IJMMS 2004:25, 1293–1297
PII. S0161171204309191
http://ijmms.hindawi.com

A REMARK ON THE INTERSECTION OF THE CONJUGATES OF THE BASE OF QUASI-HNN GROUPS

R. M. S. MAHMOOD

Received 25 September 2003

Quasi-HNN groups can be characterized as a generalization of HNN groups. In this paper, we show that if G^* is a quasi-HNN group of base G, then either any two conjugates of G are identical or their intersection is contained in a conjugate of an associated subgroup of G.

2000 Mathematics Subject Classification: 20F65, 20E07, 20E08.

1. Introduction. In [8, Lemma 3.15, page 152], Scott and Wall proved that if $G = G_1 *_C G_2$ is a nontrivial free product with amalgamation group, then either $gG_1g^{-1} \cap G_i$ is a subgroup of a conjugate of C, or $i = 1$ and $g \in G_1$, so that $gG_1g^{-1} \cap G_i = G_1$. In this paper we generalize such a result to groups acting on trees with inversions and then apply the result we obtain to a new class of groups called quasi-HNN groups, introduced in [2]. This paper is divided into five sections. In Section 2, we give basic definitions. In Section 3, we have notations related to groups acting on trees with inversions. In Section 4, we discuss the intersections of vertex stabilizers of groups acting on trees with inversions. In Section 5, we apply the results of Section 4 to a tree product of groups and of quasi-HNN groups.

2. Groups acting on graphs. In this section, we begin by recalling some definitions taken from [3, 7]. First we give formal definitions related to groups acting on graphs with inversions. By *a graph* X we understand a pair of disjoint sets $V(X)$ called the set of *vertices* and $E(X)$ called the set of *edges*, with $V(X)$ nonempty, equipped with two maps $E(X) \to V(X) \times V(X)$, $y \to (o(y), t(y))$, and $E(X) \to E(X)$, $y \to \overline{y}$, satisfying the conditions $\overline{\overline{y}} = y$ and $o(\overline{y}) = t(y)$ for all $y \in E(X)$. The case $\overline{y} = y$ is possible for some $y \in E(X)$. For $y \in E(X)$, $o(y)$ and $t(y)$ are called the *ends* of y and $\overline{y}$ is called the *inverse* of y. There are obvious definitions of trees, morphisms of graphs, and $\mathrm{Aut}(X)$, the set of all automorphisms of the graph X which is a group under the composition of morphisms. We say that a group G *acts* on a graph X if there is a group homomorphism $\phi : G \to \mathrm{Aut}(X)$. If $x \in X$ (vertex or edge) and $g \in G$, we write $g(x)$ for $(\phi(g))(x)$. Thus if $g \in G$ and $y \in E(X)$, then $g(o(y)) = o(g(y))$, $g(t(y)) = t(g(y))$, and $g(\overline{y}) = \overline{g(y)}$. The case $g(y) = \overline{y}$ for some $g \in G$ and $y \in E(X)$ may occur. That is, G acts with inversions on X.

We have the following definitions related to the action of the group G on the graph X.

(1) If $x \in X$ (vertex or edge), define $G(x)$ to be the set $G(x) = \{g(x) : g \in G\}$. This set is called the *orbit* that contains x.

(2) If $x, y \in X$, define $G(x,y)$ to be the set $G(x,y) = \{g \in G : g(x) = y\}$, and $G(x,x) = G_x$, the stabilizer of x. Thus, $G(x,y) \neq \emptyset$ if and only if x and y are in the same orbit. If $y \in E(X)$ and $u \in \{o(y), t(y)\}$, then it is clear that $G_{\overline{y}} = G_y$ and $G_y \leq G_u$.

(3) If X is connected, then a subtree T of X is called a tree of *representatives* for the action of the group G on X if T contains exactly one vertex from each vertex orbit, and the subgraph Y of X containing T is called a *fundamental domain* if each edge of Y has at least one end in T, and Y contains exactly one edge y from each edge orbit such that $G(y, \overline{y}) = \emptyset$, and exactly one pair $x, \overline{x}$ from each edge orbit such that $G(x, \overline{x}) \neq \emptyset$.

3. Notations. Let G be a group acting on a tree X with inversions, let T be a tree of representatives for the action of G on X, and let Y be a fundamental domain. We have the following notations.

(1) For any vertex v of X, let v^* be the unique vertex of T such that $G(v, v^*) \neq \emptyset$. That is, v and v^* are in the same vertex orbit.

(2) For each edge y of Y, define the following:

(i) $[y]$ is an element of $G(t(y), (t(y))^*)$. That is, $[y]((t(y))^*) = t(y)$ is chosen as follows:

(a) if $o(y) \in V(T)$, then $[y] = 1$ in case $y \in E(T)$, and $[y](y) = \overline{y}$ if $G(y, \overline{y}) \neq \emptyset$,

(b) if $o(y) \notin V(T)$, then $[y] = [\overline{y}]^{-1}$ if $G(y, \overline{y}) = \emptyset$, otherwise $[y] = [\overline{y}]$ if $G(y, \overline{y}) \neq \emptyset$;

(ii) $-y$ is the edge $-y = [y]^{-1}(y)$ if $o(y) \in V(T)$, otherwise $-y = y$;

(iii) $+y$ is the edge $+y = [y](-y)$. It is clear that $t(-y) = (t(y))^*$, $o(+y) = (o(y))^*$, $G_{-y} \leq G_{(t(y))^*}$, $\overline{(-y)} = +(\overline{y})$, and $G_{+y} \leq G_{(o(y))^*}$. Moreover, if $G(y, \overline{y}) \neq \emptyset$, or $y \in E(T)$, then $G_{-y} = G_{+y} = G_y$;

(iv) ϕ_y is the map $\phi_y : G_{-y} \to G_{+y}$ given by $\phi_y(g) = [y]g[y]^{-1}$;

(v) δ_y is the element $\delta_y = [y][\overline{y}]$. It is clear that ϕ_y is an isomorphism and $\delta_y = 1$ if $G(y, \overline{y}) = \emptyset$. Otherwise $\delta_y = [y]^2$.

4. On the intersection of vertex stabilizers of groups acting on trees with inversions. In this section, G will be a group acting on a tree X with inversions, T is a tree of representatives for the action of G on X, and Y is a fundamental domain. We have the following definition.

Definition 4.1. A word w of G means an expression of the form $w = g_0 \cdot y_1 \cdot g_1 \cdot y_2 \cdot g_2 \cdot \cdots \cdot y_n \cdot g_n$, $n \geq 0$, $y_i \in E(Y)$, for $i = 1, \dots, n$, such that

(1) $g_0 \in G_{(o(y_1))^*}$,

(2) $g_i \in G_{(t(y_i))^*}$ for $i = 1, \dots, n$,

(3) $(t(y_i))^* = (o(y_{i+1}))^*$ for $i = 1, \dots, n-1$.

Define $o(w) = (o(y_1))^*$ and $t(w) = (t(y_n))^*$.

If $o(w) = t(w)$, then w is called a closed word of G of type v, $v = o(w)$.

The following concepts are related to the word w defined above:

(i) n is called the *length* of w and is denoted by $|w| = n$,

(ii) w is called a *trivial* word of G if $|w| = 0$ (or $w = g_0$),

(iii) the *value* of w, denoted by $[w]$, is defined to be the element of G:

$$[w] = g_0[y_1]g_1[y_2]g_2 \cdots [y_n]g_n \tag{4.1}$$

(iv) the inverse of w, denoted by w^{-1}, is defined to be the word of G:

$$w^{-1} = g_n^{-1} \cdot \overline{y}_n \cdot \delta_{y_n}^{-1} g_{n-1}^{-1} \cdot \cdots \cdot g_2^{-1} \cdot \overline{y}_2 \cdot \delta_{y_2}^{-1} g_1^{-1} \cdot \overline{y}_1 \cdot \delta_{y_1}^{-1} g_0^{-1}, \tag{4.2}$$

(v) w is called *reduced* if w contains no subword of the form $y_i \cdot g_i \cdot \overline{y}_i$ if $g_i \in G_{-y_i}$, or $y_i \cdot g_i \cdot y_i$ if $g_i \in G_{y_i}$ if $G(y_i, \overline{y}_i) \neq \emptyset$ for $i = 1, \ldots, n$.

Lemma 4.2. *Let w be a nontrivial reduced word of G and let $a \in G_{o(w)}$ be such that $[w]^{-1}a[w] \in G_{[w](t(w))}$. Then there exists a reduced path $x_1, \ldots, x_n$ in X from $o(w)$ to $[w](t(w))$ such that $a \in G_{x_i}$ for $i = 1, \ldots, n$.*

Proof. Let $w = g_0 \cdot y_1 \cdot g_1 \cdot y_2 \cdot g_2 \cdot \cdots \cdot y_n \cdot g_n$, $n \geq 1$. By assumption, $[w]^{-1}a[w] = b$, where $b \in G_{[w](t(w))}$. Consider the word

$$\begin{aligned} w_0 = {} & g_n^{-1} \cdot \overline{y}_n \cdot \delta_{y_n}^{-1} g_{n-1}^{-1} \cdot \cdots \cdot g_2^{-1} \cdot \overline{y}_2 \cdot \delta_{y_2}^{-1} g_1^{-1} \cdot \overline{y}_1 \\ & \cdot \delta_{y_1}^{-1} g_0^{-1} a g_0 \cdot y_1 \cdot g_1 \cdot y_2 \cdot g_2 \cdot \cdots \cdot y_n \cdot g_n b^{-1}. \end{aligned} \tag{4.3}$$

Then w_0 is a nontrivial closed word of G such that $[w_0] = 1$, the identity element of G. Therefore by [4, Corollary 1], w_0 is not reduced. Since w is reduced, then w^{-1} is reduced. Therefore the only possibility that makes w_0 not reduced is $L_i^{-1} a L_i \in G_{-(\overline{y}_i)} = G_{\overline{-(\overline{y}_i)}} = G_{+y_i}$, where $L_i = g_0[y_1]g_1[y_2]g_2 \cdots [y_{i-1}]g_{i-1}$ for $i = 1, \ldots, n$ with the convention that $[y_0] = 1$. Then $a \in L_i G_{+y_i} L_i^{-1} = G_{L_i(+y_i)}$ for $i = 1, \ldots, n$. By taking $x_i = L_i(+y_i)$, we see that $a \in G_{x_i}$ for $i = 1, \ldots, n$. By the corollary of [5, Theorem 1], $x_1, \ldots, x_n$ is a reduced path in X from $o(w)$ to $[w](t(w))$. This completes the proof. $\square$

Theorem 4.3. *For any two vertices u and v of X, $G_u = G_v$ or $G_u \cap G_v$ is contained in G_x, where x is an edge in the reduced path in X joining u and v.*

Proof. If $G_u = G_v$, we are done. Let $G_u \neq G_v$ and $h \in G_u \cap G_v$. Then it is clear that $u \neq v$. We need to show that h is in G_x, where x is an edge in the reduced path in X joining u and v. We have $u = f(u^*)$ and $v = g(v^*)$, where f and g are in G and u^* and v^* are the unique vertices of T such that $G(u, u^*) \neq \emptyset$ and $G(v, v^*) \neq \emptyset$. Then $h = faf^{-1} = gbg^{-1}$, where $a \in G_{u^*}$ and $b \in G_{v^*}$. By [5, Lemma 2], there exists a reduced word $w = g_0 \cdot y_1 \cdot g_1 \cdot y_2 \cdot g_2 \cdot \cdots \cdot y_n \cdot g_n$ of G such that $o(w) = u$, $t(w) = v$, and $[w] = g \cdot w$ is nontrivial. For, if w is trivial, then $u^* = v^*$ and $f^{-1}g \in G_{u^*}$. This implies that $f^{-1}g(v^*) = u^*$, or equivalently $u = v$. This contradicts the assumption that $u \neq v$. By Lemma 4.2, there exists a reduced path $p_1, \ldots, p_n$ in X joining $o(w) = u^*$ and $[w](t(w)) = f^{-1}g(v^*)$ such that $a \in G_{p_i}$ for $i = 1, \ldots, n$. Let $x_i = f(p_i)$, $i = 1, \ldots, n$. Then it is clear that $x_1, \ldots, x_n$ is the reduced path in X joining u and v and $h \in G_{x_i}$ for $i = 1, \ldots, n$. This implies that $G_u \cap G_v \leq G_{x_i}$ for $i = 1, \ldots, n$. This completes the proof. $\square$

We have the following corollaries of Theorem 4.3.

Corollary 4.4. *For any edge x of X, $G_{o(x)} = G_{t(x)}$ or $G_{o(x)} \cap G_{t(x)} = G_x$.*

Corollary 4.5. *Let u and v be two vertices of X and let $x_1,\dots,x_n$ be the reduced path in X joining u and v such that $G_u \neq G_v$. Then $G_u \cap G_v \leq I_{i=1}^{n} G_{x_i}$.*

Corollary 4.6. *Let u and v be two vertices of X such that $G_u \neq G_v$ and let x be an edge in the reduced path in X joining u and v. Then $G_u \cap G_v \leq G_x$.*

Corollary 4.7. *Let u be a vertex of X and let v be a vertex of T. Then $G_u \cap G_v \leq G_x$, where x is an edge in the reduced path in X joining u and v, or $u^* = v$ and $G_u \cap G_v = G_v$.*

Corollary 4.8. *Let u be a vertex of X. Then $G_u \cap G_{u^*} \leq G_x$, where x is an edge in the reduced path in X joining u and u^*, or $u^* = u$ and $G_u \cap G_{u^*} = G_u$.*

Corollary 4.9. *For any edge y of Y, $G_{(o(y))^*} = G_{(t(y))^*}$, or $G_{(o(y))^*} \cap G_{(t(y))^*} \leq G_m$, where m is an edge in the reduced path in T joining $(o(y))^*$ and $(t(y))^*$.*

5. Applications. In this section Theorem 4.3 and its corollaries are applied to a nontrivial tree product of groups introduced in [1] and of quasi-HNN groups introduced in [2].

In [5, Lemma 8], Mahmood showed that if $G = \prod_{i\in I}^{*}(A_i, U_{jk} = U_{kj})$ is a nontrivial tree product of the groups A_i, $i \in I$, then there exists a tree X on which G acts without inversions such that any tree of representatives for the action of G on X equals the fundamental domain and for every vertex u of X and every edge x of X, G_u is a conjugate of A_i for some i in I and G_x is a conjugate of U_{ik} for some i,k in I.

In [6, Lemma 5.1], Mahmood and Khanfar showed that if G^* is the quasi-HNN group $G^* = \langle G, t_i, t_j \mid \mathrm{rel}\, G,\ t_iA_it_i^{-1} = B_i,\ t_jC_jt_j^{-1} = C_j,\ t_j^2 = c_j,\ i \in I,\ j \in J\rangle$, then there exists a tree X on which G^* acts with inversions such that G^* is transitive on $V(X)$ and for every vertex v of X and every edge x of X, G_v^* is a conjugate of G andG_x^* is a conjugate of A_i, $i \in I$, or a conjugate of C_j, $j \in J$.

Then by Theorem 4.3, the following two propositions hold.

Proposition 5.1. *Let $G = \prod_{i\in I}^{*}(A_i, U_{jk} = U_{kj})$ be a nontrivial tree product of the groups A_i, $i \in J$. Then for any g in G and i and s in I, either $gA_ig^{-1} \cap A_s$ is contained in a conjugate of U_{jk} or $i = j$, $g \in A_i$, and $gA_ig^{-1} \cap A_i = A_i$. Moreover, if A_i and A_j are adjacent, then $A_i \cap A_j = U_{ij}$.*

Proposition 5.2. *Let G^* be the quasi-HNN group*

$$G^* = \langle G, t_i, t_j \mid \mathrm{rel}\, G,\ t_iA_it_i^{-1} = B_i,\ t_jC_jt_j^{-1} = C_j,\ t_j^2 = c_j,\ i \in I,\ j \in J\rangle. \tag{5.1}$$

Then for any $g \in G^*$, $gGg^{-1} \cap G$ is contained either in a conjugate of A_i, $i \in I$, or in a conjugate of C_j, $j \in J$, or $g \in G$ and $gGg^{-1} \cap G = G$.

Remark 5.3. If $J = \varnothing$, then G^* is the HNN group $G^* = \langle G, t_i \mid \mathrm{rel}\, G,\ t_iA_it_i^{-1} = B_i,\ i \in I\rangle$. Then, for any $g \in G^*$, either $gGg^{-1} \cap G$ is contained in a conjugate A_i, $i \in I$, or $g \in G$ and $gGg^{-1} \cap G = G$.

Acknowledgment. The author would like to thank the referees for their sincere evaluation and constructive comments which improved the paper considerably.

References

[1] A. Karrass and D. Solitar, *The subgroups of a free product of two groups with an amalgamated subgroup*, Trans. Amer. Math. Soc. **150** (1970), 227–255.

[2] M. I. Khanfar and R. M. S. Mahmood, *On quasi HNN groups*, Kuwait J. Sci. Engrg. **29** (2002), no. 2, 13–24.

[3] R. M. S. Mahmood, *Presentation of groups acting on trees with inversions*, Proc. Roy. Soc. Edinburgh Sect. A **113** (1989), no. 3-4, 235–241.

[4] ______, *The normal form theorem of groups acting on trees with inversions*, J. Univ. Kuwait Sci. **18** (1991), 7–16.

[5] ______, *The subgroup theorem for groups acting on trees*, Kuwait J. Sci. Engrg. **25** (1998), no. 1, 17–33.

[6] R. M. S. Mahmood and M. I. Khanfar, *Subgroups of quasi-HNN groups*, Int. J. Math. Math. Sci. **31** (2002), no. 12, 731–743.

[7] J.-P. Serre, *Arbres, Amalgames,* SL_2, Société Mathématique de France, Paris, 1977.

[8] C. T. C. Wall, *Homological Group Theory*, London Mathematical Society Lecture Note Series, vol. 36, Cambridge University Press, Cambridge, 1979.

R. M. S. Mahmood: Ajman University of Science and Technology, P.O. Box 5102, Abu Dhabi, UAE

E-mail address: rasheedmsm@yahoo.com