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UNIQUENESS OF SEMILINEAR ELLIPTIC INVERSE PROBLEM

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We consider the uniqueness of the inverse problem for a semilinear elliptic differential equation with Dirichlet condition. The necessary and sufficient condition of a unique solution is obtained. We improved the results obtained by Isakov and Sylvester (1994) for the same problem.

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1. Introduction. Isakov and Sylvester considered in [3] the problem of uniquely determining a in the following semilinear elliptic Dirichlet problem:

$$-\Delta u + a(x,u) = 0, \quad x \in \Omega, \tag{1.1}$$

$$u|_{\partial\Omega} = g \in W^{2-1/p,p}(\partial\Omega), \tag{1.2}$$

where $\Omega \subset \mathbb{R}^n$ $(n \geq 3)$ is a bounded domain and its boundary $\partial\Omega \in C^{2,\alpha}$. Denote $u(x,g)$ as the solution of (1.1) and (1.2). Under the assumptions

$$a_s(x,s) \geq 0, \quad a(x,s), a_s(x,s), a_{ss}(x,s) \in L^\infty(\Omega \times [s,s]), \tag{1.3}$$

they proved the following theorem.

Theorem 1.1. *Denote the mapping $\Lambda_a : g \to \partial u/\partial\mu|_{\partial\Omega}$. If $a_1(x,0) = a_2(x,0) = 0$ and $\Lambda_{a_1} = \Lambda_{a_2}$, then $a_1(x,s) = a_2(x,s)$ on E, where $E = \{(x,s) : \min(u_{1*}, u_{2*}) < s < \max(u_1*, u_2*),\ x \in \Omega\}$, $u_i* = \sup_{g\in W^{2-1/p,p}(\partial\Omega)} u(x,g)$, and $u_{i*} = \inf_{g\in W^{2-1/p,p}(\partial\Omega)} u(x,g)$, $i = 1,2$.*

Later, Nakamura, in [4], attempted to improve the above result by claiming that the same results can be obtained only by assuming the following conditions on a:

$$a(x,s) \in L^\infty(\bar{\Omega} \times \mathbb{R}), \qquad a_s \geq 0, \qquad a_1(x,0) = a_2(x,0). \tag{1.4}$$

The result of [4] does not hold because the key Lemma 2.1 in [4] applied in its proof is incorrect.

In this paper, we consider a general strong elliptic equation

$$-\sum_{i,j=1}^{n} c_{ij}\partial_{ij}u + a(x,u) = 0, \tag{1.5}$$

where c_{ij} are constants and $\sum_{i,j=1}^{n} c_{ij}\xi_i\xi_j \geq c_0 > 0$ for any $(\xi_1,\xi_2,\dots,\xi_n) \in \mathbb{R}^n$. The following two theorems are our main results.

Theorem 1.2. *Suppose that $a_1(x,s)$ and $a_2(x,s)$ satisfy the conditions*

$$a_1, a_2, a_{1s}, a_{2s} \in L^\infty(\bar{\Omega}\times\mathbb{R}) \cap C^{0,\alpha}(\Omega\times R), \quad a_{1s}, a_{2s} \geq 0. \tag{1.6}$$

If $\Lambda_{a_1} = \Lambda_{a_2}$, then $a_1(x,s) = a_2(x,s)$ on $\Omega\times\mathbb{R}$ if and only if there exists a constant θ_0 such that $u_1(x,\theta_0) = u_2(x,\theta_0)$, where $u_1(x,\theta_0)$ and $u_2(x,\theta_0)$ are both a solution of (1.1) and (1.2) with boundary data θ_0 and

$$\Lambda_a = \frac{\partial u}{\partial \mu}\bigg|_{\partial\Omega} = \sum_{i,j=1}^{n} c_{ij}\frac{\partial u}{\partial x_i}\cos(n,x_i)\bigg|_{\partial\Omega}. \tag{1.7}$$

The following theorem is a consequence of Theorem 1.2 and it improves the result in [3].

Theorem 1.3. *Suppose that conditions (1.6) are satisfied. If $\Lambda_{a_1} = \Lambda_{a_2}$ for $a_1, a_2 \in E = \{a(x,s) \in C^1(\Omega\times\mathbb{R})$, there exists an $s \in \mathbb{R}$ such that $a(x,s) = 0$ for all $x\}$, then $a_1(x,s) = a_2(x,s)$ on $\Omega\times\mathbb{R}$.*

Remark 1.4. In our result, we obtain a necessary and sufficient condition for the uniqueness of a. Moreover, the condition in Theorem 1.2 is weaker than that in [3].

Remark 1.5. It is significant to consider a general elliptic equation (1.5) although the equation can be transferred to a Laplace equation (1.1) through some transform. The reason is that, to determine the term a, we rely on a Dirichlet → Neumann mapping (defined in Section 2) totally, which may be defined for the general elliptic equation, but the transferred version may or may not be defined for the resulting Laplace equation.

2. The global uniqueness of the inverse problem.

Let Ω be a bounded domain in $\mathbb{R}^n$ with $C^{2,\alpha}$-boundary $\partial\Omega$.

First we state an existence result.

Lemma 2.1. *Suppose that $a(x,s), a_s(x,s) \in L^\infty$ and $a_s(x,s) \geq 0$. There exists a unique solution, $u \in W^{2,p}$, of the following Dirichlet problem:*

$$\begin{gathered} -\sum_{i,j=1}^{n} c_{ij}\partial_{ij}u + a(x,u) = 0, \quad x \in \Omega, \\ u|_{\partial\Omega} = g \in W^{2-1/p,p}(\partial\Omega). \end{gathered} \tag{2.1}$$

Proof. We first consider (2.1), with $\phi(x) \in C^{2,\alpha}(\bar{\Omega})$ as the boundary condition. The existence of a solution u of the problem is well known (cf. [1]). Then we take a sequence of functions: $\phi_n \in C^{2,\alpha}(\bar{\Omega})$ in such a way that $\phi_n \to \phi$ in $W^{2,p}$ and $\phi|_{\Omega} = g \in W^{2-1/p,p}(\partial\Omega)$. For each boundary term ϕ_n, there exists a solution $u_n \in C^{2,\alpha}(\bar{\Omega})$. By establishing a priori estimates and applying embedding theorem and maximum principle, we can show that u_n is a Cauchy sequence. Therefore, a subsequence of u_n will converge to a function u in $W^{2,p}$ and it can be shown that this limit, u, is the unique solution of (2.1). □

Given $g \in W^{2-1/p,p}(\partial\Omega)$, with the corresponding solution from Lemma 2.1, we define the Dirichlet → Neumann mapping $(W^{2-1/p,p}(\partial\Omega) \to W^{1-1/p,p}(\partial\Omega))$:

$$\Lambda_a : g \longrightarrow \frac{\partial u}{\partial \mu}\bigg|_{\partial\Omega} = \sum_{i,j=1}^{n} c_{ij}\frac{\partial u}{\partial x_i}\cos(n, x_i)\bigg|_{\partial\Omega}. \tag{2.2}$$

Following the notations in [3], for each $g \in W^{2-1/p,p}(\partial\Omega)$, we denote

$$a^*(x,g) = \frac{\partial a}{\partial u}(x, u(x,g)). \tag{2.3}$$

For

$$-\sum_{ij=1}^{n} c_{ij}\partial_{ij}v + a^*(x,g)v = 0, \tag{2.4}$$

we denote the Dirichlet → Neumann mapping as $\Lambda_{a^*(x,g)}$.

Lemma 2.2. *Suppose that a_1, a_2 satisfy conditions (1.6) and $\Lambda_{a_1} = \Lambda_{a_2}$. Then, for each $g \in W^{2-1/p,p}(\partial\Omega)$,*

$$\Lambda_{a_1^*(x,g)} = \Lambda_{a_2^*(x,g)}. \tag{2.5}$$

Proof. By definition,

$$\begin{aligned}\Lambda_a(g+\tau g*) &= \frac{\partial u(x, g+\tau g*)}{\partial \mu}\bigg|_{\partial\Omega},\\ \Lambda_a(g) &= \frac{\partial u(x,g)}{\partial \mu}\bigg|_{\partial\Omega}.\end{aligned} \tag{2.6}$$

For $g* \in W^{2-1/p,p}(\partial\Omega)$,

$$\frac{\Lambda_a(g+\tau g*) - \Lambda_a(g)}{\tau} = \frac{\partial}{\partial \mu}\frac{u(x,g+\tau g*) - u(x,g)}{\tau}\bigg|_{\partial\Omega}. \tag{2.7}$$

Since $u(x,g+\tau g*)$ and $u(x,g)$ are, respectively, solutions of the Dirichlet problems

$$\begin{gathered}
-\sum_{i,j=1}^{n} c_{ij}\partial_{ij}u + a(x,u(x,g+\tau g*)) = 0, \quad x \in \Omega, \\
u(x,g+\tau g*)\,|_{\partial\Omega} = g+\tau g* \in W^{2-1/p,p}(\partial\Omega); \\
-\sum_{i,j=1}^{n} c_{ij}\partial_{ij}u + a(x,u(x,g)) = 0, \quad x \in \Omega, \\
u(x,g)|_{\partial\Omega} = g \in W^{2-1/p,p}(\partial\Omega),
\end{gathered} \tag{2.8}$$

the difference $v(\tau) = (u(x,g+\tau g*) - u(x,g))/\tau$ satisfies the equation

$$\begin{aligned}
&-\sum_{i,j=1}^{n} c_{ij}\partial_{ij}v + v(\tau)\frac{\partial a}{\partial s}(x,u(x,g)) \\
&\quad = -v(\tau)\int_0^1 \left(\frac{\partial a}{\partial s}(x,\sigma u(x,g+\tau g*)-(1-\sigma)u(x,g)) - \frac{\partial a}{\partial s}(x,u(x,g))\right)d\sigma.
\end{aligned} \tag{2.9}$$

The maximum principle implies that

$$||v(\tau)||_{L^\infty(\Omega)} \le \max_{x\in\partial\Omega} |g*(x)| \quad \forall \tau \in \mathbb{R}. \tag{2.10}$$

Applying the L^p-estimate theorem for the solution of elliptic equation, we then obtain that

$$||u(x,g+\tau g*) - u(x,g)||_{W^{2,p}(\Omega)} \le c|\tau|\,\|g*\|_{W^{2-1/p,p}(\partial\Omega)} \longrightarrow 0 \quad \text{as } \tau \longrightarrow 0. \tag{2.11}$$

Embedding theorem shows that

$$||u(x,g+\tau g*) - u(x,g)||_{C(\Omega)} \longrightarrow 0 \quad \text{as } \tau \longrightarrow 0. \tag{2.12}$$

From the assumption that $a_s \in L^\infty(\bar{\Omega}\times\mathbb{R}) \cap C^{0,\alpha}(\Omega\times\mathbb{R})$ and (2.9), we see that

$$\left\|-\sum_{i,j=1}^{n} c_{ij}\partial_{ij}v + v(\tau)\frac{\partial a}{\partial s}(x,u(x,g))\right\|_{L^P(\Omega)} \longrightarrow 0 \quad \text{as } \tau \longrightarrow 0. \tag{2.13}$$

Now we can show that $v(\tau) \to v(0)$ in $W^{2,P}(\Omega)$. In fact,

$$\begin{aligned}
&\sum_{i,j=1}^{n} c_{ij}\partial_{ij}(v(\tau)-v(0)) \\
&\quad = (v(\tau)-v(0))\frac{\partial a}{\partial s}(x,u(x,g)) - \frac{\partial a}{\partial s}(x,u(x,g)) \\
&\qquad - v(\tau)\int_0^1 \left(\frac{\partial a}{\partial s}(x,\sigma u(x,g+\tau g*) - (1-\sigma)u(x,g))\right)d\sigma, \\
&\qquad\qquad (v(\tau)-v(0))\,|_{\partial\Omega} = 0.
\end{aligned} \tag{2.14}$$

Therefore,

$$\begin{aligned}||v(\tau)-v(0)|| \\ \le C\|g*\|_{W^{2-1/p,p}(\partial\Omega)} \max_{x\in\Omega,\sigma\in[0,1]} \bigg| \frac{\partial a}{\partial s}(x,\sigma u(x,g+\tau g*) \\ -(1-\sigma)u(x,g)) - \frac{\partial a}{\partial s}(x,u(x,g))\bigg|.\end{aligned} \tag{2.15}$$

The fact that $a_s \in C^{0,\alpha}(\Omega\times\mathbb{R})$ implies that $v(\tau)\to v(0)$ in $W^{2,P}(\Omega)$, that is,

$$\frac{u(x,g+\tau g*)-u(x,g)}{\tau} \longrightarrow v(0) \quad \text{in } W^{2,P}(\Omega). \tag{2.16}$$

Applying the trace theorem, we obtain that

$$\frac{\partial v(\tau)}{\partial\mu}\bigg|_{\partial\Omega} \longrightarrow \frac{\partial v(0)}{\partial\mu}\bigg|_{\partial\Omega} = \Lambda_{a^*(x,g)}g* \tag{2.17}$$

or

$$\lim_{\tau\to 0}\frac{\Lambda_{a(x,g+\tau g*)}-\Lambda_{a(x,g)}}{\tau} = \Lambda_{a^*(x,g)}. \tag{2.18}$$

The assumption that $\Lambda_{a_1}=\Lambda_{a_2}$ implies (2.5). □

Lemma 2.3 [2]. *Consider a linear equation of order m with constant coefficients*

$$(P_j(-i\partial)+c_j)u_j = 0. \tag{2.19}$$

Let Σ_0 be a nonempty open set in $\mathbb{R}^n$. Suppose that, for any $\xi(0)\in\Sigma_0$ and any constant R, there exists a solution $\xi(j)$ of the following algebraic equation:

$$\xi(1)+\xi(2)=\xi(0), \quad P_j(\xi(j))=0, \ |\xi(j)|>R. \tag{2.20}$$

Also suppose that there exists a constant C such that, for all $\zeta\in\mathbb{R}^n$,

$$\frac{1}{|\xi(j)|} \le C\tilde{P}_j(\zeta+\xi(j)), \tag{2.21}$$

where $\tilde{P}(\zeta) = (\sum_{|\alpha|\le m}|\partial_\zeta^\alpha P(\zeta)|^2)^{1/2}$. If $f\in L^1(\Omega)$ and for all L^2 solution, u_j, it holds that

$$\int_\Omega f u_1 u_2\,dx = 0, \tag{2.22}$$

then $f=0$.

In our case, we take

$$P(\partial u) = -\sum_{i,j=1}^n a_{ij}\partial_{ij}u + a(x)u = 0, \tag{2.23}$$

where $\sum_{i,j=1}^{n} c_{ij}\xi_i\xi_j \geq c_0|\xi|$ for any $\xi \in \mathbb{R}^n$. It can be shown algebraically that, for the differential operator defined in (2.23), all the conditions in Lemma 2.3 are satisfied.

Now, we apply the result of Lemma 2.3 to prove the following lemma.

Lemma 2.4. *Under the assumptions of Lemma 2.2, for any $g \in W^{2-1/p,p}(\partial\Omega)$ and any $x \in \Omega$, it holds that*

$$a_1^*(x,g) = a_2^*(x,g). \tag{2.24}$$

Proof. From Lemma 2.2, for any $g^* \in W^{2-1/p,p}(\partial\Omega)$,

$$\Lambda_{a_1^*(x,g)} g^* = \Lambda_{a_2^*(x,g)} g^*. \tag{2.25}$$

That is, if $v_1(x,g*)$ and $v_2(x,g*)$ satisfy, respectively, the equations

$$-\sum_{ij=1}^{n} c_{ij}\partial_{ij}v_1(x,g*) + a_1^*(x,g)v_1(x,g*) = 0, \tag{2.26}$$

$$v_1(x,g*)\big|_{\partial\Omega} = g*; \tag{2.27}$$

$$-\sum_{ij=1}^{n} c_{ij}\partial_{ij}v_2(x,g*) + a_2^*(x,g)v_2(x,g*) = 0, \tag{2.28}$$

$$v_2(x,g*)\big|_{\partial\Omega} = g*, \tag{2.29}$$

then

$$\frac{\partial v_1(x,g*)}{\partial \mu}\bigg|_{\partial\Omega} = \frac{\partial v_2(x,g*)}{\partial \mu}\bigg|_{\partial\Omega}. \tag{2.30}$$

We can easily prove the conclusion of the lemma by multiplying (2.26) by v_2 and (2.28) by v_1, integrating the difference of the two equations over Ω, and applying Lemma 2.3. □

Lemma 2.5. *If there is a constant θ_0 such that $u_1(x,\theta_0) = u_2(x,\theta_0)$, then $\Lambda_{a_1} = \Lambda_{a_2}$ implies that $a_1(x,u_1(x,\theta_0)) = a_2(x,u_2(x,\theta_0))$.*

Proof. Applying Green's formula, we obtain, for any $v \in C^0(\bar{\Omega}) \cap C^2(\Omega)$,

$$\begin{aligned}
&\int_\Omega \sum_{i,j=1}^{n} a_{ij}\partial_i v\partial_j u_1\,dx - \int_{\partial\Omega}\sum_{i,j=1}^{n} a_{ij}\partial_j u_1 v\cos(n,x_i)\,ds + \int_\Omega a_1(x,u_1)v\,dx \\
&\quad = \int_\Omega \sum_{i,j=1}^{n} a_{ij}\partial_i v\partial_j u_1\,dx - \int_{\partial\Omega} v\frac{\partial u_1}{\partial\mu}ds + \int_\Omega a_1(x,u_1)v\,dx \\
&\quad = 0.
\end{aligned} \tag{2.31}$$

Similarly, for $u_2(x,\theta_0)$, we have

$$\int_\Omega \sum_{i,j=1}^n a_{ij}\partial_i v \partial_j u_2 dx - \int_{\partial\Omega} v \frac{\partial u_2}{\partial \mu} ds + \int_\Omega a_2(x,u_2) v\, dx = 0. \tag{2.32}$$

Therefore,

$$\int_\Omega [a_1(x,u_1) - a_2(x,u_2)] v\, dx = 0, \tag{2.33}$$

which then implies that $a_1(x,u_1(x,\theta_0)) = a_2(x,u_2(x,\theta_0))$. □

Lemma 2.6. *Suppose that* $\Lambda_{a_1} = \Lambda_{a_2}$. *There exists a number* $\theta^* > 0$ *such that, when* $|\theta - \theta_0| < \theta^*$,

$$u_1(x,\theta) = u_2(x,\theta). \tag{2.34}$$

Proof. Let $v = u_2(x,\theta) - u_1(x,\theta)$. Then v satisfies equations

$$\begin{gathered} -\sum_{i,j=1}^n c_{ij}\partial_{ij} v + v \int_0^1 \frac{\partial a}{\partial s}(x, \sigma u_2 + (1-\sigma) u_1) = a_1(x,u_1) - a_2(x,u_2), \quad x \in \Omega, \\ v|_{\partial\Omega} = 0. \end{gathered} \tag{2.35}$$

It results from the maximum principle that

$$\|v\|_{L^\infty(\Omega)} \le C \| a_1(x,u_1) - a_2(x,u_2) \|_{L^p(\Omega)}. \tag{2.36}$$

Since $(\partial a_1/\partial s)(x,u_1) = (\partial a_2/\partial s)(x,u_2)$,

$$\left| \frac{\partial a_1}{\partial s}(x,u_1) - \frac{\partial a_2}{\partial s}(x,u_1) \right| = \left| \frac{\partial a_2}{\partial s}(x,u_2) - \frac{\partial a_2}{\partial s}(x,u_1) \right| \le C |u_1 - u_2|^\alpha. \tag{2.37}$$

From Lemma 2.4, we know that, for $\theta > \theta_0$,

$$\begin{aligned} &\| a_1(x,u_1) - a_2(x,u_1) \| \\ &\quad = \left| \int_{\theta_0}^\theta \left(\frac{\partial a_1}{\partial s}(x,u_1(x,\tau)) - \frac{\partial a_2}{\partial s}(x,u_1(x,\tau)) \right) \frac{\partial u_1}{\partial \tau} d\tau \right| \\ &\quad \le C|\theta - \theta_0| \sup_{\theta_0 \le \tau \le \theta, x \in \Omega} |u_1(x,\tau) - u_2(x,\tau)|^\alpha. \end{aligned} \tag{2.38}$$

Substituting it in (2.36) yields

$$\| u_1(x,\theta) - u_2(x,\theta) \|_{L^\infty(\Omega)} \le C|\theta - \theta_0| \sup_{\theta_0 \le \tau \le \theta, x \in \Omega} |u_1(x,\tau) - u_2(x,\tau)|^\alpha. \tag{2.39}$$

Therefore, there exists θ^* such that, when $|\theta - \theta_0| < \theta^*$,

$$u_1(x,\theta) = u_2(x,\theta). \tag{2.40}$$

□

Lemma 2.7. *Assume that a_1, a_2 satisfy all the conditions in Lemma 2.2 and that $\Lambda_{a_1} = \Lambda_{a_2}$. Then $u_1(x,\theta) = u_2(x,\theta)$ for all $\theta \in \mathbb{R}$.*

Proof. Again, let $v = u_2(x,\theta) - u_1(x,\theta)$. From the proof of Lemma 2.6, we obtain that

$$\left| \frac{\partial a_1}{\partial s}(x,u_1) - \frac{\partial a_2}{\partial s}(x,u_1) \right| \le C|u_1 - u_2|^{\alpha}. \tag{2.41}$$

Thus,

$$\frac{\partial a_1}{\partial s}(x,u_1(x,\theta)) - \frac{\partial a_2}{\partial s}(x,u_1(x,\theta)) = 0 \quad \forall |\theta - \theta_0| \le \theta^*. \tag{2.42}$$

Then we have

$$\begin{aligned}
&||a_1(x,u_1) - a_2(x,u_1)|| \\
&\quad = \left| \int_{\theta_0}^{\theta} \left(\frac{\partial a_1}{\partial s}(x,u_1(x,\tau)) - \frac{\partial a_2}{\partial s}(x,u_1(x,\tau)) \right) \frac{\partial u_1}{\partial \tau} d\tau \right| \\
&\quad = \left| \int_{\theta_0+\theta^*}^{\theta} \left(\frac{\partial a_1}{\partial s}(x,u_1(x,\tau)) - \frac{\partial a_2}{\partial s}(x,u_1(x,\tau)) \right) \frac{\partial u_1}{\partial \tau} d\tau \right| \\
&\quad \le C|\theta - \theta_0 - \theta^*| \sup_{\theta_0+\theta^* \le \tau \le \theta, x \in \Omega} |u_1(x,\tau) - u_2(x,\tau)|^{\alpha}.
\end{aligned} \tag{2.43}$$

Therefore,

$$\begin{aligned}
&\sup_{x\in\Omega} |a_1(x,u_1) - a_2(x,u_1)| \\
&\quad \le C|\theta - \theta_0 - \theta^*| \sup_{\theta_0+\theta^* \le \tau \le \theta, x \in \Omega} |u_1(x,\tau) - u_2(x,\tau)|^{\alpha},
\end{aligned} \tag{2.44}$$

which implies that there exists $h_1 > 0$ such that, when $\theta_0 + \theta^* < \theta \le \theta_0 + \theta^* + h_1$, $a_1(x,u_1) - a_2(x,u_1) = 0$. Similarly, there exists $h_2 > 0$ such that, when $\theta_0 + \theta^* - h_2 < \theta \le \theta_0 + \theta^*$, $a_1(x,u_1) - a_2(x,u_1) = 0$. Note that

$$||u_1(x,\theta) = u_2(x,\theta)||_{L^{\infty}(\Omega)} \le C||a_1(x,u_1) - a_2(x,u_2)||_{L^p(\Omega)}. \tag{2.45}$$

Therefore, there exists a common h such that, when $|\theta - \theta_0 - \theta^*| < h$,

$$u_1(x,\theta) = u_2(x,\theta). \tag{2.46}$$

Repeating the above process, we can extend the interval each time by the length of h. Eventually, we have $u_1(x,\theta) = u_2(x,\theta)$ for all $\theta \in \mathbb{R}$. □

Now we state and prove the first main result of this paper.

Theorem 2.8. *If a_1, a_2 satisfy all the conditions in Lemma 2.2 and $\Lambda_{a_1} = \Lambda_{a_2}$, then $a_1(x,s) = a_2(x,s)$ if and only if there exists θ_0 such that $u_1(x,\theta_0) = u_2(x,\theta_0)$.*

Proof. Applying Lemmas 2.4 and 2.7, we have, for all $v \in C^0(\bar{\Omega}) \cap C^2(\Omega)$,

$$\int_\Omega [a_1(x, u_1(x,\theta)) - a_2(x, u_1(x,\theta))] v \, dx = 0. \tag{2.47}$$

Therefore, for all $\theta \in \mathbb{R}$,

$$a_1(x, u_1(x,\theta)) = a_2(x, u_1(x,\theta)). \tag{2.48}$$

It can be shown that

$$\lim_{\theta \to \pm\infty} u_1(x,\theta) = \pm\infty. \tag{2.49}$$

Since $u_1(x,\theta)$ depends on θ continuously, when θ changes from $-\infty$ to ∞, $u_1(x,\theta)$ changes from $-\infty$ to ∞. The result of this theorem then follows. □

The result in [3] is a special case of Theorem 2.8. We put it as the following corollary.

Corollary 2.9. *Suppose that a_1, a_2 satisfy all the conditions in Lemma 2.2 and that $\Lambda_{a_1} = \Lambda_{a_2}$. If $a_1(x,0) = a_2(x,0) = 0$, then $a_1(x,s) = a_2(x,s)$.*

Proof. Condition $a_2(x,0) = a_2(x,0) = 0$ implies that $u_1(x,0) = u_2(x,0) = 0$. According to Theorem 2.8, $a_1(x,s) = a_2(x,s)$. □

Next, we give another necessary and sufficient condition for the uniqueness of a.

Corollary 2.10. *Assume that a_1, a_2 satisfy all the conditions in Lemma 2.2. Then $a_1(x,s) = a_2(x,s)$, for all $s \in \mathbb{R}$ and $x \in \Omega$, if and only if there exists a θ_0 such that $a_1(x, u_1(x,\theta_0)) = a_2(x, u_2(x,\theta_0))$.*

Proof. Assume that u_1, u_2 satisfy, respectively, the problems

$$\begin{gathered} -\sum_{i,j=1}^n c_{ij}\partial_{ij}u_1 + a(x, u_1(x,\theta_0)) = 0, \\ u_1(x,\theta_0)\big|_{\partial\Omega} = \theta_0 \in W^{2-1/p,p}(\partial\Omega); \end{gathered} \tag{2.50}$$

$$\begin{gathered} -\sum_{i,j=1}^n c_{ij}\partial_{ij}u_2 + a(x, u_2(x,\theta_0)) = 0, \\ u_2(x,\theta_0)\big|_{\partial\Omega} = \theta_0 \in W^{2-1/p,p}(\partial\Omega). \end{gathered} \tag{2.51}$$

It is clear that, when $a_1(x, u_1(x,\theta_0)) = a_2(x, u_2(x,\theta_0))$, the difference $u_1 - u_2$ satisfies the problem

$$\begin{gathered} -\sum_{i,j=1}^n c_{ij}\partial_{ij}(u_1 - u_2) = 0, \quad x \in \Omega, \\ (u_1 - u_2)\big|_{\partial\Omega} = 0, \end{gathered} \tag{2.52}$$

which implies that $u_1(x,\theta_0) = u_2(x,\theta_0)$. Theorem 2.8 then assures that $a_1(x,s) = a_2(x,s)$. The inverse result follows clearly from Theorem 2.8. □

The following corollary can also be easily proven.

Corollary 2.11. *Suppose that a_1, a_2 satisfy all the conditions in Lemma 2.2 and there exists s_0 such that $a_1(x,s_0) = a_2(x,s_0) = 0$. If $\Lambda_{a_1} = \Lambda_{a_2}$, then $a_1(x,s) = a_2(x,s)$ for all $s \in \mathbb{R}$, $x \in \Omega$.*

Denote $E = \{a(x,t) \in C^1(\bar{\Omega} \times \mathbb{R}) :$ there exists an $s \in \mathbb{R}$ such that $a(x,s) = 0$ for all $x\}$. The following theorem improves the result in [3].

Theorem 2.12. *Let $a_1, a_2 \in E$. If a_1, a_2 satisfy all the conditions in Lemma 2.2 and $\Lambda_{a_1} = \Lambda_{a_2}$, then $a_1(x,s) = a_2(x,s)$ for all $s \in \mathbb{R}$, $x \in \Omega$.*

Proof. Suppose that $a_1(x,s_1) = 0$ and $a_2(x,s_2) = 0$. We will show that $\Lambda_{a_1} = \Lambda_{a_2}$ implies $s_1 = s_2$. Then the theorem follows from Corollary 2.11. In fact, since $u_1 = s_1$ satisfies

$$-\sum_{i,j=1}^{n} c_{ij}\partial_{ij}u_1 + a_1(x,s_1) = 0, \quad u_1|_{\partial\Omega} = s_1, \tag{2.53}$$

we have $\Lambda_{a_1}s_1 = \Lambda_{a_2}s_1 = 0$. That is, u_2 satisfies

$$-\sum_{i,j=1}^{n} c_{ij}\partial_{ij}u_2 + a_2(x,u_2) = 0, \quad u_2|_{\partial\Omega} = s_1. \tag{2.54}$$

Therefore, $u_1 - u_2$ satisfies

$$-\sum_{i,j=1}^{n} c_{ij}\partial_{ij}(u_1-u_2) + (u_1-u_2)\int_0^1 \frac{\partial a_2}{\partial s}(x,\sigma u_2 + (1-\sigma)s_2) = 0, \tag{2.55}$$

$$(u_2 - s_2)|_{\partial\Omega} = s_1 - s_2, \quad \frac{\partial(u_1-u_2)}{\partial\mu}\bigg|_{\partial\Omega} = 0. \tag{2.56}$$

Multiplying both sides of (2.55) by $(u_2 - s_2)$ and integrating over Ω yields $u_2 - s_2 = 0$. Therefore, $s_1 = s_2$. Corollary 2.11 and the fact that $a_2(x,s_1) = 0$ and $a_1(x,s_1) = 0$ assure that $a_1(x,s) = a_2(x,s)$ for all $s \in \mathbb{R}$, $x \in \Omega$. □

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