IJMMS 2003:7, 397–403
PII. S0161171203107090
http://ijmms.hindawi.com

HILBERT SERIES AND APPLICATIONS TO GRADED RINGS

SELMA ALTINOK

Received 23 July 2001 and in revised form 14 March 2002

This paper contains a number of practical remarks on Hilbert series that we expect to be useful in various contexts. We use the fractional Riemann-Roch formula of Fletcher and Reid to write out explicit formulas for the Hilbert series $P(t)$ in a number of cases of interest for singular surfaces (see Lemma 2.1) and 3-folds. If X is a $\mathbb{Q}$-Fano 3-fold and $S \in |-K_X|$ a $K3$ surface in its anticanonical system (or the general elephant of X), polarised with $D = \mathbb{O}_S(-K_X)$, we determine the relation between $P_X(t)$ and $P_{S,D}(t)$. We discuss the denominator $\prod(1-t^{a_i})$ of $P(t)$ and, in particular, the question of how to choose a reasonably small denominator. This idea has applications to finding $K3$ surfaces and Fano 3-folds whose corresponding graded rings have small codimension. Most of the information about the anticanonical ring of a Fano 3-fold or $K3$ surface is contained in its Hilbert series. We believe that, by using information on Hilbert series, the classification of $\mathbb{Q}$-Fano 3-folds is too close. Finding $K3$ surfaces are important because they occur as the general elephant of a $\mathbb{Q}$-Fano 3-fold.

2000 Mathematics Subject Classification: 14Q10, 14Q15, 32S25, 13A02.

1. Introduction. We work with graded rings $R = \bigoplus_{n\geq 0} R_n$ that are finitely generated over an algebraically closed field k of characteristic 0 and satisfy $R_0 = k$. The *Hilbert function* of R is the numerical function $P_n = \dim R_n$ for $n \geq 0$; the *Hilbert series* $P(t)$ or $P_R(t)$ of R is the formal power series defined by $P(t) = \sum P_n t^n$. It is elementary and well known that $P(t)$ is a rational function of t. In fact, if $x_1,\dots,x_d$ are homogeneous elements of weight $\operatorname{wt} x_i = a_i$ generating R (or more generally, generating a subring over which R is finite), then $\prod(1-t^{a_i})P(t) = Q(t)$ is a polynomial.

2. Fractional Riemann-Roch formula

2.1. Surfaces with Du Val singularities. We use the definitions and notation of Reid [9] for singularities. If S is a projective surface with Du Val singularities and D a Weil divisor on S, then some multiple rD is Cartier, and there is a formula [9, Theorem 9.1]

$$\chi(S,\mathbb{O}_S(D)) = \chi(\mathbb{O}_S) + \frac{1}{2}(D^2 - DK_S) + \sum_P c_P(D), \tag{2.1}$$

where $c_P(D) \in \mathbb{Q}$ is a fractional contribution due to the singularity of S and

$\mathcal{O}_S(D)$ at P. Here, $DK_S \in \mathbb{Z}$ is the intersection number with the canonical class, and $D^2 \in \mathbb{Q}$ the self-intersection of the $\mathbb{Q}$-Cartier divisor D. Moreover, $c_P(D)$ can be written as a sum

$$c_P(D) = -\sum_{(r,a)\in\mathcal{B}} \frac{a(r-a)}{2r} \tag{2.2}$$

over a *basket* $\mathcal{B} = \{(r,a)\}$ with $0 < a < r$ and each a is coprime to r. Here, the basket appears simply as a list of combinatorial data for computing the right-hand side of (2.2); Reid [9, Section 9] interprets (2.1) and (2.2) as the singularity of S, and $\mathcal{O}_S(D)$ at P has the same effect on $\chi(S,\mathcal{O}_S(D))$ as a basket $\mathcal{B}$ of virtual cyclic quotient singularities of type $(1/r)(1,-1)$, at which $\mathcal{O}_S(D)$ is locally isomorphic to the eigensheaf of ε^a (see [9, Chapter III] for definition).

It follows from this interpretation and the proof of [9, Theorem 9.1] that for $n \in \mathbb{Z}$,

$$c_P(nD) = -\sum_{(r,a)\in\mathcal{B}} \frac{\overline{na}(r-\overline{na})}{2r}, \tag{2.3}$$

where, for each $(r,a) \in \mathcal{B}$, the bar stands for the smallest positive residue modulo r.

Lemma 2.1. *Let S be a surface with Du Val singularities and D a Weil divisor. We assume that $H^i(S,\mathcal{O}_S(nD)) = 0$ for all $i > 0$ and for all $n \geq 1$. Then, the graded ring $R(S,D) = \bigoplus_{n\geq 0} H^0(S,\mathcal{O}_S(nD))$ has Hilbert series*

$$\begin{aligned} P_S(t) = \frac{1+(\chi-1)t}{1-t} + \frac{t+t^2}{2(1-t)^3}D^2 - \frac{t}{2(1-t)^2}DK_S \\ - \sum_{(r,a)\in\mathcal{B}} \frac{\sum_{n=1}^{r-1}\overline{an}(r-\overline{an})t^n}{2r(1-t^r)}. \end{aligned} \tag{2.4}$$

Here, $\chi = \chi(\mathcal{O}_S)$.

Proof. The first three terms of (2.4) expand out as

$$\begin{aligned} 1+\chi t+\chi t^2+\cdots, \quad \frac{(t+2^2t^2+3^2t^3+\cdots)D^2}{2}, \\ -\frac{(t+2t^2+3t^2+\cdots)DK_S}{2}, \end{aligned} \tag{2.5}$$

respectively, corresponding to the sum over nD of first three terms of (2.1). For each element of the basket, the denominator $1-t^r$ has the effect of repeating the contribution $-\overline{na}(r-\overline{na})/2r$ from (2.3) periodically over the intervals $[0,r],[r,2r],\ldots$ (zero at the endpoints), giving the last term of (2.1). ☐

2.2. 3-folds with canonical singularities. Let X be a projective 3-fold with canonical singularities and A a Weil divisor. To use the formulas of [9, Section 10], assume that, at every singular point $P \in X$, we have $\mathcal{O}_X(A) \cong \mathcal{O}_X(lK_X)$ for some l (possibly depending on P). Then, [9, Theorem 10.2] states that

$$\chi(X,A) = \chi(\mathcal{O}_X) + \frac{1}{6}A^3 - \frac{1}{4}A^2K_X + \frac{1}{12}A(K_X^2 + c_2) + \sum c_Q(A), \tag{2.6}$$

where $c_Q(A)$ is a sum $\sum_{\mathcal{B}} c(r,a,l)$ taken over a basket $\mathcal{B} = \{(r,a,l)\}$, with a and l coprime to r, and the contributions are

$$c(r,a,l) = -\frac{r^2-1}{12r}l + \sum_{i=1}^{l-1} \frac{\overline{ai}(r-\overline{ai})}{2r} \tag{2.7}$$

$$= \frac{r^2-1}{12r}(r-l) - \sum_{i=l}^{r-1} \frac{\overline{ai}(r-\overline{ai})}{2r}. \tag{2.8}$$

The interpretation here is that $c(r,a,l)$ is the contribution from a singularity of type $(1/r)(a,1,-1)$ at which A is locally the ε^{al} eigensheaf.

We can write out the other terms in the Hilbert series by analogy with Lemma 2.1 (see the following section). The contribution made by each element $(r,a,l) \in \mathcal{B}$ to the Hilbert series is thus

$$\sum_{n=1}^{\infty} c(r,a,nl)t^n = \frac{1}{1-t^r} \sum_{n=1}^{r-1} c(r,a,nl)t^n. \tag{2.9}$$

Remark 2.2. The two formulas (2.7) and (2.8) are equal because

$$\sum_{i=1}^{r-1} \overline{ai}(r-\overline{ai}) = \sum_{i=1}^{r-1} i(r-i) = \frac{1}{6}r(r^2-1); \tag{2.10}$$

moreover, they also hold for l not in $[0,r]$ (the expression only depends on l modulo r).

In dealing with a single divisor A, we may assume that l is coprime to r; but the proof of [9, Section 10] is valid for any l. Thus, we can use formulas (2.7) and (2.8) for the contribution $c(r,a,nl)$ for any n in calculating $\chi(nA)$.

3. Fano 3-folds and $K3$ surfaces

3.1. Fano 3-folds. Let X be a Fano 3-fold, that is, a 3-fold with canonical singularities and ample anticanonical class $A = -K_X$. Standard use of vanishing gives

$$P_n = h^0(X,nA) = \chi(\mathcal{O}(nA)), \quad \text{for } n \geq 0. \tag{3.1}$$

Now, $\chi(\mathcal{O}_X) = 1$ and the basket of X is $\mathcal{B} = \{(r,a,-1)\}$, and by [9, Corollary 10.3], we have

$$\frac{1}{12}Ac_2 = -\frac{1}{12}K_Xc_2 = 2 - \sum_{\mathcal{B}} \frac{r^2-1}{12r}. \tag{3.2}$$

Using all this, formulas (2.6) and (2.7) specialise to give

$$P_n = 2n+1+\frac{1}{12}n(n+1)(2n+1)A^3+\sum_{\mathcal{B}}\Big(-\frac{r^2-1}{12r}n+c(r,a,-n)\Big). \tag{3.3}$$

Now, consider $P_n - P_{n-1}$. To handle the second term, we use

$$n(n+1)(2n+1)-(n-1)(n)(2n-1) = 6n^2. \tag{3.4}$$

By (2.8), the bracketed expression inside the sum over $\mathcal{B}$ equals

$$-\sum_{i=-n}^{r-1}\frac{\overline{ai}(r-\overline{ai})}{2r}, \tag{3.5}$$

and the difference from n to $n-1$ is just one term of the sum. We obtain

$$P_n - P_{n-1} = 2+\frac{1}{2}n^2A^3-\sum_{(r,a,-1)\in\mathcal{B}}\frac{\overline{an}(r-\overline{an})}{2r}. \tag{3.6}$$

Since $P_n - P_{n-1}$ is the coefficient of t^n in $(1-t)P_X(t)$, arguing as in Lemma 2.1 gives the following result.

Corollary 3.1. *The Hilbert series of a Fano* 3*-fold* X *is*

$$P_X(t) = \frac{1+t}{(1-t)^2}+\frac{t(1+t)}{2(1-t)^4}A^3-\sum_{(r,a,-1)\in\mathcal{B}}\frac{\sum_{n=1}^{r-1}\overline{an}(r-\overline{an})t^n}{2r(1-t)(1-t^r)}. \tag{3.7}$$

3.2. $K3$ **surfaces.** For a $K3$ surface S with Du Val singularities and a Weil divisor D, Lemma 2.1 specialises to give the Hilbert series of $P_n = h^0(S,nD)$ in the form

$$P_S(t) = \frac{1+t}{1-t}+\frac{t(1+t)}{2(1-t)^3}D^2-\sum_{(r,a)\in\mathcal{B}}\frac{\sum_{n=1}^{r-1}\overline{an}(r-\overline{an})t^n}{2r(1-t^r)}. \tag{3.8}$$

Corollary 3.2. *If* X *is a Fano* 3*-fold polarised by* $A=-K_X$ *and* $S\in|-K_X|$ *a* $K3$ *surface polarised by* $D=A_{|S}$*, then*

$$P_S(t) = (1-t)P_X(t). \tag{3.9}$$

This result follows, of course, from the restriction exact sequence

$$0\longrightarrow \mathbb{O}_X((n-1)A)\longrightarrow \mathbb{O}_X(nA)\longrightarrow \mathbb{O}_S(nD)\longrightarrow 0. \tag{3.10}$$

The point, however, is that the corollary gives a formula for the Hilbert series $P_X(t)$ of the Fano 3-fold in terms of simpler data for a $K3$ surface. This formula is valid even if there is no $K3$ surface $S\in|-K_X|$, for example, if $P_1(X)=0$ so that $|-K_X|=\varnothing$. Compare Corti, Pukhlikov, and Reid [5, Remark 7.2.3].

4. Applications. In [2], we studied polarised $K3$ surfaces S in terms of their graded rings $R(S,D)$. We also studied Fano 3-folds X in a similar way. The first step in our strategy consisted of finding suitable weights a_0, a_1, and a_2 so that

$$(1-t^{a_0})(1-t^{a_1})(1-t^{a_2})P(t) \tag{4.1}$$

is a polynomial with positive coefficients and a_0, a_1, and a_2 are “fairly small.” This is a combinatorial analogue of finding a polynomial subring $k[x_0,x_1,x_2] \subset R(S,D)$ over which $R(S,D)$ is a finite-free module of “fairly small rank.” If $\mathcal{B} = \{(r,a)\}$ is the basket of S, then each $(1-t^r)$ appears in the denominator so that a first necessary condition for this is that each r divides some a_i.

We have the obvious bound $\sum n \le 19$ for the number and types of Du Val singularities A_n, D_n, and E_n on a $K3$ surface S. This implies the bound $\sum(r-1) \le 19$ for the basket $\mathcal{B} = \{(r,a)\}$ on a $K3$ surface (see [6, Theorem III.9.20] and [7, Theorem II.8.21]). For a Fano 3-fold X, a similar bound on the basket $\{(r,a,-1)\}$ is provided by an argument of Kawamata [8] who first proves that $Ac_2 > 0$; then (3.2) implies

$$\sum\left(r-\frac{1}{r}\right) < 24. \tag{4.2}$$

Thus, there are only finitely many possibilities for the basket $\mathcal{B}$. An easy calculation shows that in either case, at most, 5 distinct values of r occur.

In [2], we develop a procedure based on these ideas to find all possible Hilbert series for $K3$ surfaces and Fano 3-folds with graded ring of given codimension. We give explicit lists [1] of codimension 3 and 4 cases (comparable to the lists of hypersurfaces and codimension 2 in Fletcher [7]) and, in most cases, settle the question of the existence of the varieties.

The *numerical data* of a $K3$ surface S with D is $P_1 = h^1(S,D)$, where $0 \le P_1 \le 3+\operatorname{codim} S$ and the basket $\mathcal{B} = \{(r,a)\}$. We can rewrite D^2 from the formula of P_1 in terms of P_1 and the basket $\mathcal{B}$,

$$D^2 = 2(P_1-2)+\sum\frac{a(r-a)}{r}. \tag{4.3}$$

We produced the lists of codimension 3 and 4 [1] by searching all possible $K3$ surfaces of given numerical data. We give an example below to show how this search is carried out. More details of applications are given in the Singapore paper [3].

Example 4.1. A typical example is

$$D^2 = -2+\frac{1}{2}+\frac{1}{2}+\frac{3}{4}+\frac{4}{5}. \tag{4.4}$$

That is, the numerical data are the basket $\mathcal{B} = \{(2,1),(2,1),(4,1),(5,1)\}$ and $P_1 = 1$ (see formula (4.3)). This is #39 in the codimension 3 list [1]. Since $P_1 = 1$, there is a generator of degree 1, say x. From formula (3.8),

$$\begin{aligned} P_S(t) &= \frac{1+t}{1-t} + \frac{t(1+t)}{2(1-t)^3}\frac{11}{20} - 2\times\frac{t}{2\cdot 2(1-t^2)} \\ &\quad - \frac{3t+4t^2+3t^3}{2\cdot 4(1-t^4)} - \frac{4t+6t^2+6t^3+4t^4}{2\cdot 5(1-t^5)}. \end{aligned} \tag{4.5}$$

To kill its denominators, we must have at least two more generators, say t_1 and v, whose degrees are divisible by 4 and 5, respectively. Therefore, the smallest choice of (a_1,a_2,a_3) is $(1,4,5)$. After simplifying, we obtain

$$(1-t)(1-t^4)(1-t^5)P_S(t) = t^{10}+t^8+t^7+2t^6+t^5+2t^4+t^3+t^2+1. \tag{4.6}$$

This looks like the Hilbert series of an Artinian ring with further generators y, z, t_2 of degrees 2, 3, 4 so that the possible candidates S is in the weighted projective space $\mathbb{P}(1,2,3,4,4,5)$ (see [2, 6, 7] for definition). To find the structure of its resolution, multiply (3.6) again by $(1-t^2)(1-t^3)(1-t^4)$,

$$\begin{aligned} &(1-t)(1-t^2)(1-t^3)(1-t^4)^2(1-t^5)P_S(t) \\ &\quad = 1-t^6-t^7-2t^8-t^9+t^{10}+2t^{11}+t^{12}+t^{13}-t^{19}. \end{aligned} \tag{4.7}$$

From here, we read off the shape of the resolution of $R(S,D)$ over $A = k[x,y,z,t_1,t_2,v]$, namely,

$$\begin{aligned} 0 &\longrightarrow A(-19) \xrightarrow{P^t} A(-13)\oplus A(-12)\oplus A(-11)\oplus A(-11)\oplus A(-10) \\ &\xrightarrow{M} A(-9)\oplus A(-8)\oplus A(-8)\oplus A(-7)\oplus A(-6) \xrightarrow{P} A \longrightarrow R \longrightarrow 0, \end{aligned} \tag{4.8}$$

where P is a 5×1 vector and M is a 5×5 skew-symmetric matrix. In other words, we expect 5 relations in degrees 6, 7, 8, 8, and 9, and 5 syzygies in degrees 10, 11, 11, 12, and 13. Note that $n = 1+2+3+4+4+5$ corresponds to the canonical class of $\mathbb{P}(1,2,3,4,4,5)$ and hence the canonical divisor K_S of S is $\mathbb{O}(19-n)$, which is trivial. The shape of the polynomial, together with the Buchsbaum-Eisenbud theorem on Gorenstein rings in codimension 3 (see [4]), gives us the equations of the relations as the Paffian of a 5×5 skew-symmetric matrix M with degrees

$$\begin{pmatrix} 0 & 2 & 2 & 3 & 4 \\ 2 & 0 & 3 & 4 & 5 \\ 2 & 3 & 0 & 4 & 5 \\ 3 & 4 & 4 & 0 & 6 \\ 4 & 5 & 5 & 6 & 0 \end{pmatrix}. \tag{4.9}$$

Acknowledgments. This work derives mainly from the Ph.D. thesis [2], University of Warwick. I would like to thank both Miles Reid and Alessio Corti for their help and encouragement at different times.

References

[1] S. Altınok, *Lists of $K3$ surfaces in codimension* 3 *and* 4, preprint, http://www.maths.warwick.ac.uk/~miles/doctors/Selma.

[2] ______, *Graded rings corresponding to polarised $K3$ surfaces and $\mathbb{Q}$-Fano 3-folds*, Ph.D. thesis, University of Warwick, Coventry, UK, 1998.

[3] S. Altınok, G. Brown, and M. Reid, *Fano 3-folds, $K3$ surfaces, and graded rings*, Singapore International Symposium in Topology and Geometry (NUS, 2001) (A. J. Berrick, M. C. Leung, and X. W. Xu, eds.), Contemp. Math., American Mathematical Society, Rhode Island, 2002, to appear.

[4] D. A. Buchsbaum and D. Eisenbud, *Algebra structures for finite free resolutions, and some structure theorems for ideals of codimension* 3, Amer. J. Math. **99** (1977), no. 3, 447–485.

[5] A. Corti, A. Pukhlikov, and M. Reid, *Fano 3-fold hypersurfaces*, Explicit Birational Geometry of 3-Folds (A. Corti and M. Reid, eds.), London Mathematical Society Lecture Note Series, vol. 281, Cambridge University Press, Cambridge, 2000, pp. 175–258.

[6] A. R. Iano-Fletcher, *Plurigenera of 3-folds and weighted hypersurfaces*, Ph.D. thesis, University of Warwick, Coventry, UK, 1988.

[7] ______, *Working with weighted complete intersections*, Explicit Birational Geometry of 3-Folds (A. Corti and M. Reid, eds.), London Mathematical Society Lecture Note Series, vol. 281, Cambridge University Press, Cambridge, 2000, pp. 101–173.

[8] Y. Kawamata, *Boundedness of $\mathbf{Q}$-Fano threefolds*, Proceedings of the International Conference on Algebra, Part 3 (Novosibirsk, 1989), Contemp. Math., vol. 131, American Mathematical Society, Rhode Island, 1992, pp. 439–445.

[9] M. Reid, *Young person's guide to canonical singularities*, Algebraic Geometry, Bowdoin, 1985 (Brunswick, Maine, 1985), Proc. Sympos. Pure Math., vol. 46, American Mathematical Society, Rhode Island, 1987, pp. 345–414.

Selma Altinok: Department of Mathematics, Arts and Science Faculty, Adnan Menderes University, Aydın 09010, Turkey
E-mail address: `saltinok43@hotmail.com`

Mathematical Problems in Engineering

Special Issue on
Modeling Experimental Nonlinear Dynamics and Chaotic Scenarios

Call for Papers

Thinking about nonlinearity in engineering areas, up to the 70s, was focused on intentionally built nonlinear parts in order to improve the operational characteristics of a device or system. Keying, saturation, hysteretic phenomena, and dead zones were added to existing devices increasing their behavior diversity and precision. In this context, an intrinsic nonlinearity was treated just as a linear approximation, around equilibrium points.

Inspired on the rediscovering of the richness of nonlinear and chaotic phenomena, engineers started using analytical tools from "Qualitative Theory of Differential Equations," allowing more precise analysis and synthesis, in order to produce new vital products and services. Bifurcation theory, dynamical systems and chaos started to be part of the mandatory set of tools for design engineers.

This proposed special edition of the *Mathematical Problems in Engineering* aims to provide a picture of the importance of the bifurcation theory, relating it with nonlinear and chaotic dynamics for natural and engineered systems. Ideas of how this dynamics can be captured through precisely tailored real and numerical experiments and understanding by the combination of specific tools that associate dynamical system theory and geometric tools in a very clever, sophisticated, and at the same time simple and unique analytical environment are the subject of this issue, allowing new methods to design high-precision devices and equipment.

Authors should follow the Mathematical Problems in Engineering manuscript format described at http://www.hindawi.com/journals/mpe/. Prospective authors should submit an electronic copy of their complete manuscript through the journal Manuscript Tracking System at http://mts.hindawi.com/ according to the following timetable:

Manuscript Due	February 1, 2009
First Round of Reviews	May 1, 2009
Publication Date	August 1, 2009

Guest Editors

José Roberto Castilho Piqueira, Telecommunication and Control Engineering Department, Polytechnic School, The University of São Paulo, 05508-970 São Paulo, Brazil; piqueira@lac.usp.br

Elbert E. Neher Macau, Laboratório Associado de Matemática Aplicada e Computação (LAC), Instituto Nacional de Pesquisas Espaciais (INPE), São Josè dos Campos, 12227-010 São Paulo, Brazil ; elbert@lac.inpe.br

Celso Grebogi, Department of Physics, King's College, University of Aberdeen, Aberdeen AB24 3UE, UK; grebogi@abdn.ac.uk

Hindawi Publishing Corporation
http://www.hindawi.com