_{\alpha \in E} c_\alpha r^\alpha \phi_\alpha(\theta), \tag{1.5}$$

$E$ is the set of solutions of the equation in $\alpha$

$$\sin^2(\alpha-1)\omega = \left(\frac{1-\nu}{3+\nu}\right)^2 (\alpha-1)^2 \sin^2\omega, \quad \operatorname{Re}\alpha > 1. \tag{1.6}$$

For the study of the solutions of (1.6), see, for example, Blum and Rannacher [1] and Grisvard [2].

We are going to calculate the coefficients $c_\alpha$ of development (1.5). These calculations have already been done by Tcha-Kondor [3] for the Dirichlet's boundary conditions. He has established, thanks to the Green's formula, a relation of biorthogonality between the functions $\phi_\alpha$ and $\phi_\beta$ allowing him to calculate the coefficients $c_\beta$. We follow the same approach. This needs the writing in the domain $V$ of an appropriate Green formula. Using this formula, we establish a relation of biorthogonality between the functions $\phi_\alpha$ and $\phi_\beta$, which is reduced under some conditions to the simple relation obtained by Tcha-Kondor, which enables us to calculate the coefficients $c_\beta$ in the particular case of the crack $(\omega = 2\pi)$.

**2. Separation of variables.** Replacing $u$ by $r^\alpha \phi_\alpha(\theta)$ in problem (1.3) leads us to the boundary value problem

$$\phi_\alpha^{(4)}(\theta) + [\alpha^2 + (\alpha-2)^2]\phi_\alpha''(\theta) + \alpha^2(\alpha-2)^2\phi_\alpha(\theta) = 0, \tag{2.1}$$

$$[\nu\alpha^2 + (1-\nu)\alpha]\phi_\alpha + \phi_\alpha'' = 0, \quad \theta = 0,\ \theta = \omega, \tag{2.2}$$

$$[(2-\nu)\alpha^2 - 3(1-\nu)\alpha + 2(1-\nu)]\phi_\alpha' + \phi_\alpha^{(3)} = 0, \quad \theta = 0,\ \theta = \omega. \tag{2.3}$$

The relation similar to orthogonality for the biharmonic operator is given by the following theorem.

**Theorem 2.1.** *Let $\phi_\alpha$ and $\phi_\beta$ be solutions of (2.1) with $\alpha$ and $\beta$ solutions of (1.6). So, for $\alpha \neq \overline{\beta}$, we have the following relation:*

$$\begin{aligned}[\phi_\alpha, \phi_\beta] = \int_0^\omega \Bigg\{ &\left[(\alpha^2 - 2\alpha)\phi_\alpha - \frac{\nu(\alpha+\overline{\beta}) + (3-\nu) - 2\alpha}{\alpha - \overline{\beta}}\phi_\alpha''\right]\overline{\phi_\beta} \\ &+ \left[(\overline{\beta}^2 - 2\overline{\beta})\overline{\phi_\beta} + \frac{\nu(\alpha+\overline{\beta}) + (3-\nu) - 2\overline{\beta}}{\alpha - \overline{\beta}}\overline{\phi_\beta''}\right]\phi_\alpha \Bigg\} d\theta \\ = 0.\end{aligned} \tag{2.4}$$

**Proof.** We use the following Green formula:

$$\int_V (v\triangle^2 u - u\triangle^2 v)\,dx = \int_\Gamma \left\{\left(uTv + \frac{\partial u}{\partial n}Mv\right) - \left(vTu + \frac{\partial v}{\partial n}Mu\right)\right\} d\sigma, \tag{2.5}$$

($\Gamma$ is the boundary of $V$). For two functions $u, v$ solutions of (1.3), we get the Green's formula in the following form:

$$\int_{\Sigma} \left\{ \left( uTv + \frac{\partial u}{\partial n} Mv \right) - \left( vTu + \frac{\partial v}{\partial n} Mu \right) \right\} d\sigma = 0. \tag{2.6}$$

On $\Sigma$, we have, for the function $u_\alpha = r^\alpha \phi_\alpha$,

$$\begin{aligned} \frac{\partial u_\alpha}{\partial n} &= \frac{\partial u_\alpha}{\partial r} = \alpha r^{\alpha-1} \phi_\alpha, \\ Mu_\alpha &= r^{\alpha-2}\{[\alpha^2 - (1-\nu)\alpha]\phi_\alpha + \nu \phi''_\alpha\}, \\ Tu_\alpha &= r^{\alpha-3}\{-\alpha^2(\alpha-2)\phi_\alpha + [(\nu-2)\alpha + (3-\nu)]\phi''_\alpha\}. \end{aligned} \tag{2.7}$$

The theorem results from the application of formula (2.6) to the biharmonic functions $u_\alpha = r^\alpha \phi_\alpha$ and $\overline{u}_\beta = r^{\overline{\beta}} \overline{\phi}_\beta$, and by using relations (2.7). □

**Remark 2.2.** This relation between the functions $\phi_\alpha$ and $\phi_\beta$ is similar to the relation of biorthogonality obtained when the functions $\phi_\alpha$ and $\phi_\beta$ fulfill (2.1) with the Dirichlet's boundary conditions $\phi_\alpha = \phi'_\alpha = \phi_\beta = \phi'_\beta = 0$ for $\theta = 0$ and $\theta = \omega$. In this case, the relation is simplified because we have

$$\int_0^\omega \phi_\alpha \phi''_\beta \, d\theta = \int_0^\omega \phi''_\alpha \phi_\beta \, d\theta. \tag{2.8}$$

**Remark 2.3.** By a double integration by parts, we get

$$\int_0^\omega \phi_\alpha \phi''_\beta \, d\theta = \int_0^\omega \phi''_\alpha \phi_\beta \, d\theta + [\phi_\alpha, \phi'_\beta]_0^\omega - [\phi'_\alpha, \phi_\beta]_0^\omega. \tag{2.9}$$

**Corollary 2.4.** *Let $\phi_\alpha$ and $\phi_\beta$ be solutions of (2.1) with $\alpha$ and $\beta$ solutions of (1.6); in addition,*

$$[\phi_\alpha, \phi'_\beta]_0^\omega - [\phi'_\alpha, \phi_\beta]_0^\omega = 0. \tag{2.10}$$

*So, for $\alpha \neq \overline{\beta}$, we have the following relation:*

$$[\phi_\alpha, \phi_\beta] = \int_0^\omega \left\{ [(\alpha^2 - 2\alpha)\phi_\alpha + \phi''_\alpha]\overline{\phi}_\beta + \left[ (\overline{\beta}^2 - 2\overline{\beta})\overline{\phi}_\beta + \overline{\phi}''_\beta \right] \phi_\alpha \right\} d\theta = 0. \tag{2.11}$$

**Remark 2.5.** For $u_\alpha = r^\alpha \phi_\alpha$, we have

$$\triangle u_\alpha - \frac{2}{r} \frac{\partial u_\alpha}{\partial r} = r^{\alpha-2}[(\alpha^2 - 2\alpha)\phi_\alpha + \phi''_\alpha]. \tag{2.12}$$

Let $P$ be the operator $P=\triangle-(2/r)(\partial/\partial r)$. From Corollary 2.4 and Remark 2.5, we deduce the following corollary.

**Corollary 2.6.** *Set $u_\alpha = r^\alpha \phi_\alpha(\theta)$ and $\overline{u}_\beta = r^{\overline{\beta}}\,\overline{\phi}_\beta$, where $\phi_\alpha$ and $\phi_\beta$ are solutions of (2.1) with $\alpha$ and $\beta$ solutions of (1.6); in addition,*

$$[\phi_\alpha, \phi'_\beta]_0^\omega - [\phi'_\alpha, \phi_\beta]_0^\omega = 0. \tag{2.13}$$

*If $\alpha \neq \overline{\beta}$, we have the following relation:*

$$\int_\Sigma (Pu_\alpha \overline{u}_\beta + u_\alpha P\overline{u}_\beta)\, d\sigma = 0. \tag{2.14}$$

Now, using Corollary 2.6, we calculate the coefficients $c_\alpha$ of the development of the solution $u$ of (1.3). The calculations will be done in the case of the crack $(\omega = 2\pi)$, which is a very important case of singularity of domains. The explicit knowledge of the roots manifestly simplifies the calculations.

**3. Case of a crack.** The crack corresponds to $\omega = 2\pi$; if we replace this value in (1.6), we find that solutions $\alpha$ of (1.6) are the real values $k/2$. In this case, all the roots are of multiplicity 2.

We are going to represent $u$ as follows:

$$\begin{gathered} u = \sum_{\alpha\in E} c_\alpha u_\alpha + \sum_{\alpha\in E} d_\alpha v_\alpha, \quad E = \left\{\frac{k}{2},\ k > 2\right\}, \\ u_\alpha = r^\alpha \varphi_\alpha, \qquad v_\alpha = r^\alpha \psi_\alpha, \end{gathered} \tag{3.1}$$

$\varphi_\alpha$ are the even solutions in $\theta$

$$\varphi_\alpha(\theta) = r^\alpha\left[\cos(\alpha-2)\theta + \frac{4-(1-\nu)\alpha}{(1-\nu)\alpha}\cos\alpha\theta\right], \tag{3.2}$$

and $\psi_\alpha$ the odd solutions in $\theta$

$$\psi_\alpha(\theta) = r^\alpha\left[\sin(\alpha-2)\theta - \frac{4+(1-\nu)(\alpha-2)}{(1-\nu)\alpha}\sin\alpha\theta\right]. \tag{3.3}$$

In this case $(\omega = 2\pi)$, we have $\alpha = k/2$, then

$$\varphi'_\alpha(\omega) = \varphi'_\alpha(0) = 0, \qquad \psi_\alpha(\omega) = \psi_\alpha(0) = 0; \tag{3.4}$$

hence, $[\varphi'_\alpha,\varphi_\beta]_0^\omega = [\varphi_\alpha,\varphi'_\beta]_0^\omega = 0$ and $[\psi'_\alpha,\psi_\beta]_0^\omega = [\psi_\alpha,\psi'_\beta]_0^\omega = 0$. But $[\varphi'_\alpha,\psi_\beta]_0^\omega = 0$ and $[\varphi_\alpha,\psi'_\beta]_0^\omega \neq 0$. From here comes the idea of decomposing the solution $u$ of (1.3) to its even and odd parts with respect to $k$

$$\begin{gathered} u = u_1 + u_2, \\ u_i = \sum_{\alpha\in E_i}(c_\alpha u_\alpha + d_\alpha v_\alpha), \quad i = 1,2, \\ E_1 = \{n,\ n > 1\}, \qquad E_2 = \left\{\frac{2n+1}{2},\ 2n > 1\right\}. \end{gathered} \tag{3.5}$$

**3.1. Calculation of $c_\beta$ and $d_\beta$.** We consider the integrals

$$\int_\Sigma (Pu_i u_\beta + u_i P u_\beta)\,d\sigma, \qquad \int_\Sigma (Pu_i v_\beta + u_i P v_\beta)\,d\sigma, \tag{3.6}$$

$$\begin{aligned} &\text{if } \alpha\in E_1, \text{ then } \varphi_\alpha(\omega) = \varphi_\alpha(0), \quad \psi'_\alpha(\omega) = \psi'_\alpha(0), \\ &\text{if } \alpha\in E_2, \text{ then } \varphi_\alpha(\omega) = -\varphi_\alpha(0), \quad \psi'_\alpha(\omega) = -\psi'_\alpha(0). \end{aligned} \tag{3.7}$$

Equations (3.4) and (3.7) allow us to apply Corollary 2.6 to functions $u_\alpha$ and $u_\beta$ (resp., $u_\alpha$, $v_\beta$ and $v_\alpha$, $v_\beta$); then, we obtain

$$\begin{aligned} \int_\Sigma (Pu_i u_\beta + u_i P u_\beta)\,d\sigma &= 2c_\beta\int_\Sigma u_\beta P u_\beta\,d\sigma + d_\beta\int_\Sigma (Pv_\beta u_\beta + v_\beta P u_\beta)\,d\sigma, \\ \int_\Sigma (Pu_i v_\beta + u_i P v_\beta)\,d\sigma &= c_\beta\int_\Sigma (Pu_\beta v_\beta + u_\beta P v_\beta)\,d\sigma + 2d_\beta\int_\Sigma v_\beta P v_\beta\,d\sigma. \end{aligned} \tag{3.8}$$

Direct calculation gives us

$$\begin{aligned} \int_\Sigma (Pu_\beta v_\beta + u_\beta P v_\beta)\,d\sigma &= 0, \\ \int_\Sigma u_\beta P u_\beta\,d\sigma &= \frac{2\rho^{2\beta-1}\omega}{(1-\nu)^2\beta}[\beta(1-\nu)(3+\nu)-8], \\ \int_\Sigma v_\beta P v_\beta\,d\sigma &= -\frac{2\rho^{2\beta-1}\omega}{(1-\nu)^2\beta}[(1-\nu)(3+\nu)(\beta-2)+8]. \end{aligned} \tag{3.9}$$

So, we have just established the following proposition.

**Proposition 3.1.** *Let $u$ be the solution of (1.3) written in the form*

$$u = u_1 + u_2, \tag{3.10}$$

*where*

$$u_i = \sum_{\alpha\in E_i}(c_\alpha u_\alpha + d_\alpha v_\alpha). \tag{3.11}$$

*Suppose that the series that gives* $u_i$ *is uniformly convergent; so, if* $\beta \in E_i$, $i = 1,2$, *then*

$$c_\beta = \frac{(1-\nu)^2\beta\rho^{1-2\beta}}{4\omega[(1-\nu)(3+\nu)\beta-8]}\int_\Sigma (Pu_iu_\beta + u_iPu_\beta)\,d\sigma,$$
$$d_\beta = \frac{-(1-\nu)^2\beta\rho^{1-2\beta}}{4\omega[(1-\nu)(3+\nu)(\beta-2)+8]}\int_\Sigma (Pu_iv_\beta + u_iPv_\beta)\,d\sigma. \tag{3.12}$$

**Remark 3.2.** We have $\zeta \in \widetilde{H}^{3/2}(\Sigma)$, the trace of $u$ on $\Sigma$ and $\chi \in H^{1/2}(\Sigma)$, the trace of $Pu$ on $\Sigma$. If $\zeta$ belongs to the space $H^4(]0,2\pi[)$ and $\chi$ to $H^2(]0,2\pi[)$, then we have a uniform convergence of the series in $\overline{V}_{\rho_0}$ for all $\rho_0 \le \rho$, [3].

**3.2. Independence of the coefficients.** We are going to prove that the coefficients $c_\beta$ (resp., $d_\beta$) of the development of the solution $u$ of (1.3) are independent from $\rho$.

We have the following result.

**Theorem 3.3.** *The coefficients* $c_\beta$ *and* $d_\beta$ *are independent from* $\rho$.

**Proof.** In order to prove that $c_\beta$ is independent from $\rho$, we are going to show that its derivative with respect to $\rho$ is null, and by observing the expression of $c_\beta$ (Proposition 3.1), we just have to prove that

$$\gamma_\beta = \rho^{1-2\beta}\int_\Sigma (Pu_iu_\beta + u_iPu_\beta)\,d\sigma \tag{3.13}$$

has the null derivative with respect to $\rho$. □

By derivation in regard to $r$, we have

$$\gamma'_\beta = \int_0^\omega \left\{\frac{\partial\triangle u_i}{\partial r}r^{2-\beta}\varphi_\beta + \left[(2-\beta)\triangle u_i - 2\frac{\partial^2 u_i}{\partial r^2} + (\beta^2-2)\frac{1}{r}\frac{\partial u_i}{\partial r}\right]r^{1-\beta}\varphi_\beta + \frac{\partial u_i}{\partial r}r^{-\beta}\varphi''_\beta - \beta u_i r^{-\beta-1}[(\beta^2-2\beta)\varphi_\beta + \varphi''_\beta]\right\}d\theta. \tag{3.14}$$

On $\Sigma$, we have

$$\frac{\partial\triangle u_i}{\partial r} = -Tu_i + (1-\nu)\left(\frac{1}{r^3}\frac{\partial^2 u_i}{\partial\theta^2} - \frac{1}{r^2}\frac{\partial^3 u_i}{\partial r\partial\theta^2}\right),$$
$$(2-\beta)\triangle u_i - 2\frac{\partial^2 u_i}{\partial r^2} = -\beta Mu_i + [2-(1-\nu)\beta]\left[\frac{1}{r}\frac{\partial u_i}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u_i}{\partial\theta^2}\right]. \tag{3.15}$$

Reinjecting these formulas in the expression of $\gamma'_\beta$, we obtain

$$\begin{aligned}\gamma'_\beta = &-\int_0^\omega \left(Mu_i\beta r^{1-\beta}\varphi_\beta + Tu_i r^{2-\beta}\varphi_\beta\right) d\theta \\ &+\int_0^\omega \left\{([\beta^2-(1-\nu)\beta]\varphi_\beta+\varphi''_\beta)\frac{\partial u_i}{\partial r}-(1-\nu)\frac{\partial^3 u_i}{\partial r\partial\theta^2}\varphi_\beta\right\} r^{-\beta}\, d\theta \\ &+\int_0^\omega \left\{[2-(1-\nu)(\beta-1)]\frac{\partial^2 u_i}{\partial\theta^2}\varphi_\beta-\beta u_i[(\beta^2-2\beta)\varphi_\beta+\varphi''_\beta]\right\} r^{-1-\beta}\, d\theta.\end{aligned} \tag{3.16}$$

By a double integration by parts, we verify that

$$\int_0^\omega \frac{\partial^2 u_i}{\partial\theta^2}\varphi_\beta\, d\theta = \int_0^\omega u_i\varphi''_\beta\, d\theta, \tag{3.17}$$

$$\int_0^\omega \frac{\partial^3 u_i}{\partial r\partial\theta^2}\varphi_\beta\, d\theta = \int_0^\omega \frac{\partial u_i}{\partial r}\varphi''_\beta\, d\theta. \tag{3.18}$$

Reinjecting in the expression of $\gamma'_\beta$ and putting $\rho^{1-2\beta}\cdot\rho$ in factor, we obtain

$$\begin{aligned}\gamma'_\beta = &-\rho^{1-2\beta}\int_0^\omega \left[Mu_i(\beta r^{\beta-1}\varphi_\beta)+Tu_i(r^\beta\varphi_\beta)\right]\rho\, d\theta \\ &+\rho^{1-2\beta}\int_0^\omega \left[([\beta^2-(1-\nu)\beta]\varphi_\beta+\nu\varphi''_\beta)r^{\beta-2}\right]\frac{\partial u_i}{\partial r}\rho\, d\theta \\ &+\rho^{1-2\beta}\int_0^\omega \left\{-\beta^2(\beta-2)\varphi_\beta+[-(2-\nu)\beta+(3-\nu)]\varphi''_\beta\right\} r^{\beta-3}u_i\rho\, d\theta.\end{aligned} \tag{3.19}$$

By taking account of (2.7), whose expressions appear explicitly in $\gamma'_\beta$, we obtain

$$\gamma'_\beta = \rho^{1-2\beta}\left\{-\int_\Sigma \left(Mu_i\frac{\partial u_\beta}{\partial n}+Tu_i u_\beta\right) d\sigma + \int_\Sigma \left(Mu_\beta\frac{\partial u_i}{\partial n}+Tu_\beta u_i\right) d\sigma\right\} = 0 \tag{3.20}$$

since we come back to the Green's formula (2.6) applied to $u_i$ and $u_\beta$.

We follow the same analysis to prove the independence of $d_\beta$ with respect to $\rho$.

**3.3. Convergence of the series.** We write $c_\alpha$ and $d_\alpha$ in the form

$$c_\alpha = I_i\rho^{-\alpha}, \qquad d_\alpha = j_i\rho^{-a}, \tag{3.21}$$

where

$$I_i = \rho\frac{(1-\nu)^2\alpha}{4\omega[(1-\nu)(3+\nu)\alpha-8]}\int_\Sigma\{Pu_i\varphi_\alpha+u_i[(\alpha^2-2\alpha)\varphi_\alpha+\varphi''_\alpha]\rho^{-2}\}\,d\sigma,$$
$$J_i = -\rho\frac{(1-\nu)^2\alpha}{4\omega[(1-\nu)(3+\nu)(\alpha-2)+8]}\int_\Sigma\{Pu_i\psi_\alpha+u_i[(\alpha^2-2\alpha)\psi_\alpha+\psi''_\alpha]\rho^{-2}\}\,d\sigma. \tag{3.22}$$

The solution $u$ of (1.3) is then written as follows:

$$u = u_1+u_2, \tag{3.23}$$
$$u_i = \sum_{\alpha\in E_i}\left[\left(\frac{r}{\rho}\right)^\alpha I_i\varphi_\alpha+\left(\frac{r}{\rho}\right)^\alpha J_i\psi_\alpha\right]. \tag{3.24}$$

We have the following result.

**Theorem 3.4.** *The series (3.24) converges as soon as $r<\rho$.*

**Proof.** Set

$$N_{i,\alpha} = \int_0^\omega\{Pu_i\varphi_\alpha+u_i[(\alpha^2-2\alpha)\varphi_\alpha+\varphi''_\alpha]\rho^{-2}\}\,d\theta$$
$$= \int_0^\omega Pu_i\varphi_\alpha\,d\theta+(\alpha^2-2\alpha)\rho^{-2}\int_0^\omega u_i\varphi_\alpha\,d\theta+\rho^{-2}\int_0^\omega u_i\varphi''_\alpha\,d\theta. \tag{3.25}$$

We show that $N_{i,\alpha}$ is a product of $1/\alpha$ by limited term for $\alpha$ large.

According to (3.17), we have

$$\int_0^\omega u_i\varphi''_\alpha\,d\theta = \int_0^\omega u''_i\varphi_\alpha\,d\theta. \tag{3.26}$$

Replacing $\varphi_\alpha$ by its expression and integrating by parts, we get

$$\int_0^\omega u''_i\varphi_\alpha\,d\theta = \frac{1}{\alpha}\left\{\frac{-\alpha}{\alpha-2}\int_0^\omega u'''_i\sin(\alpha-2)\theta\,d\theta-\frac{4-(1-\nu)\alpha}{(1-\nu)\alpha}\int_0^\omega u'''_i\sin\alpha\theta\,d\theta\right\}. \tag{3.27}$$

On the other hand, by a triple integration by parts, we have

$$(\alpha^2-2\alpha)\int_0^\omega u_i\varphi_\alpha\,d\theta = \frac{1}{\alpha}\left\{\frac{\alpha^2}{(\alpha-2)^2}\int_0^\omega u'''_i\sin(\alpha-2)\theta\,d\theta + \frac{\alpha-2}{\alpha}\frac{4-(1-\nu)\alpha}{(1-\nu)\alpha}\int_0^\omega u'''_i\sin\alpha\theta\,d\theta\right\}. \tag{3.28}$$

Also, by an integration by parts, we get

$$\int_0^\omega \left(\Delta u_i - \frac{2}{r}\frac{\partial u_i}{\partial r}\right)\varphi_\alpha \, d\theta = -\frac{1}{\alpha-2}\int_0^\omega \left(\frac{\partial \Delta u_i}{\partial \theta} - \frac{2}{r}\frac{\partial^2 u_i}{\partial r \partial \theta}\right)\sin(\alpha-2)\theta \, d\theta$$
$$-\frac{1}{\alpha}\frac{4-(1-\nu)\alpha}{(1-\nu)\alpha}\int_0^\omega \left(\frac{\partial \Delta u_i}{\partial \theta} - \frac{2}{r}\frac{\partial^2 u_i}{\partial r \partial \theta}\right)\sin \alpha\theta \, d\theta. \tag{3.29}$$

Then, we deduce the existence of a constant $C_0$ so as

$$|N_{i,\alpha}| \leq \frac{C_0}{\alpha}. \tag{3.30}$$

Using this last inequality and remarking that $\varphi_\alpha$ is limited, as well as the term

$$\frac{(1-\nu)^2\alpha}{4\omega[(1-\nu)(\nu+3)\alpha-8]} \tag{3.31}$$

for large $\alpha$, we deduce the existence of a constant $C$ so as

$$\left|\sum_{\alpha \in E_i} c_\alpha r^\alpha \varphi_\alpha\right| \leq \sum_{\alpha \in E_i} \frac{C}{\alpha}\left(\frac{r}{\rho}\right)^\alpha, \tag{3.32}$$

which converges as soon as $r < \rho$.

In the same way, we prove the convergence of the series $\sum_{\alpha \in E_i} d_\alpha r^\alpha \psi_\alpha$. □

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