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RANK AND k-NULLITY OF CONTACT MANIFOLDS

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We prove that the dimension of the 1-nullity distribution $N(1)$ on a closed Sasakian manifold M of rank l is at least equal to $2l-1$ provided that M has an isolated closed characteristic. The result is then used to provide some examples of K-contact manifolds which are not Sasakian. On a closed, $2n+1$-dimensional Sasakian manifold of positive bisectional curvature, we show that either the dimension of $N(1)$ is less than or equal to $n+1$ or $N(1)$ is the entire tangent bundle TM. In the latter case, the Sasakian manifold M is isometric to a quotient of the Euclidean sphere under a finite group of isometries. We also point out some interactions between k-nullity, Weinstein conjecture, and minimal unit vector fields.

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1. Introduction. Contact, non-Sasakian manifolds whose characteristic vector field lies in the k-nullity distribution have been fully classified by Boeckx [7]. One of the main goals of the present paper is to describe the leaves of the 1-nullity distribution and the topology of the Sasakian manifolds using the notion of "rank" of a K-contact manifold. After collecting some preliminaries on contact metric geometry in Section 2, we define the rank of a closed K-contact manifold in Section 3.

In Section 4, we define the k-nullity distribution of a Riemannian manifold and prove Theorem 4.3.

Relying on a construction of Yamazaki [25], we use Theorem 4.3 in Section 5, where we exhibit examples of five-dimensional manifolds whose K-contact structures are not Sasakian.

Section 6 deals with Sasakian manifolds with positive bisectional curvature. Using variational calculus techniques, we prove Theorem 6.2.

A conjecture of Weinstein asserts that any compact contact manifold should have at least one closed characteristic. In Section 7, we point out how this conjecture holds true in the case, where the characteristic vector field belongs to the k-nullity distribution, and the contact metric manifold carries a nonsingular Killing vector field.

We conclude our paper by an observation relating k-nullity and the existence of minimal unit vector fields in Section 8. It is shown here that if the characteristic vector field belongs to the k-nullity distribution, then one can deform the contact metric in such a way that the same characteristic vector field becomes a critical point of the volume functional which is defined on the space of unit vector fields.

2. Preliminaries. A *contact form* on a $2n+1$-dimensional manifold M is a 1-form α such that $\alpha \wedge (d\alpha)^n$ is a volume form on M. There is always a unique vector field

Z, the characteristic vector field of α, which is determined by the equations $\alpha(Z) = 1$ and $d\alpha(Z, X) = 0$ for arbitrary X. The distribution $D_p = \{V \in T_pM : \alpha(V) = 0\}$ is called the contact distribution of α. Clearly, D is a symplectic vector bundle with symplectic form $d\alpha$.

On a contact manifold (M, α, Z), there is also a nonunique Riemannian metric g and a partial complex operator J adapted to α in the sense that the identities

$$2g(X, JY) = d\alpha(X, Y), \qquad J^2X = -X + \alpha(X)Z, \tag{2.1}$$

hold for any vector fields X, Y on M. We have adopted the convention for exterior derivative so that

$$d\alpha(X, Y) = X\alpha(Y) - Y\alpha(X) - \alpha([X, Y]). \tag{2.2}$$

The tensors α, Z, J, and g are called contact metric structure tensors and the manifold M with such a structure will be called a *contact metric manifold* [6]. We will use the notation (M, α, Z, J, g) to denote a contact metric manifold M with specified structure tensors. Assuming that (M, g) is a complete Riemannian manifold, let ψ_t, $t \in \mathbb{R}$, denote the 1-parameter group of diffeomorphism generated by Z. The group ψ_t preserves the contact form α, that is, $\psi_t^*\alpha = \alpha$. If ψ_t is also a 1-parameter group of isometries of g, then the contact metric manifold is called a *K-contact manifold*. By ∇ we will denote the Levi-Civita covariant derivative operator of g. On a K-contact manifold, one has the identity

$$\nabla_X Z = -JX \tag{2.3}$$

valid for any tangent vector X. On a general contact metric manifold, the identity

$$\nabla_X Z = -JX - JhX \tag{2.4}$$

is satisfied, where $hX = (1/2)L_Z JX$. If the identity

$$(\nabla_X J)Y = g(X, Y)Z - \alpha(Y)X \tag{2.5}$$

is satisfied for any vector fields X and Y on M, then the contact metric structure (M, α, Z, J, g) is called a *Sasakian* structure. A submanifold N in a contact manifold (M, α, Z, J, g) is said to be invariant if Z is tangent to N and JX is tangent to N whenever X is. An invariant submanifold is a contact submanifold.

3. Rank of K-contact manifolds.

On a compact K-contact metric manifold (M, α, Z, g, J), the closure of the 1-parameter group ψ_t in the isometry group of (M, g) is a torus group T^l for some nonzero integer l. A K-contact manifold with the action of such a torus T^l is said to be of *rank* l [24]. The K-contact manifolds of rank 1 are those whose 1-parameter group ψ_t is periodic, that is, the integral curves of Z are all circles. It is shown in [15] or [25] that the rank l of a K-contact manifold is at most equal to $n+1$ if the manifold is $2n+1$ dimensional. Also, from [14], a closed K-contact manifold of dimension $2n+1$ carries at least $n+1$ closed characteristics, that is, $n+1$ closed orbits of the flow ψ_t. Each one of these closed characteristics is a 1-dimensional orbit of the action of a circle subgroup of the torus T^l, where l is the rank of the K-contact manifold.

4. k-nullity distribution.

For a real number k, the *k-nullity distribution* of a Riemannian manifold (M,g) is the subbundle $N(k)$ defined at each point $p \in M$ by

$$N_p(k) = \{H \in T_pM \mid R(X,Y)H = k\big(g(Y,H)X - g(X,H)Y\big);\ \forall X,Y \in T_pM\}, \tag{4.1}$$

where R denotes the Riemann curvature tensor given by the formula

$$R(X,Y)H = \nabla_X \nabla_Y H - \nabla_Y \nabla_X H - \nabla_{[X,Y]} H, \tag{4.2}$$

for arbitrary vector fields X, Y, and H on M. If H lies in $N(k)$, then the sectional curvatures of all plane sections containing H are equal to k.

The interaction between rank and dimension of 1-nullity distribution of Sasakian manifolds can be described thanks to the following simple observation.

Proposition 4.1. *The k-nullity distribution of a Riemannian manifold (M,g) is left invariant by any isometry of (M,g).*

Proof. If $H \in N(k)$ and φ is an isometry of (M,g), then, for any vector fields X, Y on M, one has

$$\begin{aligned} R(\varphi_* X, \varphi_* Y)\varphi_* H &= \varphi_* R(X,Y)H \\ &= \varphi_*\big(k\big(g(Y,H)X - g(X,H)Y\big)\big) \\ &= k\big[g(\varphi_* Y, \varphi_* H)\varphi_* X - g(\varphi_* X, \varphi_* H)\varphi_* Y\big]. \end{aligned} \tag{4.3}$$

Since φ_* is an automorphism of the tangent bundle of M, the above identity shows that $\varphi_* H \in N(k)$. □

By R_k we denote the tensor field defined for arbitrary vector fields X, Y, H by

$$R_k(X,Y)H = R(X,Y)H - k\{g(Y,H)X - g(X,H)Y\}. \tag{4.4}$$

R_k satisfies similar identities as the curvature tensor R, mainly,

(i) $g(R_k(X,Y)H,V) = -g(R_k(X,Y)V,H)$,

(ii) $g(R_k(X,Y)H,V) = g(R_k(X,H)Y,V)$,

(iii) $\nabla_X R_k(Y,H)V + \nabla_Y R_k(H,X)V + \nabla_H R_k(X,Y)V = 0$.

Now, let X, Y, V be any tangent vectors at $p \in M$. Extend X, Y and V into local vector fields such that at p one has $\nabla X = 0 = \nabla Y = \nabla V$. Let H, W be two vector fields in the nullity distribution of R_k, that is,

$$R_k(X,Y)H = 0 = R_k(X,Y)W, \tag{4.5}$$

for any X, Y on M. Using identity (iii), one obtains

$$\begin{aligned} 0 &= g(\nabla_H R_k(X,Y)V + \nabla_X R_k(Y,H)V + \nabla_Y R_k(H,X)V, W) \\ &= g\Big(\nabla_H(R_k(X,Y)V) + \nabla_X(R_k(Y,H)V) + \nabla_Y(R_k(H,X)V) \\ &\qquad - R_k(Y,\nabla_X H)V - R_k(\nabla_Y H, X)V + \text{Others}, W\Big) \\ &= Zg(R_k(X,Y)V,W) - g(R_k(X,Y)V,\nabla_H W) + Xg(R_k(Y,H)V,W) \\ &\quad - g(R_k(Y,H)V,\nabla_X W) + Yg(R_k(H,X)V,W) - g(R_k(H,X)V,\nabla_Y W) \\ &\quad - g(R_k(Y,\nabla_X H)V,W) - g(R_k(\nabla_Y H,X)V,W) + g(\text{Others},W). \end{aligned} \tag{4.6}$$

"Others" stands for terms vanishing at p. Applying identities (i) and (ii), and evaluating at p, we obtain

$$0 = g(R_k(X,Y)\nabla_H W, V), \tag{4.7}$$

for arbitrary X, Y, and V. This means that $\nabla_H W$ also belongs to the k-nullity distribution whenever H and W do. The above argument proves that $N(k)$ is an integrable subbundle with totally geodesic leaves of constant curvature k [20]. Hence, if $k > 0$ and $\dim N(k) > 1$, then each leaf of $N(k)$ is a compact manifold [13, Corollary 19.5].

On a contact metric $2n+1$-dimensional manifold M, $n > 1$, Blair and Koufogiorgos showed that if the characteristic vector field Z lies in $N(k)$, then $k \le 1$. If $k < 1$ and $k \ne 0$, then the dimension of $N(k)$ is equal to 1 [1]. The corresponding result for $n = 1$ is due to Sharma [19]. If $k = 0$, then M is locally $E^{n+1} \times \mathbf{S}^n(4)$ and Z is tangent to the Euclidean factor giving that the dimension of $N(0)$ is equal to $n+1$ [5]. If $k = 1$, the contact metric structure is Sasakian and we wish to investigate the dimension of $N(1)$ on a Sasakian manifold. Contact, non-Sasakian manifolds whose characteristic vector field lies in the k-nullity distribution have been fully classified by Boeckx in [7]. First, we will describe the leaves of the 1-nullity distribution on a Sasakian manifold.

Proposition 4.2. *Let (M,α,Z,J,g) be a closed Sasakian manifold. If the dimension of $N(1)$ is bigger than 1, then each leaf of $N(1)$ is a closed Sasakian submanifold which is isometric to a quotient of a Euclidean sphere under a finite group of isometries.*

Proof. Let N be such a leaf of $N(1)$. Since the leaf is a totally geodesic submanifold and Z is tangent to it, one has that $JX = -\nabla_X Z$ is tangent to the leaf for any X tangent to it. So, N is an invariant contact submanifold of the Sasakian manifold M and therefore it is also Sasakian. Since N is complete of constant curvature 1, it is isometric to a quotient of a Euclidean sphere under a finite group of Euclidean isometries [23]. □

To simplify notations, we will denote the dimension of $N(k)$ by $\dim N(k)$. As a consequence of Proposition 4.2 and the work in [14], we obtain the following theorem.

Theorem 4.3. *Let M be a closed Sasakian $2n+1$-dimensional manifold of rank l, with structure tensors α, Z, J, and g. The following hold.*

(1) *If $\dim N(1) > 1$ and M has an isolated closed characteristic, then $\dim N(1) \ge 2l-1$. In particular, if $l = n+1$, then $\dim N(1) = 2n+1$ and M is isometric to the quotient of a Euclidean $2n+1$-sphere under a finite group of Euclidean isometries.*

(2) *If M has a finite number of closed characteristics, then again* $\dim N(1) = 2n+1$, *and M is isometric to the quotient of a Euclidean $2n+1$-sphere under a finite group of Euclidean isometries.*

Proof. Under the hypothesis, one has $l \geq 2$. There is then a torus T^l acting on M by isometric strict contact diffeomorphisms. Let $Z_1, \dots, Z_l$ be a basis of periodic Killing vector fields for the Lie algebra of T^l. Any isolated closed characteristic of α is a common orbit of all the Z_i and Z (see [14]). Let N be a leaf of $N(1)$ containing an isolated closed characteristic, then, since by Proposition 4.1, each Z_i preserves the foliation by leaves of $N(1)$, one sees that each of Z_i preserves the leaf N. Therefore, each Z_i is tangent to N. It follows that $\dim N(1) \geq 2l-1$ since JZ_i is also tangent to N for each i and at most $l-1$ of the $JZ_i's$ can be linearly independent.

In the case $l = n+1$, $N(1)$ has only one leaf, the manifold M itself. In the case M has a finite number $\mathbf{S}$ of closed characteristics, then $N(1)$ cannot have more than $\mathbf{S}$ leaves because, being a closed Sasakian manifold, each leaf of $N(1)$ must contain at least one closed characteristic. It follows again that there is only one leaf which must be the manifold M itself. In any of the above two cases, M is a closed manifold of constant curvature 1 and it is well known that any compact, constant curvature-1 manifold is isometric to a quotient of a Euclidean sphere under a finite group of Euclidean isometries. □

5. K-Contact, non-Sasakian manifolds.

In dimension 3, a K-contact manifold is automatically Sasakian, not so in higher dimensions. We will provide 5-dimensional examples documenting the existence of K-contact structures which are not Sasakian. One well-known way of obtaining K-contact structures which are not Sasakian is as follows. Let S be a closed manifold admitting a symplectic form but not Kähler form. Examples of such manifolds may be found for instance in [21] or [12]. Let ω be a symplectic form on S whose cohomology class $[\omega]$ lies in $H^2(S,\mathbb{Z})$, and let $\pi : E \to S$ be the Boothby-Wang fibration associated with ω [8]. If g and J are a metric and an almost complex operator adapted to ω, then E carries a K-contact structure whose tensors (α, Z, J^*, g^*) are naturally derived from (ω, J, g). The contact form α is just the connection 1-form of the S^1-bundle, $d\alpha = \pi^*\omega$, $\pi_* J^* = J\pi_*$, and $g^* = \pi^* g + \alpha \otimes \alpha$. The characteristic vector field Z is, up to a sign, the unit tangent vector field along the fibers of π. That the above K-contact manifold E is not Sasakian follows from the well-known result of Hatakeyama which states that a regular contact manifold with structure tensors (α, Z, J, g) is Sasakian if and only if the space of orbits of Z is a Kähler manifold with projected tensors [10]. As a consequence of Theorem 4.3, we derive other examples of K-contact structures which are not Sasakian. These come as simply connected, 5-dimensional K-contact manifolds of maximum rank 3.

In [25], closed simply connected K-contact manifolds of dimension 5 and rank 3 have been classified. Let M denote $\mathbf{S}^2 \times \mathbf{S}^3$ and N denote the nontrivial oriented $\mathbf{S}^3$ bundle over $\mathbf{S}^2$. Let r be an integer, $r > 3$. In [25], Yamazaki showed that the connected sum $Q = \#_{r-3} M$ of $r-3$ copies of M carries a K-contact structure of rank 3 with exactly r closed characteristics. Also, he showed that the connected sum $W = N \#_{r-4} M$ of N with $r-4$ copies of M carries a K-contact structure of rank 3 with exactly r closed characteristics. None of the manifolds Q and W above is homeomorphic to $\mathbf{S}^5$, therefore, As an immediate consequence of Theorem 4.3 in this note, none of the K-contact structures on Q and W is Sasakian.

6. Sasakian manifolds of positive bisectional curvature. On compact Sasakian manifolds, one has the following lemma due to Binh and Tamássy [4].

Lemma 6.1. *Let (M,α,Z,J,g) be a closed $2n+1$-dimensional Sasakian manifold and $N \subset M$, a $2r+1$-dimensional invariant submanifold. Let $\gamma(t)$ be a normal geodesic issuing from $\gamma(0) = x \in N$ in a direction perpendicular to N. Then, there exist orthonormal vectors $E_i \in T_xN$, $i = 1,2,\dots,r$, such that their parallel translated $E_i(t)$ along $\gamma(t)$ completed with $JE_i(t)$ form a vector system which is orthonormal and parallel along $\gamma(t)$.*

Proof. Let $E_1,\dots,E_r$ be an orthonormal system of vectors in T_xN such that Z,E_1, $Je_1,\dots,E_r,JE_r$ is an orthonormal system of vectors tangent to N at x. We translate E_i parallel along $\gamma(t)$ to obtain $E_i(t)$, $E_i(0) = E_i$. We claim that $JE_i(t)$ is also parallel along $\gamma(t)$. Indeed, denoting $\dot{\gamma}$ by V and applying identity (2.5), one obtains

$$\nabla_V(JE_i(t)) = (\nabla_V J)E_i(t) = -g(Z,E_i(t))V. \tag{6.1}$$

Hence, $JE_i(t)$ will be parallel along γ if and only if $f_i(t) = g(Z,E_i(t)) = 0$ for all t. We will show that $f_i(t)$ satisfies the linear differential equations

$$\begin{gathered} f_i''(t) = -f_i(t), \\ f_i'(t) = Vg(E_i,Z) = -g(E_i(t),JV), \\ f_i''(t) = -Vg(E_i(t),JV) = -g(E_i(t),Z) = -f_i(t). \end{gathered} \tag{6.2}$$

Moreover, $f_i(0) = g(E_i,Z) = 0$ and $f_i'(0) = -g(E_i,JV) = 0$, because N is invariant and so is the normal bundle of N in M. The initial value problem

$$f_i'' + f_i = 0, \qquad f_i(0) = 0, \qquad f_i'(0) = 0 \tag{6.3}$$

has the unique solution $f_i(t) = 0$ for all t. □

Given two unit tangent vectors X and Y such that $\alpha(X) = 0 = \alpha(Y)$ on a contact $2n+1$-dimensional manifold (M,α,Z,J,g), the bisectional curvature $H(X \wedge Y)$ of the plane spanned by X and Y is defined by

$$H(X \wedge Y) = g(R(X,Y)Y,X) + g(R(X,JY)JY,X). \tag{6.4}$$

The J-sectional curvature is by definition the sectional curvature of a plane spanned by X and JX.

For closed, Sasakian manifolds of constant J-sectional curvature, Sharma [19] has shown that the dimension of $N(1)$ is either 1 or $2n+1$. Assuming only that the manifold has positive bisectional curvature, we prove the following result.

Theorem 6.2. *Let (M,α,Z,J,g) be a closed $2n+1$-dimensional Sasakian manifold of positive bisectional curvature. Then $\dim N(1) \le n+1$ or $\dim N(1) = 2n+1$.*

Proof. Suppose that N_1 and N_2 are two distinct $2l-1$-dimensional leaves of $N(1)$, where $2l-1>1$. Denoting by T the distance between N_1 and N_2, there exists a minimal geodesic $c(t)$, $t\in[0,T]$, from N_1 to N_2 such that $c(0)\in N_1$, $c(T)\in N_2$, and $c(t)$ is the shortest such curve. Let $V(t)$ be the unit tangent vector to the geodesic $c(t)$. Then $V(0)$ is orthogonal to N_1 and $V(T)$ is orthogonal to N_2. Let E_i, JE_i, $i=1,2,\dots,l-1$, be an orthonormal basis for the contact distribution at $c(0)\in N_1$ (recall N_1 is a contact submanifold). Let $E_i(t)$ denote the parallel translation of E_i from $c(0)$ to $c(t)$. Then $E_i(t)$, $JE_i(t)$, $i=1,2,\dots,l-1$, is a parallel orthonormal frame field along $c(t)$ as was shown in Lemma 6.1 (see also [4]). Suppose now that $2l-1>n+1$. Then the span of E_i, JE_i, $i=1,2,\dots,l-1$, has dimension $2l-2$ which is bigger than $2n-(2l-2)$, the fiber dimension of the normal bundle of N_2. Consequently, one can find a unit vector $F\in T_{c(0)}N_1$ which is a linear combination of the $E_i(0)$, $JE_i(0)$, $i=1,2,\dots,l-1$, such that its parallel translated $F(T)\in T_{c(T)}N_2$. Since N_1 and N_2 are invariant contact submanifolds, one has also $JF(T)\in T_{c(T)}N_2$. The vector fields $F(t)$ and $JF(t)$ along $c(t)$ provide variations $c_s(t)$ of the geodesic $c(t)$ with endpoints in N_1 and N_2. Let $V_s(t)$ denote the tangent vector to the curves in such a variation. Then the arclength functional $L(s)$ is given by

$$L(s)=\int_0^T ||V_s(t)||dt. \tag{6.5}$$

One has $L'(0)=0$ because $c(t)$ is a minimal geodesic. Furthermore, by Singe formula for the second variation [9], one has

$$L''_F(0)=\sigma_{N_2}(F,F)(T)-\sigma_{N_1}(F,F)(0)-\int_0^T g(R(F,V)V,F)(t)dt, \tag{6.6}$$

where σ_{N_i} is the second fundamental form of the submanifold N_i and $g(R(F,V)V,F)$ is the sectional curvature of the plane spanned by F and V. Similarly,

$$L''_{JF}(0)=\sigma_{N_2}(JF,JF)(T)-\sigma_{N_1}(JF,JF)(0)-\int_0^T g(R(JF,V)V,JF)(t)dt. \tag{6.7}$$

Adding the two second variations and recalling that N_i, $i=1,2$, is totally geodesic and the bisectional sectional curvature $H(V\wedge F)$ is positive by assumption, one has that $\sigma_{N_1}(F,F)(0)=0=\sigma_{N_2}(F,F)(T)$ and

$$L''_F(0)+L''_{JF}(0)=-\int_0^T H(V\wedge F)(t)dt<0. \tag{6.8}$$

Therefore, at least one of the second variations at $c(t)$ is strictly negative, contradicting the minimality of the geodesic $c(t)$. Thus, we have established that if $\dim N(1)>n+1$, then $N(1)$ has only one leaf, which has to be the manifold M itself. □

7. Weinstein conjecture. Let (M,α,Z,J,g) be a closed contact manifold. Weinstein conjecture [22] asserts that the characteristic vector field Z of α should have at least one closed orbit. If Z belongs to the 1-nullity distribution $N(1)$, then (M,α,Z,J,g) is Sasakian, and Weinstein conjecture has been settled in that case [15]. In the non-Sasakian case, one has the following.

Theorem 7.1. *Let (M,α,Z,J,g) be a closed contact metric structure such that Z belongs to the k-nullity distribution $N(k)$, $0<k<1$. Suppose that there is a nonsingular, Killing vector field on (M,g). Then Z has at least two closed characteristics.*

Proof. Since $0<k<1$, $\dim N(k)=1$. Let C be a nonsingular Killing vector field on M; we may assume C to be a periodic vector field. Since C preserves the k-nullity distribution $N(k)$, one has

$$[C,Z]=fZ \tag{7.1}$$

for some smooth function f on M. But also, using identity (2.4), one obtains the following:

$$\alpha([C,Z])=g(Z,[C,Z])=-g(Z,JC+JhC)-g(Z,\nabla_Z C). \tag{7.2}$$

But since C is Killing, $g(Z,\nabla_Z C)=-g(\nabla_Z C,Z)$, thus $g(\nabla_Z C,Z)=0$ and

$$f=\alpha([C,Z])=g(\nabla_Z C,Z)=0, \tag{7.3}$$

from which follows the identity

$$[C,Z]=0. \tag{7.4}$$

Moreover, for an arbitrary vector field X on M,

$$\begin{aligned} L_C\alpha(X)&=C\alpha(X)-\alpha([C,X])\\ &=Cg(Z,X)-g(Z,[C,X])\\ &=g([C,Z],X)\\ &=0. \end{aligned} \tag{7.5}$$

Therefore, one has

$$L_C\alpha=0. \tag{7.6}$$

As in [3], one defines a smooth function S on M by

$$S=i_C\alpha. \tag{7.7}$$

S is a basic function relative to Z, indeed,

$$dS(Z)=i_Z di_C\alpha=i_Z(L_C\alpha-i_C d\alpha)=0. \tag{7.8}$$

The differential of S is given by

$$dS=di_C\alpha=L_C\alpha-i_C d\alpha=-i_C d\alpha. \tag{7.9}$$

A point $p\in M$ is a critical point of S if and only if $C(p)$ is proportional to $Z(p)$. Moreover, since S is basic relative to Z and $[C,Z]=0$, any C orbit containing a critical point p is itself a critical manifold and coincides with the Z orbit containing p. Thus, it is a closed orbit of Z. Since S must have at least two critical points on two distinct Z orbits, we conclude that Z must have at least two closed orbits. □

8. Minimal unit vector fields. Given a contact metric manifold (M,α,Z,J,g), one can look at Z as an embedding

$$Z : M \longrightarrow T^1M, \tag{8.1}$$

where T^1M is the unit tangent sphere bundle endowed with the Sasaki metric. One can then ask when is Z a minimal unit vector field, that is, when is Z a critical point for the volume functional defined on the space of unit vector fields on M.

The manifold T^1M is also equipped with a natural contact metric structure whose characteristic vector field generates the well-known geodesic flow of M [17]. If $Z(M) \subset T^1M$ is a contact metric submanifold, then it is a minimal unit vector field [11]. Some examples of minimal unit vector fields whose images in T^1M are also contact submanifolds have been presented in [16]. Here, we prove the following.

Theorem 8.1. *Let (M,α,Z,J,g) be a closed contact metric manifold with Z belonging to the k-nullity distribution, $k<1$. Let g_l denote the deformation of g given by*

$$g_l = lg + (1-l)\alpha\otimes\alpha, \tag{8.2}$$

with

$$l = \frac{1}{\sqrt{2-k}} \tag{8.3}$$

and let T^1M be endowed with the Sasaki metric induced by g_l. Then $Z : M \to T^1M$ is a minimal unit vector field and $Z(M) \subset T^1M$ is a contact submanifold.

Proof. Since Z belongs to the k-nullity distribution, the operator h^2 restricted to the contact distribution acts as multiplication by $\lambda^2 = (1-k)$ [2]. The condition on l then implies that $\lambda^2 = 1/l^2 - 1$, and by [18, Theorem 4.1], Z is a minimal unit vector field whose image $Z(M) \subset T^1M$ is a contact submanifold. □

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