

A BASIC INEQUALITY FOR SUBMANIFOLDS IN A COSYMPLECTIC SPACE FORM

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For submanifolds tangent to the structure vector field in cosymplectic space forms, we establish a basic inequality between the main intrinsic invariants of the submanifold, namely, its sectional curvature and scalar curvature on one side; and its main extrinsic invariant, namely, squared mean curvature on the other side. Some applications, including inequalities between the intrinsic invariant δ_M and the squared mean curvature, are given. The equality cases are also discussed.

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1. Introduction. To find simple relationships between the main extrinsic invariants and the main intrinsic invariants of a submanifold is one of the natural interests of the submanifold theory. Let M be an n -dimensional Riemannian manifold. For each point $p \in M$, let $(\inf K)(p) = \inf\{K(\pi) : \text{plane sections } \pi \subset T_p M\}$. Then, the well-defined intrinsic invariant δ_M of M introduced by Chen [4] is

$$\delta_M(p) = \tau(p) - (\inf K)(p), \quad (1.1)$$

where τ is the scalar curvature of M (see also [6]).

In [3], Chen established the following basic inequality involving the intrinsic invariant δ_M and the squared mean curvature for n -dimensional submanifolds M in a real space form $R(c)$ of constant sectional curvature c :

$$\delta_M \leq \frac{n^2(n-2)}{2(n-1)} \|H\|^2 + \frac{1}{2}(n+1)(n-2)c. \quad (1.2)$$

The above inequality is also true for anti-invariant submanifolds in complex space forms $\tilde{M}(4c)$ as remarked in [7]. In [5], he proved a general inequality for an arbitrary submanifold of a dimension greater than 2 in a complex space form. Applying this inequality, he showed that (1.2) is also valid for arbitrary submanifolds in the complex hyperbolic space $\text{CH}^m(4c)$. He also established the basic inequality for a submanifold in a complex projective space CP^m .

A submanifold normal to the structure vector field ξ of a contact manifold is anti-invariant. Thus, the C -totally real submanifolds in a Sasakian manifold are anti-invariant as they are normal to ξ . An inequality similar to (1.2) for C -totally

real submanifolds in a Sasakian space form $\tilde{M}(c)$ of constant φ -sectional curvature c is given in [8]. In [9], for submanifolds in a Sasakian space form $\tilde{M}(c)$ tangential to the structure vector field ξ , a basic inequality, along with some applications, is presented.

There is another interesting class of almost contact metric manifolds, namely, cosymplectic manifolds [10]. In this paper, submanifolds tangent to the structure vector field ξ in cosymplectic space forms are studied. Section 2 contains the necessary details of submanifolds and cosymplectic space forms for further use. In Section 3, for submanifolds tangent to the structure vector field ξ in cosymplectic space forms, we establish a basic inequality between the main intrinsic invariants, namely, its sectional curvature function K and its scalar curvature function τ of the submanifold on the one side, and its main extrinsic invariant, namely, its mean curvature function $\|H\|$ on the other side. In Section 4, we give some applications including inequalities between the intrinsic invariant δ_M and the extrinsic invariant $\|H\|$. We also discuss the equality cases.

2. Preliminaries. Let \tilde{M} be a $(2m + 1)$ -dimensional almost contact manifold [2] endowed with an almost contact structure (φ, ξ, η) , that is, φ is a $(1, 1)$ tensor field, ξ is a vector field, and η is 1-form such that

$$\varphi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1. \tag{2.1}$$

Then, $\varphi(\xi) = 0$ and $\eta \circ \varphi = 0$.

Let g be a compatible Riemannian metric with (φ, ξ, η) , that is, $g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y)$ or, equivalently, $g(X, \varphi Y) = -g(\varphi X, Y)$ and $g(X, \xi) = \eta(X)$ for all $X, Y \in T\tilde{M}$. Then, \tilde{M} becomes an almost contact metric manifold equipped with an almost contact metric structure (φ, ξ, η, g) . An almost contact metric manifold is *cosymplectic* [2] if $\tilde{\nabla}_X \varphi = 0$, where $\tilde{\nabla}$ is the Levi-Civita connection of the Riemannian metric g . From the formula $\tilde{\nabla}_X \varphi = 0$, it follows that $\tilde{\nabla}_X \xi = 0$.

A plane section σ in $T_p \tilde{M}$ of an almost contact metric manifold \tilde{M} is called a φ -section if $\sigma \perp \xi$ and $\varphi(\sigma) = \sigma$. The $(2m + 1)$ -dimensional almost contact manifold \tilde{M} is of the *constant φ -sectional curvature* if the sectional curvature $\tilde{K}(\sigma)$ does not depend on the choice of the φ -section σ of $T_p \tilde{M}$ and the choice of a point $p \in \tilde{M}$. A cosymplectic manifold \tilde{M} is of the constant φ -sectional curvature c if and only if its curvature tensor \tilde{R} is of the form [10]

$$\begin{aligned} 4\tilde{R}(X, Y, Z, W) = & c \{ g(X, W)g(Y, Z) - g(X, Z)g(Y, W) \\ & + g(X, \varphi W)g(Y, \varphi Z) - g(X, \varphi Z)g(Y, \varphi W) \\ & - 2g(X, \varphi Y)g(Z, \varphi W) \\ & - g(X, W)\eta(Y)\eta(Z) + g(X, Z)\eta(Y)\eta(W) \\ & - g(Y, Z)\eta(X)\eta(W) + g(Y, W)\eta(X)\eta(Z) \}. \end{aligned} \tag{2.2}$$

Let M be an $(n + 1)$ -dimensional submanifold of a manifold \tilde{M} equipped with a Riemannian metric g . The Gauss and Weingarten formulae are given, respectively, by

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X, Y), \quad \tilde{\nabla}_X N = -A_N X + \nabla_X^\perp N, \tag{2.3}$$

for all $X, Y \in TM$ and $N \in T^\perp M$, where $\tilde{\nabla}$, ∇ , and ∇^\perp , respectively, are the Riemannian, induced Riemannian, and induced normal connections in \tilde{M} , M , and the normal bundle $T^\perp M$ of M , respectively, and h is the second fundamental form related to the shape operator A by $g(h(X, Y), N) = g(A_N X, Y)$.

Let $\{e_1, \dots, e_{n+1}\}$ be an orthonormal basis of the tangent space $T_p M$. The mean curvature vector $H(p)$ at $p \in M$ is

$$H(p) = \frac{1}{n+1} \sum_{i=1}^{n+1} h(e_i, e_i). \tag{2.4}$$

The submanifold M is *totally geodesic* in \tilde{M} if $h = 0$ and *minimal* if $H = 0$. We put

$$h_{ij}^r = g(h(e_i, e_j), e_r), \quad \|h\|^2 = \sum_{i,j=1}^{n+1} g(h(e_i, e_j), h(e_i, e_j)), \tag{2.5}$$

where $\{e_{n+2}, \dots, e_{2m+1}\}$ is an orthonormal basis of $T_p^\perp M$ and $r = n + 2, \dots, 2m + 1$.

3. A basic inequality. Let M be a submanifold of an almost contact metric manifold. For $X \in TM$, let

$$\varphi X = PX + FX, \quad PX \in TM, \quad FX \in T^\perp M. \tag{3.1}$$

Thus, P is an endomorphism of the tangent bundle of M and satisfies

$$g(X, PY) = -g(PX, Y), \quad X, Y \in TM. \tag{3.2}$$

For a plane section $\pi \subset T_p M$ at a point $p \in M$,

$$\alpha(\pi) = g(e_1, Pe_2)^2, \quad \beta(\pi) = (\eta(e_1))^2 + (\eta(e_2))^2 \tag{3.3}$$

are real numbers in the closed unit interval $[0, 1]$, which are independent of the choice of the orthonormal basis $\{e_1, e_2\}$ of π .

We recall the following lemma from [3].

LEMMA 3.1. *If a_1, \dots, a_{n+1}, a are $n + 2$ ($n \geq 1$) real numbers such that*

$$\left(\sum_{i=1}^{n+1} a_i\right)^2 = n\left(\sum_{i=1}^{n+1} a_i^2 + a\right), \tag{3.4}$$

then $2a_1a_2 \geq a$, with equality holding if and only if $a_1 + a_2 = a_3 = \dots = a_{n+1}$.

Now, we prove the following theorem.

THEOREM 3.2. *Let M be an $(n + 1)$ -dimensional ($n \geq 2$) submanifold isometrically immersed in a $(2m + 1)$ -dimensional cosymplectic space form $\tilde{M}(c)$ such that the structure vector field ξ is tangent to M . Then, for each point $p \in M$ and each plane section $\pi \subset T_pM$, we have*

$$\begin{aligned} \tau - K(\pi) &\leq \frac{(n + 1)^2(n - 1)}{2n} \|H\|^2 \\ &+ \frac{c}{8} (3\|P\|^2 - 6\alpha(\pi) + 2\beta(\pi) + (n + 1)(n - 2)). \end{aligned} \tag{3.5}$$

The equality in (3.5) holds at $p \in M$ if and only if there exists an orthonormal basis $\{e_1, \dots, e_{n+1}\}$ of T_pM and an orthonormal basis $\{e_{n+2}, \dots, e_{2m+1}\}$ of $T_p^\perp M$ such that

- (a) $\pi = \text{Span}\{e_1, e_2\}$,
- (b) *the forms of the shape operators $A_r \equiv A_{e_r}$, $r = n + 2, \dots, 2m + 1$, become*

$$\begin{aligned} A_{n+2} &= \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & (\lambda + \mu)I_{n-1} \end{pmatrix}, \\ A_r &= \begin{pmatrix} h_{11}^r & h_{12}^r & 0 \\ h_{12}^r & -h_{11}^r & 0 \\ 0 & 0 & 0_{n-1} \end{pmatrix}, \quad r = n + 3, \dots, 2m + 1. \end{aligned} \tag{3.6}$$

PROOF. In view of the Gauss equation and (2.2), the scalar curvature and the mean curvature of M are related by

$$2\tau = \frac{c}{4} (3\|P\|^2 + n(n - 1)) + (n + 1)^2 \|H\|^2 - \|h\|^2, \tag{3.7}$$

where $\|P\|^2$ is given by

$$\|P\|^2 = \sum_{i,j=1}^{n+1} g(e_i, Pe_j)^2 \tag{3.8}$$

for any local orthonormal basis $\{e_1, e_2, \dots, e_{n+1}\}$ for T_pM . We introduce

$$\rho = 2\tau - \frac{(n+1)^2(n-1)}{n} \|H\|^2 - \frac{c}{4} (3\|P\|^2 + n(n-1)). \tag{3.9}$$

From (3.7) and (3.9), we get

$$(n+1)^2 \|H\|^2 = n(\|h\|^2 + \rho). \tag{3.10}$$

Let p be a point of M and let $\pi \subset T_pM$ be a plane section at p . We choose an orthonormal basis $\{e_1, e_2, \dots, e_{n+1}\}$ for T_pM and $\{e_{n+2}, \dots, e_{2m+1}\}$ for the normal space $T_p^\perp M$ at p such that $\pi = \text{Span}\{e_1, e_2\}$ and the mean curvature vector $H(p)$ is parallel to e_{n+2} ; then from (3.10), we get

$$\left(\sum_{i=1}^{n+1} h_{ii}^{n+2}\right)^2 = n\left(\sum_{i=1}^{n+1} (h_{ii}^{n+2})^2 + \sum_{i \neq j} (h_{ij}^{n+2})^2 + \sum_{r=n+3}^{2m+1} \sum_{i,j=1}^{n+1} (h_{ij}^r)^2 + \rho\right). \tag{3.11}$$

Using Lemma 3.1, from (3.11) we obtain

$$h_{11}^{n+2} h_{22}^{n+2} \geq \frac{1}{2} \left\{ \sum_{i \neq j} (h_{ij}^{n+2})^2 + \sum_{r=n+3}^{2m+1} \sum_{i,j=1}^{n+1} (h_{ij}^r)^2 + \rho \right\}. \tag{3.12}$$

From the Gauss equation and (2.2), we also have

$$K(\pi) = \frac{c}{4} (1 + 3\alpha(\pi) - \beta(\pi)) + h_{11}^{n+2} h_{22}^{n+2} - (h_{12}^{n+2})^2 + \sum_{r=n+3}^{2m+1} (h_{11}^r h_{22}^r - (h_{12}^r)^2). \tag{3.13}$$

Thus, we have

$$\begin{aligned} K(\pi) \geq & \frac{c}{4} (1 + 3\alpha(\pi) - \beta(\pi)) + \frac{1}{2} \rho + \sum_{r=n+2}^{2m+1} \sum_{j>2} \left\{ (h_{1j}^r)^2 + (h_{2j}^r)^2 \right\} \\ & + \frac{1}{2} \sum_{i \neq j > 2} (h_{ij}^{n+2})^2 + \frac{1}{2} \sum_{r=n+3}^{2m+1} \sum_{i,j>2} (h_{ij}^r)^2 + \frac{1}{2} \sum_{r=n+3}^{2m+1} (h_{11}^r + h_{22}^r)^2, \end{aligned} \tag{3.14}$$

or

$$K(\pi) \geq \frac{c}{4} (1 + 3\alpha(\pi) - \beta(\pi)) + \frac{1}{2} \rho, \tag{3.15}$$

which, in view of (3.9), yields (3.5).

If the equality in (3.5) holds, then the inequalities given by (3.12) and (3.14) become equalities. In this case, we have

$$\begin{aligned}
 h_{1j}^{n+2} = 0, \quad h_{2j}^{n+2} = 0, \quad h_{ij}^{n+2} = 0, \quad i \neq j > 2; \\
 h_{1j}^r = h_{2j}^r = h_{ij}^r = 0, \quad r = n + 3, \dots, 2m + 1; \quad i, j = 3, \dots, n + 1; \\
 h_{11}^{n+3} + h_{22}^{n+3} = \dots = h_{11}^{2m+1} + h_{22}^{2m+1} = 0.
 \end{aligned}
 \tag{3.16}$$

Furthermore, we may choose e_1 and e_2 so that $h_{12}^{n+2} = 0$. Moreover, by applying Lemma 3.1, we also have

$$h_{11}^{n+2} + h_{22}^{n+2} = h_{33}^{n+2} = \dots = h_{n+1n+1}^{n+2}.
 \tag{3.17}$$

Thus, choosing a suitable orthonormal basis $\{e_1, \dots, e_{2m+1}\}$, the shape operator of M becomes of the form given by (3.6). The converse is straightforward. \square

4. Some applications. For the case $c = 0$, from (3.5) we have the following pinching result.

PROPOSITION 4.1. *Let M be an $(n + 1)$ -dimensional ($n > 1$) submanifold isometrically immersed in a $(2m + 1)$ -dimensional cosymplectic space form $\tilde{M}(c)$ with $c = 0$ such that $\xi \in TM$. Then,*

$$\delta_M \leq \frac{(n + 1)^2(n - 1)}{2n} \|H\|^2.
 \tag{4.1}$$

A submanifold M of an almost contact metric manifold \tilde{M} with $\xi \in TM$ is called a *semi-invariant submanifold* [1] of \tilde{M} if $TM = \mathcal{D} \oplus \mathcal{D}^\perp \oplus \{\xi\}$, where $\mathcal{D} = TM \cap \varphi(TM)$ and $\mathcal{D}^\perp = TM \cap \varphi(T^\perp M)$. In fact, the condition $TM = \mathcal{D} \oplus \mathcal{D}^\perp \oplus \{\xi\}$ implies that the endomorphism P is an *f-structure* [12] on M with $\text{rank}(P) = \dim(\mathcal{D})$. A semi-invariant submanifold of an almost contact metric manifold becomes an *invariant* or *anti-invariant submanifold* according as the anti-invariant distribution \mathcal{D}^\perp is $\{0\}$ or the invariant distribution \mathcal{D} is $\{0\}$ [1, 12].

Now, we establish two inequalities in the following theorems, which are analogous to that of (1.2).

THEOREM 4.2. *Let M be an $(n + 1)$ -dimensional ($n > 1$) submanifold isometrically immersed in a $(2m + 1)$ -dimensional cosymplectic space form $\tilde{M}(c)$ such that the structure vector field ξ is tangent to M . If $c < 0$, then*

$$\delta_M \leq \frac{(n + 1)^2(n - 1)}{2n} \|H\|^2 + \frac{1}{2}(n + 1)(n - 2)\frac{c}{4}.
 \tag{4.2}$$

The equality in (4.2) holds if and only if M is a semi-invariant submanifold with $\dim(\mathfrak{D}) = 2$.

PROOF. Since $c < 0$, in order to estimate δ_M , we minimize $3\|P\|^2 - 6\alpha(\pi) + 2\beta(\pi)$ in (3.5). For an orthonormal basis $\{e_1, \dots, e_{n+1}\}$ of T_pM with $\pi = \text{span}\{e_1, e_2\}$, we write

$$\|P\|^2 - 2\alpha(\pi) = \sum_{i,j=3}^{n+1} g(e_i, \varphi e_j)^2 + 2 \sum_{j=3}^{n+1} \{g(e_1, \varphi e_j)^2 + g(e_2, \varphi e_j)^2\}. \tag{4.3}$$

Thus, we see that the minimum value of $3\|P\|^2 - 6\alpha(\pi) + 2\beta(\pi)$ is zero provided $\pi = \text{span}\{e_1, e_2\}$ is orthogonal to ξ and $\text{span}\{\varphi e_j \mid j = 3, \dots, n\}$ is orthogonal to the tangent space T_pM . Thus, we have (4.2) with equality case holding if and only if M is semi-invariant such that $\dim(\mathfrak{D}) = 2$ with $\beta = 0$. □

THEOREM 4.3. *Let M be an $(n + 1)$ -dimensional $(n > 1)$ submanifold isometrically immersed in a $(2m + 1)$ -dimensional cosymplectic space form $\tilde{M}(c)$ such that $\xi \in TM$. If $c > 0$, then*

$$\delta_M \leq \frac{(n + 1)^2(n - 1)}{2n} \|H\|^2 + \frac{1}{2}n(n + 2)\frac{c}{4}. \tag{4.4}$$

The equality in (4.4) holds if and only if M is an invariant submanifold.

PROOF. Since $c > 0$, in order to estimate δ_M , we maximize $3\|P\|^2 - 6\alpha(\pi) + 2\beta(\pi)$ in (3.5). We observe that the maximum of $3\|P\|^2 - 6\alpha(\pi) + 2\beta(\pi)$ is attained for $\|P\|^2 = n$, $\alpha(\pi) = 0$, and $\beta(\pi) = 1$, that is, M is an invariant and $\xi \in \pi$. Thus, we obtain (4.4) with equality case if and only if M is invariant with $\beta = 1$. □

In last, we prove the following theorem.

THEOREM 4.4. *If M is an $(n + 1)$ -dimensional $(n > 1)$ submanifold isometrically immersed in a $(2m + 1)$ -dimensional cosymplectic space form $\tilde{M}(c)$ such that $c > 0$, $\xi \in TM$ and*

$$\delta_M = \frac{(n + 1)^2(n - 1)}{2n} \|H\|^2 + \frac{1}{2}n(n + 2)\frac{c}{4}, \tag{4.5}$$

then M is a totally geodesic cosymplectic space form $M(c)$.

PROOF. In view of Theorem 4.3, M is an odd-dimensional invariant submanifold of the cosymplectic space form $\tilde{M}(c)$. For every point $p \in M$, we can choose an orthonormal basis $\{e_1 = \xi, e_2, \dots, e_{n+1}\}$ for T_pM and $\{e_{n+2}, \dots, e_{2m+1}\}$ for $T_p^\perp M$ such that A_r ($r = n + 2, \dots, 2m + 1$) take the form (3.6). Since M is an

invariant submanifold of a cosymplectic manifold, therefore, it is minimal and $A_r\varphi + \varphi A_r = 0$, $r = n + 2, \dots, 2m + 1$ [11]. Thus, all the shape operators take the form

$$A_r = \begin{pmatrix} c_r & d_r & 0 \\ d_r & -c_r & 0 \\ 0 & 0 & 0_{n-1} \end{pmatrix}, \quad r = n + 2, \dots, 2m + 1. \quad (4.6)$$

Since $A_r\varphi e_1 = 0$, $r = n + 2, \dots, 2m + 1$, from $A_r\varphi + \varphi A_r = 0$, we get $\varphi A_r e_1 = 0$. Applying φ to this equation, we obtain $A_r e_1 = \eta(A_r e_1)\xi = \eta(A_r e_1)e_1$; thus, $d_r = 0$, $r = n + 2, \dots, 2m + 1$. This implies that $A_r e_2 = -c_r e_2$. Applying φ to both sides, in view of $A_r\varphi + \varphi A_r = 0$ we get $A_r\varphi e_2 = c_r\varphi e_2$. Since φe_2 is orthogonal to ξ and e_2 and φ has a maximal rank, the principal curvature c_r is zero. Hence, M becomes totally geodesic. As in [12, Proposition 1.3, page 313], it is easy to show that M is a cosymplectic manifold of the constant φ -sectional curvature c . \square

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