

## THE EIGENVALUE PROBLEM FOR THE $p$ -LAPLACIAN-LIKE EQUATIONS

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We consider the eigenvalue problem for the following  $p$ -Laplacian-like equation:  $-\operatorname{div}(a(|Du|^p)|Du|^{p-2}Du) = \lambda f(x, u)$  in  $\Omega$ ,  $u = 0$  on  $\partial\Omega$ , where  $\Omega \subset \mathbb{R}^n$  is a bounded smooth domain. When  $\lambda$  is small enough, a multiplicity result for eigenfunctions are obtained. Two examples from nonlinear quantized mechanics and capillary phenomena, respectively, are given for applications of the theorems.

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**1. Introduction.** This paper is devoted to the study of the eigenvalue problem for the  $p$ -Laplacian-like equation

$$\begin{aligned} -\operatorname{div}(a(|Du|^p)|Du|^{p-2}Du) &= \lambda f(x, u), & x \in \Omega, \\ u(x) &= 0, & x \in \partial\Omega, \end{aligned} \tag{1.1}$$

where  $\lambda > 0$  is a real parameter,  $1 < p < n$ ,  $\Omega$  is a bounded smooth domain in  $\mathbb{R}^n$ , and  $Du$  denotes the gradient of  $u$ ,  $f \in C(\overline{\Omega} \times \mathbb{R}, \mathbb{R})$ ,  $a \in C(\mathbb{R}^+, \mathbb{R})$ .

We call  $\lambda$  an eigenvalue of (1.1) provided (1.1), for this  $\lambda$ , has a nontrivial weak solution, say  $u_\lambda$ , which is then called an eigenfunction corresponding to  $\lambda$ . Denote

$$A(r) = \int_0^r a(s) ds, \quad F(x, t) = \int_0^t f(x, s) ds. \tag{1.2}$$

We look for nontrivial solutions of (1.1), and this question is reduced to show, for some  $\lambda \in \mathbb{R}$ , the existence of critical points for the functional

$$I_\lambda(u) = \frac{1}{p} \int_\Omega A(|Du|^p) dx - \lambda \int_\Omega F(x, u) dx, \quad u \in E = W_0^{1,p}(\Omega). \tag{1.3}$$

In [5], Pielichowski discussed the existence and nonnegativity of the first eigenvalue and eigenfunction, in a weak sense, of the  $p$ -Laplace equations with some kind of nonlinear terms below

$$-\operatorname{div}(|Du|^{p-2}Du) + a(x)|u|^{p-2}u = \lambda m(x)|u|^{p-2}u. \tag{1.4}$$

Under the assumption that  $A(\sigma) \leq \phi(\sigma(x))/|\psi(\sigma(x))| \leq B(\sigma)$  where  $A(\sigma)$ ,  $B(\sigma)$  are constants, Garcia-Huidobro et al. [4] proved the existence of eigenvalues and eigenfunctions for the *p-Laplacian-like* equation in the radial form

$$\begin{aligned} [r^{n-1}\phi(u')]'+\lambda r^{n-1}\psi(u) &= 0, \quad r \in (0,R), \\ u'(0) = 0, \quad u(R) &= 0. \end{aligned} \tag{1.5}$$

They used the fixed-point theorem and continuation to techniques. Recently, Boccardo [2] showed the existence of positive eigenfunctions to a kind of *p-Laplace-like* equations

$$\begin{aligned} -\operatorname{div}(M(x,u)Du) &= \lambda u, \quad x \in \Omega, \\ u &> 0, \quad x \in \Omega, \\ \|u\|_{L^2(\Omega)} &= r \quad r \in \mathbb{R}^+. \end{aligned} \tag{1.6}$$

We are especially interested in Ubilla’s paper [7], which studied the solvability of the boundary value problem for *p-Laplacian-like* equation in the radial form

$$\begin{aligned} -\left(a(|u'(r)|^p)|u'(r)|^{p-2}u'(r)\right)' &= f(u(r)) \quad r \in I = (0,1) \\ u(0) = u(1) &= 0. \end{aligned} \tag{1.7}$$

Under the assumption that

$$a(|t|^p)|t|^{p-2}t \in C^1(R \setminus \{0\}, R) \cap C(R, R), \quad (a(|t|^p)|t|^{p-2}t)' > 0, \quad \forall t \neq 0, \tag{1.8}$$

a multiplicity result was obtained by using energy relations and the shooting method. The key of our trick is to change this assumption into that the mapping  $r \mapsto A(|r|^p)$  defined in (1.2) is strictly convex, and then consider the eigenvalue problem (1.1). Also, the method we used, the mountain pass theorem and the minimax principle, is different from [7] and some other related papers (see [7] and the references therein). We got the existence of two eigenfunctions  $u_\lambda, v_\lambda$  not necessarily radial ones. In addition, we found that the behaviors of these two eigenfunctions near  $\lambda = 0$  are much different as  $\lim_{\lambda \rightarrow 0^+} \|u_\lambda\|_E = +\infty, \lim_{\lambda \rightarrow 0^+} \|v_\lambda\|_E = 0$ . Our idea comes partially from [1].

**2. Main results.** Assume that

- (A1) the mapping  $r \mapsto B(r) = A(|r|^p)$  is strongly convex;
- (A2) there exist constants  $c_0 > 0, T > 0$  such that  $A(t) \geq c_0t$ , for all  $t \geq 0$  and  $a(s) \leq T$ , for all  $s \geq 0$ ;
- (A3) there exist constants  $b_0 > 0, b_1 > 0$  such that for all  $x \in \Omega$ ,

$$|f(x, u)| \leq b_0|u|^{r-1} + b_1|u|^{q-1}, \quad \text{for } 1 < q < p < r < p^*, \quad p^* = \frac{np}{n-p}; \tag{2.1}$$

(A4) there exist constants  $t_0, \theta$  such that  $0 < \theta < c_0/pT$  where  $c_0, T$  are constants as in (A2), and

$$\theta f(x, t) t > F(x, t) > 0, \quad \forall x \in \bar{\Omega}, 0 < t_0 < |t|; \tag{2.2}$$

(A5) for all  $x \in \bar{\Omega}, t \geq 0, f(x, t) \geq 0$ , it holds that

$$\lim_{t \rightarrow 0^+} \frac{F(x, t)}{t^p} = +\infty. \tag{2.3}$$

Then we have the main results.

**THEOREM 2.1.** *Under assumptions (A1) to (A5), there exists a number  $\lambda^* > 0$  such that for each  $\lambda \in (0, \lambda^*)$ , there exists an eigenfunction  $u_\lambda$  of (1.1) satisfying  $\lim_{\lambda \rightarrow 0} \|u_\lambda\|_E = +\infty$ .*

**THEOREM 2.2.** *Assume (A1) to (A5) and  $f(x, t) \geq 0$ , then there is a number  $\lambda^* > 0$  such that for each  $\lambda \in (0, \lambda^*)$ , (1.1) has one eigenfunction  $u_\lambda$  behaving  $\lim_{\lambda \rightarrow 0^+} \|u_\lambda\|_E = 0$ .*

**3. Proof of the main results**

**LEMMA 3.1.** *Assume (A1) to (A4), then  $I_\lambda$  defined in (1.3) belongs to  $C^1(E, R)$ .*

**PROOF.** Denote

$$I_A(u) = \frac{1}{p} \int_{\Omega} A(|Du|^p) dx, \quad I_F(u) = \lambda \int_{\Omega} F(x, u) dx, \quad u \in E \tag{3.1}$$

so  $I_\lambda(u) = I_A(u) - I_F(u)$ . We will then complete the proof by the following two claims.

**CLAIM 1** ( $I_A \in C^1(E, R)$ ). In fact, by (A1), for all  $\lambda \in (0, 1), \varphi \in E$ , we have

$$\begin{aligned} & \int_0^{|\lambda Du + (1-\lambda)(Du+D\varphi)|^p} a(s) ds \\ & \leq \lambda \int_0^{|Du|^p} a(s) ds + (1-\lambda) \int_0^{|Du+D\varphi|^p} a(s) ds, \end{aligned} \tag{3.2}$$

that is,

$$\begin{aligned} & \int_0^{|Du+(1-\lambda)D\varphi|^p} a(s) ds - \int_0^{|Du|^p} a(s) ds \\ & \leq (1-\lambda) \left( \int_0^{|Du+D\varphi|^p} a(s) ds - \int_0^{|Du|^p} a(s) ds \right). \end{aligned} \tag{3.3}$$

Set, in the above inequality,  $1 - \lambda = t$ , we then have

$$\begin{aligned} \frac{I_A(u + t\varphi) - I_A(u)}{t} &= \frac{1}{tp} \int_{\Omega} \left( \int_0^{|Du+tD\varphi|^p} a(s) ds - \int_0^{|Du|^p} a(s) ds \right) dx \\ &\leq \frac{1}{p} \int_{\Omega} \left( \int_0^{|Du+D\varphi|^p} a(s) ds - \int_0^{|Du|^p} a(s) ds \right) dx \quad (3.4) \\ &< +\infty, \end{aligned}$$

which is independent of  $t$ . Hence, we can apply the Lebesgue dominated convergence theorem to the equality

$$\begin{aligned} \frac{I_A(u + t\varphi) - I_A(u)}{t} &= \frac{1}{p} \int_{\Omega} a(|Du|^p + \eta(|Du + tD\varphi|^p - |Du|^p)) \\ &\quad \cdot \frac{1}{t} (|Du + tD\varphi|^p - |Du|^p) dx, \quad \text{for some } \eta \in (0, 1), \end{aligned} \quad (3.5)$$

and letting  $t \rightarrow 0$ , we then get

$$I'_A(u)\varphi = \int_{\Omega} a(|Du|^p) |Du|^{p-2} \cdot D\varphi dx. \quad (3.6)$$

Next, we show that  $I'_A$  is continuous in  $u$ . In the following, the constant  $C$  may vary line by line.

Suppose  $\{u_m\} \subset E$  satisfying  $\|u_m - u\|_E \rightarrow 0$  as  $m \rightarrow \infty$ . We then claim that  $\|I'_A(u_m) - I'_A(u)\| \rightarrow 0$ . In fact,

$$\begin{aligned} & \|I'_A(u_m) - I'_A(u)\| \\ &= \sup_{\varphi \in E} \frac{\left| \int_{\Omega} \left( a(|Du_m|^p) |Du_m|^{p-2} Du_m \cdot D\varphi - a(|Du|^p) |Du|^{p-2} Du \cdot D\varphi \right) dx \right|}{\|\varphi\|_E} \\ &\leq \frac{1}{p} \|B'(Du_m) - B'(Du)\|_{L^{p'}(\Omega)}, \end{aligned} \quad (3.7)$$

where

$$B'(r) \equiv DB(r) = pa(|r|^p) |r|^{p-2} r, \quad r \in \mathbb{R}^n, \quad p' = \frac{p}{p-1}. \quad (3.8)$$

Because  $u_m \rightarrow u$  in  $E$ , by Egorov theorem, for any  $\eta > 0$  there exists  $\Omega_{\eta} \subset \Omega$  such that  $|\Omega \setminus \Omega_{\eta}| < \eta$  and  $u_m, Du_m$  converge uniformly to  $u, Du$ , respectively,

in  $\Omega_\eta$ . Also,  $\Omega_\eta$  can be chosen large enough so that the following holds as well

$$\int_{\Omega \setminus \Omega_\eta} |Du|^p dx < \frac{\varepsilon}{2} \tag{3.9}$$

for any given  $\varepsilon > 0$ . By virtue of  $Du_m \rightarrow Du$  in  $L^p(\Omega)$ , when  $m$  is large enough,

$$\begin{aligned} \int_{\Omega \setminus \Omega_\eta} |Du_m|^p dx &< C \left( \int_{\Omega \setminus \Omega_\eta} |Du_m - Du|^p dx + \int_{\Omega \setminus \Omega_\eta} |Du|^p dx \right) \\ &< C \left( \int_{\Omega} |Du_m - Du|^p dx + \varepsilon/2 \right) < C\varepsilon. \end{aligned} \tag{3.10}$$

Then, by (A2), (3.8), and (3.10), when  $m$  is large enough, we obtain

$$\begin{aligned} &\left( \int_{\Omega \setminus \Omega_\eta} |B'(Du_m)|^{p/(p-1)} dx \right)^{(p-1)/p} \\ &\leq p \left( \int_{\Omega \setminus \Omega_\eta} (T|Du_m|^{p-1})^{p/(p-1)} dx \right)^{(p-1)/p} \leq T(C\varepsilon)^{(p-1)/p}, \end{aligned} \tag{3.11}$$

that is,  $\|B'(Du_m)\|_{L^{p'}(\Omega \setminus \Omega_\eta)}^{p'} \leq C\varepsilon$ . Similarly,

$$\|B'(Du)\|_{L^{p'}(\Omega \setminus \Omega_\eta)}^{p'} \leq C\varepsilon. \tag{3.12}$$

Noticing that

$$\begin{aligned} \|B'(Du_m) - B'(Du)\|_{L^{p'}(\Omega)}^{p'} &\leq \|B'(Du_m) - B'(Du)\|_{L^{p'}(\Omega_\eta)}^{p'} \\ &\quad + \|B'(Du_m)\|_{L^{p'}(\Omega \setminus \Omega_\eta)}^{p'} + \|B'(Du)\|_{L^{p'}(\Omega \setminus \Omega_\eta)}^{p'}. \end{aligned} \tag{3.13}$$

We then get  $\|I'_A(u_m) - I'_A(u)\| \rightarrow 0$  as  $m \rightarrow \infty$ . Therefore,  $I'_A$  is continuous at the point  $u$ , that is,  $I_A \in C^1(E, R)$ .

**CLAIM 2** ( $I_F \in C^1(E, R)$ ). The proof is similar to Claim 1 and we then omit it. This completes the proof of Lemma 3.1. □

**LEMMA 3.2.** Assume (A1) to (A4), then  $I_\lambda$  satisfies (PS) condition.

**PROOF.** From Lemma 3.1, we know that

$$I'_\lambda(u)\varphi = \int_{\Omega} [a(|Du|^p)|Du|^{p-2}Du \cdot D\varphi - \lambda f(x, u)\varphi] dx \quad \forall u, v \in E. \tag{3.14}$$

Suppose that  $S = \{u_m\} \subset E$  satisfies that for some  $M > 0$ ,

$$I_\lambda(u_m) \leq M, \quad \forall u_m \in S, \tag{3.15}$$

$$I'_\lambda(u_m) \rightarrow 0. \tag{3.16}$$

We prove below that there exists a subsequence of  $\{u_m\}$  converging strongly in  $E$ .

(a) At first, we show that  $S$  is bounded in  $E$ . From (3.16), for all  $\varphi \in E$ , it holds that

$$\int_\Omega \left[ a(|Du_m|^p) |Du_m|^{p-2} Du_m \cdot D\varphi - \lambda f(x, u_m) \varphi \right] dx = o(1) \|\varphi\|_E. \tag{3.17}$$

Using (A4) and (A2), we have

$$\begin{aligned} I_\lambda(u_m) - \theta I'_\lambda(u_m) u_m &= \frac{1}{p} \int_\Omega A(|Du_m|^p) dx - \theta \int_\Omega a(|Du_m|^p) |Du_m|^p dx \\ &\quad + \lambda \int_\Omega [\theta f(x, u_m) u_m - F(x, u_m)] dx \\ &> \frac{1}{p} \int_\Omega A(|Du_m|^p) dx - \theta \int_\Omega a(|Du_m|^p) |Du_m|^p dx \\ &> \frac{c_0}{p} \int_\Omega |Du_m|^p dx - \theta \int_\Omega T |Du_m|^p dx. \end{aligned} \tag{3.18}$$

Combining this with (3.17) yields

$$\left( \frac{c_0}{p} - \theta T \right) \int_\Omega |Du_m|^p dx < M + o(1) \theta \|u_m\|_E, \tag{3.19}$$

which implies

$$\|u_m\|_E \leq C. \tag{3.20}$$

Hence, there exists a subsequence of  $S$ , still denoted by  $\{u_m\}$ , such that  $u_m \rightharpoonup u$  in  $E$  and hence  $Du_m \rightharpoonup Du$  in  $L^p(\Omega)$ ,  $u_m \rightarrow u$  in  $L^s(\Omega)$ ,  $1 < s < p^*$ .

(b) Set

$$p_m(x) \equiv \left( a(|Du_m|^p) |Du_m|^{p-2} Du_m - a(|Du|^p) |Du|^{p-2} Du \right) (Du_m - Du), \tag{3.21}$$

then

$$\begin{aligned}
 I_m &\equiv \int_{\Omega} p_m(x) \, dx \\
 &= \int_{\Omega} a(|Du_m|^p) |Du_m|^{p-2} Du_m (Du_m - Du) \, dx \\
 &\quad - \int_{\Omega} a(|Du|^p) |Du|^{p-2} Du (Du_m - Du) \, dx \\
 &\equiv I_m^{(1)} + I_m^{(2)}.
 \end{aligned}
 \tag{3.22}$$

We show below that  $p_m(x) \rightarrow 0$  a.e. in  $\Omega$ . As  $Du_m \rightarrow Du$  in  $L^p(\Omega)$ , it is obvious that  $I_m^{(2)} \rightarrow 0$ . We choose in (3.17)  $\varphi = u_m - u$ , then

$$I_m^{(1)} = \lambda \int_{\Omega} f(x, u_m) (u_m - u) \, dx + o(1) \|u_m - u\|_E.
 \tag{3.23}$$

By (A3) and the Sobolev imbedding theorem,

$$\begin{aligned}
 \left| \int_{\Omega} f(x, u_m) (u_m - u) \, dx \right| &\leq \|f(x, u_m)\|_{r'} \|u_m - u\|_r, \quad r' = r/r - 1 \\
 &\leq (b_0 \|u_m^{r-1}\|_{r'} + b_1 \|u_m^{q-1}\|_{r'}) \|u_m - u\|_r \\
 &\leq c (\|u_m\|_E^{r-1} + \|u_m\|_E^{q-1}) \|u_m - u\|_r \\
 &\rightarrow 0, \quad \text{as } m \rightarrow \infty.
 \end{aligned}
 \tag{3.24}$$

Therefore, from (3.23),  $I_m^{(1)} \rightarrow 0$  and so  $I_m \rightarrow 0$  as  $m \rightarrow \infty$ . Because  $B(r)$  is strictly convex, then for all  $r_1, r_2 \in \mathbb{R}^n$ , it holds that

$$(B'(r_1) - B'(r_2)) \cdot (r_1 - r_2) \geq 0,
 \tag{3.25}$$

where the equality sign holds if and only if  $r_1 = r_2$ . From this and the definition of  $p_m(x)$ , we then get  $p_m(x) \geq 0$ , which with  $I_m \rightarrow 0$  gives  $p_m(x) \rightarrow 0$ , a.e.  $x \in \Omega$ . So we can find  $\Omega_0 \subset \Omega$  such that  $\text{meas}(\Omega - \Omega_0) = 0$ ,  $u_m(x) \rightarrow u(x)$  and  $p_m(x) \rightarrow 0$  on  $\Omega_0$ .

(c) Based on (3.25) and the fact that  $p_m(x) \geq 0$ , very similar to the first part of the proof of [3, Lemma 1], we can get  $Du_m(x) \rightarrow Du(x)$ , for all  $x \in \Omega_0$ .

(d) At last, we prove  $\|u_m - u\|_E \rightarrow 0$ . From the step (c),  $Du_m \rightarrow Du$ , a.e.  $x \in \Omega$ . By Egorov theorem, for any  $\delta > 0$ , there exists  $\Omega_\delta \subset \Omega$  such that  $\text{meas}(\Omega - \Omega_\delta) < \delta$  and  $Du_m$  converges uniformly to  $Du$  on  $\Omega_\delta$ . Because  $B(r)$  is convex, then for any  $r_1, r_2 \in \mathbb{R}^n$  we have

$$B'(r_1) \cdot (r_1 - r_2) \geq B(r_1) - B(r_2).
 \tag{3.26}$$

Choosing  $r_2 = 0$ , with  $B(0) = A(0) = 0$ , then

$$B'(r_1) \cdot r_1 \geq B(r_1) = A(|r_1|^p) \geq c_0 |r_1|^p. \tag{3.27}$$

Suppose  $\Omega' \subset \Omega$ , by (3.27) and (3.8). Using (A2) and Young's inequality, we get

$$\begin{aligned} \frac{c_0}{p} \int_{\Omega'} |Du_m(x)|^p dx &\leq \int_{\Omega'} a(|Du_m|^p) |Du_m|^p dx \\ &= \int_{\Omega'} p_m(x) dx + \int_{\Omega'} a(|Du_m|^p) |Du_m|^{p-2} Du_m \cdot Du dx \\ &\quad + \int_{\Omega'} a(|Du|^p) |Du|^{p-2} Du \cdot Du_m dx \\ &\quad - \int_{\Omega'} a(|Du|^p) |Du|^p dx \\ &\leq \int_{\Omega'} p_m(x) dx + T \int_{\Omega'} |Du_m|^{p-1} |Du| dx \\ &\quad + T \int_{\Omega'} |Du|^{p-1} |Du_m| dx + T \int_{\Omega'} |Du|^p dx \\ &\leq \int_{\Omega'} p_m(x) dx + \varepsilon_1 \int_{\Omega'} |Du_m|^p dx + C(\varepsilon_1) \int_{\Omega'} |Du|^p dx \\ &\quad + \varepsilon_2 \int_{\Omega'} |Du_m|^p dx + C(\varepsilon_2) \int_{\Omega'} |Du|^p dx \\ &\quad + T \int_{\Omega'} |Du|^p dx. \end{aligned} \tag{3.28}$$

Setting  $\varepsilon_1 = \varepsilon_2 = c_0/4p$  in the above inequality yields

$$\frac{c_0}{2p} \int_{\Omega'} |Du_m(x)|^p dx \leq \int_{\Omega'} p_m(x) dx + C \int_{\Omega'} |Du|^p dx. \tag{3.29}$$

Let  $|\Omega'|$  be small enough so that for a given  $\varepsilon > 0$  there holds

$$\int_{\Omega'} |Du|^p dx < \varepsilon. \tag{3.30}$$

Since  $I_m \rightarrow 0$  and  $p_m(x) > 0$ , then when  $m$  is large enough we have

$$\int_{\Omega'} p_m(x) dx \leq \int_{\Omega} p_m(x) dx < \varepsilon. \tag{3.31}$$

Combining this with (3.29), we get  $\int_{\Omega'} |Du_m(x)|^p dx < C\varepsilon$  when  $m$  become large enough. Noticing  $Du_m \rightarrow Du$  uniformly on  $\Omega \setminus \Omega'$ , then

$$\begin{aligned} \|Du_m - Du\|_{L^p(\Omega)} &= \|Du_m - Du\|_{L^p(\Omega \setminus \Omega')} + \|Du_m - Du\|_{L^p(\Omega')} \\ &\leq \|Du_m - Du\|_{L^p(\Omega \setminus \Omega')} + \|Du_m\|_{L^p(\Omega')} + \|Du\|_{L^p(\Omega')} \quad (3.32) \\ &\leq C\varepsilon \quad \text{as } m \text{ is large enough.} \end{aligned}$$

This completes the proof of Lemma 3.2. □

**PROOF OF THEOREM 2.1.** We complete the proof by three steps.

**STEP 1.** In fact, from (A3) we find

$$|F(x, u)| \leq \frac{b_0}{r} |u|^r + \frac{b_1}{q} |u|^q, \quad x \in \Omega. \quad (3.33)$$

Condition (A2) and the Sobolev imbedding theorem yield

$$\begin{aligned} I_\lambda(u) &\geq \frac{c_0}{p} \int_\Omega |Du|^p dx - \lambda \int_\Omega \left( \frac{b_0}{r} |u|^r + \frac{b_1}{q} |u|^q \right) dx \\ &\geq \frac{c_0}{p} \|u\|_E - k_0 \lambda \|u\|_E^r - k_1 \lambda \|u\|_E^q, \end{aligned} \quad (3.34)$$

where  $k_0 > 0, k_1 > 0$  are constants and independent of  $u$ .

Suppose  $u \in E$  satisfying that  $\|u\|_E = \lambda^{-\alpha}, 0 < \alpha < 1/(r - p)$ , then by (3.34) we have

$$I_\lambda(u) \geq \frac{c_0}{p} \lambda^{-\alpha p} - k_0 \lambda^{1-\alpha r} - k_1 \lambda^{1-\alpha q}. \quad (3.35)$$

Because  $0 < \alpha < 1/(r - p)$ , then  $\alpha_\lambda \equiv (c_0/p)\lambda^{-\alpha p} - k_0 \lambda^{1-\alpha r} - k_1 \lambda^{1-\alpha q} \rightarrow +\infty$  as  $\lambda \rightarrow 0^+$ . Hence, there exists  $\lambda^* > 0$  small enough such that  $\alpha_\lambda > 0$  for all  $\lambda \in (0, \lambda^*)$ . Then, we get

$$I_\lambda(u) \geq \alpha_\lambda > 0 \quad \text{for } \|u\|_E = \rho_\lambda, \quad (3.36)$$

where  $\rho_\lambda = \lambda^{-\alpha}$ .

**STEP 2.** Condition (A4) implies that

$$F(x, t) > d_0 t^{1/\theta} - d_1, \quad \forall (x, t) \in \bar{\Omega} \times R, \quad (3.37)$$

where  $d_0, d_1$  are positive constants. Using (3.37) and condition (A2), we find

$$\begin{aligned}
 I_\lambda(tv) &= \frac{1}{p} \int_\Omega A(t^p |Dv|^p) dx - \lambda \int_\Omega F(x, tv) dx \\
 &\leq \frac{T}{p} \int_\Omega t^p |Dv|^p dx - \lambda \int_\Omega (d_0 t^{1/\theta} v^{1/\theta} - d_1) dx \\
 &= \frac{T}{p} t^p \|v\|_E^p - \lambda d_0 t^{1/\theta} \|v\|_{1/\theta}^{1/\theta} + \lambda \tilde{d}_1.
 \end{aligned}
 \tag{3.38}$$

Condition (A2) implies that  $c_0 \leq T$ , and then by (A4) we get  $p < 1/\theta$ . Thus as  $t \rightarrow +\infty$ ,  $I_\lambda(tv) \rightarrow -\infty$ .

**STEP 3.** By Lemma 3.2,  $I_\lambda$  satisfies the (PS) condition. Then, by the results of Steps 1 and 2, we can apply the mountain pass theorem to get that there exists a nontrivial critical point  $u_\lambda$  of  $I_\lambda$  such that

$$I_\lambda(u_\lambda) = c_\lambda \geq \alpha_\lambda > 0, \tag{3.39}$$

and then

$$\begin{aligned}
 I_\lambda(u_\lambda) &\leq \frac{1}{p} \int_\Omega T |Du_\lambda|^p dx + \lambda \int_\Omega \left( \frac{b_0}{r} |u_\lambda|^r + \frac{b_1}{q} |u_\lambda|^q \right) dx \\
 &= \frac{T}{p} \|u_\lambda\|_E^p + \frac{\lambda b_0}{r} \|u_\lambda\|_r^r + \frac{\lambda b_1}{q} \|u_\lambda\|_q^q \\
 &\leq \frac{T}{p} \|u_\lambda\|_E^p + \tilde{b}_0 \|u_\lambda\|_E^r + \tilde{b}_1 \|u_\lambda\|_E^q.
 \end{aligned}
 \tag{3.40}$$

Let  $\lambda \rightarrow 0+$  in (3.40) as  $\alpha_\lambda \rightarrow +\infty$ , then we obtain  $\|u_\lambda\|_E \rightarrow +\infty$ . This completes the proof.  $\square$

**PROOF OF THEOREM 2.2.** For  $0 < \alpha < 1/p$ , let  $\|u\|_E = \lambda^\alpha$ . By (3.34), we have

$$I_\lambda(u) \geq \frac{c_0}{p} \lambda^{\alpha p} - k_0 \lambda^{1+\alpha r} - k_1 \lambda^{1+\alpha q} = \lambda \left( \frac{c_0}{p} \lambda^{\alpha p-1} - k_0 \lambda^{\alpha r} - k_1 \lambda^{\alpha q} \right). \tag{3.41}$$

As  $\alpha p - 1 < 0$ , then there exists  $\lambda^* > 0$  small enough so that  $I_\lambda(u) > 0$  for  $\lambda \in (0, \lambda^*)$ , that is,

$$I_\lambda(u) > 0, \quad \forall 0 < \lambda < \lambda^*, \quad \|u\|_E = \rho_\lambda, \tag{3.42}$$

where  $\rho_\lambda = \lambda^\alpha$ . Set  $B_{\rho_\lambda} = \{u \in E : \|u\|_E < \rho_\lambda\}$ , then for  $u \in \overline{B_{\rho_\lambda}}$ , by (3.34), we find

$$\begin{aligned}
 I_\lambda(u) &\geq \frac{c_0}{p} \|u\|_E^p - k_0 \lambda \|u\|_E^r - k_1 \lambda \|u\|_E^q \\
 &\geq -k_0 \lambda \rho_\lambda^r - k_1 \lambda \rho_\lambda^q \geq -k_0 (\lambda^*)^{1+r\alpha} - k_1 (\lambda^*)^{1+q\alpha},
 \end{aligned}
 \tag{3.43}$$

then  $I_\lambda$  is bounded below on  $\overline{B}_{\rho_\lambda}$ . Choosing  $v \in C_0^\infty(\Omega)$ ,  $0 < v < 1$ ,  $0 \leq |Dv| \leq 1$ ,  $t \geq 0$ , then

$$\begin{aligned} I_\lambda(tv) &= \frac{1}{p} \int_\Omega A(t^p |Dv|^p) dx - \lambda \int_\Omega F(x, tv) dx \\ &\leq \frac{T}{p} \int_\Omega t^p |Dv|^p dx - \lambda \int_\Omega F(x, tv) dx \\ &\leq t^p \left[ \frac{T}{p} \int_\Omega |Dv|^p dx - \lambda \frac{\inf_{x \in \overline{\Omega}} F(x, t)}{t^p} \int_\Omega \frac{F(x, tv)}{F(x, t)} dx \right]. \end{aligned} \tag{3.44}$$

From (A5), we know that  $f(x, t) \geq 0$ , for all  $x \in \overline{\Omega}$ ,  $t \geq 0$  and hence  $F(x, tv)/F(x, t) \leq 1$ . By (3.44), (A5), and applying the dominated convergence theorem to (3.44), we find that there exist  $\delta > 0$ ,  $0 < t < \delta$ ,  $tv \in B_{\rho_\lambda}$  such that

$$I_\lambda(tv) < 0. \tag{3.45}$$

Because  $I_\lambda$  satisfies the (PS) condition, the minimax theorem on  $\overline{B}_{\rho_\lambda}$  claims that  $I_\lambda$  has a nontrivial critical point  $u_\lambda \in B_{\rho_\lambda}$ , which is a local minimum and  $I_\lambda(u_\lambda) < 0$ . Then  $\|u_\lambda\|_E < \rho_\lambda = \lambda^\alpha$ ,  $\|u_\lambda\|_E \rightarrow 0$  as  $\lambda \rightarrow 0+$ . This ends the proof.  $\square$

#### 4. Examples

**EXAMPLE 4.1.** Let  $\Omega \subset \mathbb{R}^n$  be a bounded smooth domain. We consider the  $p$ -Laplacian problem from nonlinear quantized mechanics as

$$\begin{aligned} -\operatorname{div}(|Du|^{p-2}Du) &= \lambda(|u|^{q-2}u + |u|^{r-2}u), \quad x \in \Omega, \\ u(x) &= 0, \quad x \in \partial\Omega, \end{aligned} \tag{4.1}$$

where  $\lambda > 0$ ,  $1 < p < n$ ,  $\Omega \subset \mathbb{R}^n$  is a bounded smooth domain,  $1 < q < p < r < p^*$ ,  $p^* = np/(n-p)$ . In this case,  $a(s) = 1$ ,  $B(r) = |r|^p$  is strictly convex, and conditions (A2) and (A4) are satisfied for  $c_0 = T = 1$  while  $0 < \theta < 1/p$ . Obviously, (A3) also holds. These conditions have been posted directly on the given functions in some papers which dealt with the solvability of the boundary value or eigenvalue problem for the  $p$ -Laplacian equation (see [6] and the references therein). Then, by virtue of Theorems 2.1 and 2.2, when  $\lambda$  is small enough, problem (4.1) possesses at least two eigenfunctions  $u_\lambda$  and  $v_\lambda$ , and

$$\lim_{\lambda \rightarrow 0} \|u_\lambda\|_E = +\infty, \quad \lim_{\lambda \rightarrow 0} \|v_\lambda\|_E = 0. \tag{4.2}$$

**EXAMPLE 4.2.** Consider the eigenvalue problem for generalized capillarity equation originated from the capillary phenomena

$$\begin{aligned}
 -\operatorname{div} \left( \left( 1 + \frac{|Du|^p}{\sqrt{1+|Du|^{2p}}} \right) |Du|^{p-2} Du \right) &= \lambda (|u|^{q-2}u + |u|^{r-2}u), \quad x \in \Omega, \\
 u(x) &= 0, \quad x \in \partial\Omega,
 \end{aligned}
 \tag{4.3}$$

where  $\lambda > 0$ ,  $1 < q < p$ ,  $2p < r < p^*$ ,  $p^* = np/(n-p)$ . We also can check that (A1) to (A5) are satisfied. By Theorems 2.1 and 2.2, there exist two eigenfunctions  $u_\lambda$  and  $v_\lambda$  and  $\lim_{\lambda \rightarrow 0} \|u_\lambda\|_E = +\infty$ ,  $\lim_{\lambda \rightarrow 0} \|v_\lambda\|_E = 0$ .

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