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LOGARITHMIC MATRIX TRANSFORMATIONS INTO G_w

MULATU LEMMA

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We introduced the logarithmic matrix L_t and studied it as mappings into ℓ and G in 1998 and 2000, respectively. In this paper, we study L_t as mappings into G_w.

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1. Introduction. The logarithmic power series method of summability [1], denoted by L, is the following sequence-to-function transformation: if

$$\lim_{x\to 1^-}\left\{-\frac{1}{\log(1-x)}\sum_{k=0}^{\infty}\frac{1}{k+1}u_k x^{k+1}\right\}=A, \tag{1.1}$$

then u is L-summable to A. The matrix analogue of the L-summability method is the L_t matrix [2] given by

$$a_{nk}=-\frac{1}{\log(1-t_n)}\frac{1}{k+1}t_n^{k+1}, \tag{1.2}$$

where $0<t_n<1$ for all n and $\lim_n t_n=1$. Thus, the sequence u is transformed into the sequence $L_t u$ whose nth term is given by

$$(L_t u)_n=-\frac{1}{\log(1-t_n)}\sum_{k=0}^{\infty}\frac{1}{k+1}u_k t_n^{k+1}. \tag{1.3}$$

The L_t matrix is called the logarithmic matrix. Throughout this paper, t will denote such a sequence: $0<t_n<1$ for all n, and $\lim_n t_n=1$.

2. Basic notations and definitions. Let $A=(a_{nk})$ be an infinite matrix defining a sequence-to-sequence summability transformation given by

$$(Ax)_n=\sum_{k=0}^{\infty}a_{nk}x_k, \tag{2.1}$$

where $(Ax)_n$ denotes the nth term of the image sequence Ax. The sequence Ax is called the A-transform of the sequence x. Let y be a complex number sequence. Throughout this paper, we will use the following basic notations and definitions:

$$\begin{gathered}
\ell = \left\{y : \sum_{k=0}^{\infty} |y_k| \text{ is convergent}\right\}, \\
\ell(A) = \{y : Ay \in \ell\}, \\
G = \{y : y_k = O(r^k) \text{ for some } r \in (0,1)\}, \\
G_w = \{y : y_k = O(r^k) \text{ for some } r \in (0,w),\ 0 < w < 1\}, \\
G_w(A) = \{y : Ay \in G_w\}, \\
c = \{\text{the set of all convergent sequences}\}, \\
c(A) = \{y : A(y) \in c\}.
\end{gathered} \tag{2.2}$$

Definition 2.1. If X and Y are complex number sequences, then the matrix A is called an X-Y matrix if the image Au of u under the transformation A is in Y whenever u is in X.

Definition 2.2. The summability matrix A is said to be G_w-translative for a sequence u in $G_w(A)$ provided that each of the sequences T_u and S_u is in $G_w(A)$, where $T_u = \{u_1, u_2, u_3, \dots\}$ and $S_u = \{0, u_0, u_1, \dots\}$.

Definition 2.3. The matrix A is G_w-stronger than the matrix B provided that $G_w(B) \subseteq G_w(A)$.

3. Main results. Our first main result gives a necessary and sufficient condition for L_t to be G_w-G_w.

Theorem 3.1. *The logarithmic matrix L_t is a G_w-G_w matrix if and only if $-1/\log(1-t) \in G_w$.*

Proof. Since $0 < t_n < 1$, it follows that

$$\frac{|a_{nk}|}{\log(1-t_n)} \le -1, \tag{3.1}$$

for all n and k. Therefore, if $-1/\log(1-t) \in G_w$, [3, Theorem 2.3] guarantees that L_t is a G_w-G_w matrix. Conversely, if $-1/\log(1-t) \notin G_w$, then the first column of L_t is not in G_w because $a_{n,0} = -t_n/\log(1-t_n) \notin G_w$. Hence, L_t is not a G_w-G_w matrix by [3, Theorem 2.3]. □

Corollary 3.2. *If $0 < t_n < u_n < 1$ and L_t is a G_w-G_w matrix, then L_u is also a G_w-G_w matrix.*

Corollary 3.3. *Suppose $\alpha > -1$ and L_t is an G_w-G_w matrix, then $(1-t)^{\alpha+1} \in G_w$.*

Corollary 3.4. *Let $t_n = 1 - e^{-q_n}$, where $q_n = r^n$. Then L_t is a G_w-G_w matrix if and only if $r > 1/w$.*

Corollary 3.5. *If L_t is a G_w-G_w matrix, then it is a G-G matrix.*

The next result suggests that the logarithmic matrix L_t is G_w-stronger than the identity matrix. The result indicates that the L_t matrix is rather a strong method in the G_w-G_w setting.

Theorem 3.6. *If L_t is a G_w-G_w matrix and the series $\sum_{k=0}^{\infty} x_k$ has bounded partial sums, then it follows that $x \in G_w(L_t)$.*

Proof. The proof easily follows using the same techniques as in the proof of Theorem 3.10 [2]. □

Remark 3.7. Theorem 3.6 indicates that if L_t is a G_w-G_w matrix, then $G_w(L_t)$ contains the class of all conditionally convergent series. This suggests how large the size of $G_w(L_t)$ is. In fact, we can give a further indication of the size of $G_w(L_t)$ by showing that if L_t is a G_w-G_w matrix, then $G_w(L_t)$ contains also an unbounded sequence. To see this, consider the sequence x given by

$$x_k = (-1)^k (k+1)^2 (k+2)(k+3). \tag{3.2}$$

Then

$$\sum_{k=0}^{\infty} \frac{1}{k+1} x_k t_n^{k+1} = t_n \sum_{k=0}^{\infty} (-1)^k (k+1)(k+2)(k+3) t_n^k = \frac{6t_n}{(1+t_n)^4}. \tag{3.3}$$

Hence,

$$|(L_t x)_n| = \frac{6t_n}{-\log(1-t_n)(1+t_n)^4} \leq -\frac{6}{\log(1-t_n)}. \tag{3.4}$$

Thus, if L_t is an G_w-G_w matrix, then by Theorem 3.1, $-1/\log(1-t) \in G_w$, so $x \in G_w(L_t)$.

The next few results deal with the G_w-translativity of L_t. We will show that the L_t matrix is G_w-translative for some sequences in $G_w(L_t)$.

Proposition 3.8. *Every G_w-G_w L_t matrix is G_w-translative for each sequence $x \in G_w$.*

Theorem 3.9. *Suppose L_t is a G_w-G_w matrix and $\{x_k/k\}$ is a sequence such that $x_k/k = 0$ for $k = 0$, then the sequence $\{x_k/k\}$ is in $G_w(L_t)$ for each L-summable sequence x.*

Proof. Let Y be the L_t-transform of the sequence $\{x_k/k\}$. Then we have

$$|Y_n| = -\frac{1}{\log(1-t_n)} \left| \sum_{k=0}^{\infty} \frac{1}{k(k+1)} x_k t_n^{k+1} \right| \leq C_n + D_n, \tag{3.5}$$

where

$$\begin{aligned} C_n &= -\frac{|x_1| t_n}{2\log(1-t_n)} - \frac{|x_2| t_n}{6\log(1-t_n)}, \\ D_n &= -\frac{1}{\log(1-t_n)} \left| \sum_{k=3}^{\infty} \frac{1}{k(k+1)} x_k t_n^k \right|. \end{aligned} \tag{3.6}$$

By Theorem 3.1, the hypothesis that L_t is G_w-G_w implies that $C \in G_w$, and hence there remains only to show $D \in G_w$ to prove the theorem. Note that

$$\begin{aligned} D_n &= -\frac{1}{\log(1-t_n)}\left|\sum_{k=3}^{\infty}\frac{x_k}{(k+1)}\left(\int_0^{t_n} t^{k-1}dt\right)\right| \\ &= -\frac{1}{\log(1-t_n)}\left|\int_0^{t_n} dt\left(\sum_{k=3}^{\infty}\frac{1}{(k+1)}x_k t^{k-1}\right)\right|. \end{aligned} \tag{3.7}$$

The interchanging of the integral and the summation is legitimate as the radius of convergence of the power series

$$\sum_{k=3}^{\infty}\frac{1}{k+1}x_k t^{k-1} \tag{3.8}$$

is at least 1 by [2, Lemma 1] and hence the power series converges absolutely and uniformly for $0 \le t \le t_n$. Now we let

$$F(t) = \sum_{k=3}^{\infty}\frac{1}{k+1}x_k t^{k-1}. \tag{3.9}$$

Then we have

$$-\frac{F(t)}{\log(1-t)} = -\frac{1}{\log(1-t)}\sum_{k=3}^{\infty}\frac{1}{k+1}x_k t^{k-1}, \tag{3.10}$$

and the hypothesis that $x \in c(L)$ implies that

$$\lim_{t\to 1^-}\frac{F(t)}{-\log(1-t)} = A \text{ (finite)}, \quad \text{for } 0 < t < 1. \tag{3.11}$$

We also have

$$\lim_{t\to 0}\frac{F(t)}{-\log(1-t)} = 0. \tag{3.12}$$

Now (3.11) and (3.12) yield

$$\left|\frac{F(t)}{-\log(1-t)}\right| \le M, \quad \text{for some } M > 0, \tag{3.13}$$

and hence

$$|F(t)| \le -M\log(1-t), \quad \text{for } 0 < t < 1. \tag{3.14}$$

So, we have

$$
\begin{aligned}
D_n &= -\frac{1}{\log(1-t_n)}\left|\int_0^{t_n} F(t)\,dt\right| \\
&\le -\frac{1}{\log(1-t_n)}\int_0^{t_n} |F(t)|\,dt \\
&\le -\frac{M}{\log(1-t_n)}\int_0^{t_n} -\log(1-t)\,dt \\
&= -M(1-t_n) - \frac{Mt_n}{\log(1-t_n)} \\
&\le -\frac{M}{\log(1-t_n)}.
\end{aligned}
\tag{3.15}
$$

The hypothesis that L_t is G_w-G_w implies that both $-1/\log(1-t)$ and $(1-t)$ are in G_w by Theorem 3.1. Hence $D \in G_w$. □

Theorem 3.10. *Every G_w-G_w L_t matrix is G_w-translative for each L-summable sequence in $G_w(L_t)$.*

Proof. Let $x \in c(L) \cap G_w(L_t)$. Then we will show that

(i) $T_x \in G_w(L_t)$,

(ii) $S_x \in G_w(L_t)$.

We first show that (i) holds. Note that

$$
\begin{aligned}
|(L_tT_x)_n| &= -\frac{1}{\log(1-t_n)}\left|\sum_{k=0}^{\infty}\frac{1}{k+1}x_{k+1}t_n^{k+1}\right| \\
&= -\frac{1}{\log(1-t_n)}\left|\sum_{k=1}^{\infty}\frac{1}{k}x_k t_n^{k}\right| \\
&= -\frac{1}{\log(1-t_n)}\left|\sum_{k=1}^{\infty}\left(\frac{1}{k+1}+\frac{1}{k(k+1)}\right)x_k t_n^{k}\right| \\
&\le P_n + Q_n,
\end{aligned}
\tag{3.16}
$$

where

$$
\begin{aligned}
P_n &= -\frac{1}{\log(1-t_n)}\left|\sum_{k=1}^{\infty}\frac{1}{k+1}x_k t_n^{k}\right|, \\
Q_n &= -\frac{1}{\log(1-t_n)}\left|\sum_{k=1}^{\infty}\frac{1}{k(k+1)}x_k t_n^{k}\right|.
\end{aligned}
\tag{3.17}
$$

So, we have $|(L_tT_x)_n| \le P_n + Q_n$, and if we show that both P and Q are in G_w, then (i) holds. But the condition $P \in G_w$ follows from the hypothesis that $x \in G_w(L_t)$ and $Q \in G_w$ follows from Theorem 3.9.

Next we will show that (ii) holds. We have

$$\begin{aligned} |(L_t S_x)_n| &= -\frac{1}{\log(1-t_n)} \left| \sum_{k=1}^{\infty} \frac{1}{k+1} x_{k-1} t_n^{k+1} \right| \\ &= -\frac{1}{\log(1-t_n)} \left| \sum_{k=0}^{\infty} \frac{1}{k+2} x_k t_n^{k+2} \right| \\ &\le W_n + U_n, \end{aligned} \tag{3.18}$$

where

$$\begin{aligned} W_n &= -\frac{1}{\log(1-t_n)} \left| \sum_{k=0}^{\infty} \frac{1}{k+1} x_k t_n^{k+2} \right|, \\ U_n &= -\frac{1}{\log(1-t_n)} \left| \sum_{k=0}^{\infty} \frac{1}{(k+1)(k+2)} x_k t_n^{k+2} \right|. \end{aligned} \tag{3.19}$$

The hypothesis that $X \in G_w(L_t)$ implies that $W \in G_w$. We can also show that $U \in G_w$ by making a slight modification in the proof of Theorem 3.9, replacing the sequence $\{x_k/k\}$ with the sequence $\{x_k/(k+2)\}$. Hence, the theorem follows. □

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Mulatu Lemma: Department of Natural Sciences and Mathematics, Savannah State University, Savannah, GA 31406, USA

E-mail address: lemmam@savstate.edu