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CONVERGENCE OF TWO-STEP ITERATIVE SCHEME WITH ERRORS FOR TWO ASYMPTOTICALLY NONEXPANSIVE MAPPINGS

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A two-step iterative scheme with errors has been studied to approximate the common fixed points of two asymptotically nonexpansive mappings through weak and strong convergence in Banach spaces.

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1. Introduction. In 1995, Liu [4] introduced iterative schemes with errors as follows.

(a) For a nonempty subset C of a normed space E and $T : C \to C$, the sequence $\{x_n\}$ in C, iteratively defined by

$$\begin{aligned} x_1 &= x \in C, \\ x_{n+1} &= (1-a_n)x_n + a_n T y_n + u_n, \\ y_n &= (1-b_n)x_n + b_n T x_n + v_n, \quad n \geq 1, \end{aligned} \tag{1.1}$$

where $\{a_n\}$, $\{b_n\}$ are sequences in $[0,1]$ and $\{u_n\}$, $\{v_n\}$ are sequences in E satisfying $\sum_{n=1}^{\infty} \|u_n\| < \infty$, $\sum_{n=1}^{\infty} \|v_n\| < \infty$, is known as Ishikawa iterative scheme with errors.

(b) With E, C, and T as in (a), the sequence $\{x_n\}$, iteratively defined by

$$\begin{aligned} x_1 &= x \in C, \\ x_{n+1} &= (1-a_n)x_n + a_n T x_n + u_n, \quad n \geq 1, \end{aligned} \tag{1.2}$$

where $\{a_n\}$ is a sequence in $[0,1]$ and $\{u_n\}$ a sequence in E satisfying $\sum_{n=1}^{\infty} \|u_n\| < \infty$, is known as Mann iterative scheme with errors.

In 1999, Huang [2] studied the above schemes for asymptotically nonexpansive mappings. Recall that a mapping $T : C \to C$ is asymptotically nonexpansive if there is a sequence $\{k_n\} \subset [1,\infty)$ with $\lim_{n\to\infty} k_n = 1$ and $\|T^n x - T^n y\| \leq k_n \|x - y\|$ for all $x, y \in C$ and for all $n \in \mathbb{N}$, where $\mathbb{N}$ denotes the set of positive integers.

Moreover, in 2001, Khan and Takahashi [3] approximated the fixed points of two asymptotically nonexpansive mappings $S, T : C \to C$ through the sequence $\{x_n\}$ given by

$$\begin{aligned} x_1 &= x \in C, \\ x_{n+1} &= (1-a_n)x_n + a_n S^n y_n, \\ y_n &= (1-b_n)x_n + b_n T^n x_n, \end{aligned} \tag{1.3}$$

where $\{a_n\}$, $\{b_n\}$ are sequences in $[0,1]$ satisfying certain conditions.

Inspired and motivated by the study of the above schemes, we suggest a new iterative scheme $\{x_n\}$ in C constructed through a pair of asymtotically nonexpansive mappings $S, T : C \to C$ given by

$$\begin{aligned} x_1 &= x \in C, \\ x_{n+1} &= (1-a_n)x_n + a_n S^n y_n + u_n, \\ y_n &= (1-b_n)x_n + b_n T^n x_n + v_n, \quad n \geq 1, \end{aligned} \tag{1.4}$$

where $\{a_n\}$, $\{b_n\}$ are sequences in $[0,1]$ with appropriate conditions and $\{u_n\}$, $\{v_n\}$ are sequences in E with $\sum_{n=1}^{\infty} \|u_n\| < \infty$, $\sum_{n=1}^{\infty} \|v_n\| < \infty$.

It is to be noted here that each of the above schemes follows as a special case of our scheme.

2. Preliminaries. Let E be a Banach space with C as its nonempty convex subset. Throughout this paper, $\mathbb{N}$ denotes the set of positive integers and $F(T)$ the set of fixed points of the mapping T. Now we list the following definitions and results used to prove the results in the next section.

Definition 2.1. A mapping $T : C \to C$ is uniformly k-Lipschitzian if for some $k > 0$, $\|T^n x - T^n y\| \leq k\|x-y\|$ for all $x, y \in C$ and for all $n \in \mathbb{N}$.

Definition 2.2. A mapping $T : C \to C$ is completely continuous if and only if $\{Tx_n\}$ has a convergent subsequence for every bounded sequence $\{x_n\}$ in C.

Definition 2.3. E is said to satisfy Opial's condition [5] if for any sequence $\{x_n\}$ in E, $x_n \rightharpoonup x$ implies that $\limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - y\|$ for all $y \in E$ with $y \neq x$.

Definition 2.4. A mapping $T : C \to E$ is called demiclosed with respect to $y \in E$ if for each sequence $\{x_n\}$ in C and each $x \in E$, $x_n \rightharpoonup x$ and $Tx_n \to y$ imply that $x \in C$ and $Tx = y$.

Lemma 2.5 [6]. *Suppose that E is a uniformly convex Banach space and $0 < p \leq t_n \leq q < 1$ for all $n \in \mathbb{N}$. Also, suppose that $\{x_n\}$ and $\{y_n\}$ are two sequences of E such that $\limsup_{n\to\infty} \|x_n\| \leq r$, $\limsup_{n\to\infty} \|y_n\| \leq r$, and $\lim_{n\to\infty} \|t_n x_n + (1-t_n)y_n\| = r$ hold for some $r \geq 0$. Then $\lim_{n\to\infty} \|x_n - y_n\| = 0$.*

Lemma 2.6 [7]. *Let $\{r_n\}$, $\{s_n\}$, $\{t_n\}$ be three nonnegative sequences satisfying*

$$r_{n+1} \leq (1+s_n)r_n + t_n \quad \forall n \geq 1. \tag{2.1}$$

If $\sum_{n=1}^{\infty} s_n < \infty$ and $\sum_{n=1}^{\infty} t_n < \infty$, then $\lim_{n\to\infty} r_n$ exists.

Lemma 2.7 [1]. *Let E be a uniformly convex Banach space satisfying Opial's condition and let C be a nonempty closed convex subset of E. Let T be an asymptotically nonexpansive mapping of C into itself. Then $I - T$ is demiclosed with respect to zero.*

3. Approximating common fixed points. We start with the following lemma.

Lemma 3.1. *Let E be a normed space and C a nonempty bounded closed convex subset of E. Let, for $k > 0$, S and T be uniformly k-Lipschitzian mappings of C into itself. Let $\{x_n\}$ be a sequence as defined in (1.4), where $\{u_n\}$, $\{v_n\}$ are sequences in E such that $\lim_{n\to\infty}\|u_n\| = 0 = \lim_{n\to\infty}\|v_n\|$ and*

$$\lim_{n\to\infty}||x_n - S^n x_n|| = 0 = \lim_{n\to\infty}||x_n - T^n x_n||. \tag{3.1}$$

Then

$$\lim_{n\to\infty}||x_n - Sx_n|| = 0 = \lim_{n\to\infty}||x_n - Tx_n||. \tag{3.2}$$

Proof. Take $c_n = \|x_n - T^n x_n\|$ and $d_n = \|x_n - S^n x_n\|$. Consider

$$\begin{aligned}
||x_{n+1} - x_n|| &= ||a_n(S^n y_n - x_n) + u_n|| \\
&\le a_n||(S^n y_n - S^n x_n) + (S^n x_n - x_n)|| + ||u_n|| \\
&\le a_n k||(1-b_n)x_n + b_n T^n x_n + v_n - x_n|| + a_n d_n + ||u_n|| \\
&= a_n k||b_n(T^n x_n - x_n) + v_n|| + a_n d_n + ||u_n|| \\
&\le a_n b_n c_n k + a_n k||v_n|| + a_n d_n + ||u_n|| \\
&\le c_n k + d_n + k||v_n|| + ||u_n||.
\end{aligned} \tag{3.3}$$

That is,

$$||x_{n+1} - x_n|| \le c_n k + d_n + k||v_n|| + ||u_n||. \tag{3.4}$$

Next, consider

$$\begin{aligned}
||x_{n+1} - Sx_{n+1}|| &= ||(x_{n+1} - S^{n+1}x_{n+1}) + (S^{n+1}x_{n+1} - Sx_{n+1})|| \\
&\le d_{n+1} + k||(x_{n+1} - x_n) + (x_n - S^n x_n) + (S^n x_n - S^n x_{n+1})|| \\
&\le d_{n+1} + kd_n + k(k+1)||x_{n+1} - x_n|| \\
&\le d_{n+1} + kd_n + k(k+1)[c_n k + d_n + k||v_n|| + ||u_n||]
\end{aligned} \tag{3.5}$$

by (3.4). Taking limsup on both sides in the above inequality, we obtain

$$\limsup_{n\to\infty}||x_{n+1} - Sx_{n+1}|| \le 0. \tag{3.6}$$

That is,

$$\lim_{n\to\infty}||x_n - Sx_n|| = 0. \tag{3.7}$$

Similarly, we can prove that

$$\lim_{n\to\infty}||x_n - Tx_n|| = 0. \tag{3.8}$$

This completes the proof of the lemma. □

Lemma 3.2. *Let E be a uniformly convex Banach space and C its nonempty bounded closed convex subset. Let S and T be self-mappings of C satisfying*

$$\begin{aligned} ||S^n x - S^n y|| &\le k_n \|x - y\|, \\ ||T^n x - T^n y|| &\le k_n \|x - y\|, \end{aligned} \tag{3.9}$$

for all $n \in \mathbb{N}$, where $\{k_n\} \subset [1, \infty)$ such that $\sum_{n=1}^{\infty}(k_n - 1) < \infty$. Let $\{x_n\}$ be as in (1.4) with $\{a_n\}$, $\{b_n\}$ in $[\delta, 1-\delta]$ for some $\delta \in (0,1)$ and $\{u_n\}$, $\{v_n\}$ in E with $\sum_{n=1}^{\infty} \|u_n\| < \infty$, $\sum_{n=1}^{\infty} \|v_n\| < \infty$. If $F(S) \cap F(T) \neq \phi$, then $\lim_{n\to\infty} \|x_n - Sx_n\| = 0 = \lim_{n\to\infty} \|x_n - Tx_n\|$.

Proof. Let $p \in F(S) \cap F(T)$. Then

$$\begin{aligned} &||x_{n+1} - p|| \\ &\quad = ||a_n(S^n y_n - p) + (1 - a_n)(x_n - p) + u_n|| \\ &\quad \le a_n k_n ||y_n - p|| + (1 - a_n)||x_n - p|| + ||u_n|| \\ &\quad = a_n k_n ||(1 - b_n)x_n + b_n T^n x_n + v_n - p|| + (1 - a_n)||x_n - p|| + ||u_n|| \\ &\quad = a_n k_n ||b_n(T^n x_n - p) + (1 - b_n)(x_n - p) + v_n|| + (1 - a_n)||x_n - p|| + ||u_n|| \\ &\quad \le a_n b_n k_n^2 ||x_n - p|| + a_n k_n ||v_n|| + a_n(1 - b_n)k_n ||x_n - p|| + (1 - a_n)||x_n - p|| + ||u_n|| \\ &\quad = (1 + a_n b_n k_n^2 + a_n(1 - b_n)k_n - a_n)||x_n - p|| + a_n k_n ||v_n|| + ||u_n||. \end{aligned} \tag{3.10}$$

Since $\{k_n\}$ is a bounded sequence, therefore there exists $h > 0$ such that $k_n \le h$ for all $n \ge 1$ so that

$$||x_{n+1} - p|| \le [1 + a_n b_n h(k_n - 1) + a_n(k_n - 1)]||x_n - p|| + a_n h||v_n|| + ||u_n||. \tag{3.11}$$

Take $s_n = a_n b_n h(k_n - 1) + a_n(k_n - 1)$, $t_n = a_n h\|v_n\| + \|u_n\|$, and $r_n = \|x_n - p\|$. As $\sum_{n=1}^{\infty} s_n < \infty$ and $\sum_{n=1}^{\infty} t_n < \infty$, so $\lim_{n\to\infty} \|x_n - p\|$ exists by Lemma 2.6. Let $\lim_{n\to\infty} \|x_n - p\| = c$, where $c \ge 0$ is a real number. Assume that $c > 0$, as the result for the case $c = 0$ is obviously true. Now $\|T^n x_n - p\| \le k_n \|x_n - p\|$ for all $n \in \mathbb{N}$ gives $\limsup_{n\to\infty} \|T^n x_n - p\| \le c$. Also,

$$\begin{aligned} ||y_n - p|| &= ||b_n(T^n x_n - p) + (1 - b_n)(x_n - p) + v_n|| \\ &\le ||x_n - p|| + (k_n - 1)b_n ||x_n - p|| + ||v_n|| \end{aligned} \tag{3.12}$$

gives

$$\limsup_{n\to\infty} ||y_n - p|| \le c. \tag{3.13}$$

Next, consider

$$||S^n y_n - p + a_n^{-1} u_n|| \le k_n ||y_n - p|| + a_n^{-1} ||u_n|| \le k_n ||y_n - p|| + \frac{1}{\delta} ||u_n||. \tag{3.14}$$

By the above inequality and by virtue of $\|u_n\| \to 0$ and $k_n \to 1$ as $n \to \infty$, we get

$$\limsup_{n\to\infty} \|S^n y_n - p + a_n^{-1} u_n\| \le c. \tag{3.15}$$

Moreover, $c = \lim_{n\to\infty} \|x_{n+1} - p\|$ means that

$$\lim_{n\to\infty} \|a_n (S^n y_n - p + a_n^{-1} u_n) + (1 - a_n)(x_n - p)\| = c. \tag{3.16}$$

Applying Lemma 2.5,

$$\lim_{n\to\infty} \|S^n y_n - x_n + a_n^{-1} u_n\| = 0. \tag{3.17}$$

Thus

$$\|S^n y_n - x_n\| \le \|S^n y_n - x_n + a_n^{-1} u_n\| + \frac{1}{\delta}\|u_n\| \tag{3.18}$$

yields that

$$\lim_{n\to\infty} \|S^n y_n - x_n\| = 0. \tag{3.19}$$

Also, then

$$\|x_n - p\| \le \|x_n - S^n y_n\| + \|S^n y_n - p\| \le \|x_n - S^n y_n\| + k_n \|y_n - p\| \tag{3.20}$$

implies that

$$c \le \liminf_{n\to\infty} \|y_n - p\|. \tag{3.21}$$

By (3.13) and (3.21), we obtain

$$\lim_{n\to\infty} \|y_n - p\| = c. \tag{3.22}$$

That is,

$$\lim_{n\to\infty} \|b_n (T^n x_n - p + b_n^{-1} v_n) + (1 - b_n)(x_n - p)\| = c. \tag{3.23}$$

Again by Lemma 2.5, we get

$$\lim_{n\to\infty} \|T^n x_n - x_n + b_n^{-1} v_n\| = 0, \tag{3.24}$$

which finally gives that

$$\lim_{n\to\infty} \|T^n x_n - x_n\| = 0. \tag{3.25}$$

Now

$$\begin{aligned} \|S^n x_n - x_n\| &\le \|S^n x_n - S^n y_n\| + \|S^n y_n - x_n\| \\ &\le k_n b_n \|T^n x_n - x_n\| + \|v_n\| + \|S^n y_n - x_n\| \end{aligned} \tag{3.26}$$

implies, together with (3.19) and (3.25), that

$$\lim_{n\to\infty}||S^n x_n - x_n|| = 0 = \lim_{n\to\infty}||T^n x_n - x_n||. \tag{3.27}$$

Lemma 3.1 now reveals that

$$\lim_{n\to\infty}||Sx_n - x_n|| = 0 = \lim_{n\to\infty}||Tx_n - x_n||, \tag{3.28}$$

which is as desired. □

Theorem 3.3. *Let E be a uniformly convex Banach space satisfying Opial's condition and let C, S, T, and $\{x_n\}$ be as taken in Lemma 3.2. If $F(S)\cap F(T) \neq \phi$, then $\{x_n\}$ converges weakly to a common fixed point of S and T.*

Proof. Let $p \in F(S)\cap F(T)$. Then, as proved in Lemma 3.2, $\lim_{n\to\infty}\|x_n - p\|$ exists. Now we prove that $\{x_n\}$ has a unique weak subsequential limit in $F(S)\cap F(T)$. To prove this, let w_1 and w_2 be weak limits of the subsequences $\{x_{n_i}\}$ and $\{x_{n_j}\}$ of $\{x_n\}$, respectively. By Lemma 3.2, $\lim_{n\to\infty}\|x_n - Sx_n\| = 0$ and $I-S$ is demiclosed with respect to zero by Lemma 2.7; therefore, we obtain $Sw_1 = w_1$. Similarly, $Tw_1 = w_1$. Again, in the same way, we can prove that $w_2 \in F(S)\cap F(T)$. Next, we prove the uniqueness. For this, suppose that $w_1 \neq w_2$; then by Opial's condition,

$$\begin{aligned}\lim_{n\to\infty}||x_n - w_1|| &= \lim_{n_i\to\infty}||x_{n_i} - w_1|| < \lim_{n_i\to\infty}||x_{n_i} - w_2||\\ &= \lim_{n\to\infty}||x_n - w_2|| = \lim_{n_j\to\infty}||x_{n_j} - w_2||\\ &< \lim_{n_j\to\infty}||x_{n_j} - w_1|| = \lim_{n\to\infty}||x_n - w_1||,\end{aligned} \tag{3.29}$$

a contradiction. Hence the proof is over. □

Remark 3.4. If we take $u_n = v_n = 0$ for all $n \in \mathbb{N}$, the above theorem reduces to [3, Theorem 1] of Khan and Takahashi. Moreover, [6, Theorem 2.1] of Schu becomes a special case of the above theorem when $u_n = v_n = 0$ as well as $T = I$, the identity mapping.

Finally, we approximate common fixed points by the following strong convergence theorem.

Theorem 3.5. *Let E be a uniformly convex Banach space and C its bounded closed convex subset. Let S, T, and $\{x_n\}$ be as taken in Lemma 3.2. If $F(S)\cap F(T) \neq \phi$ and either S or T is completely continuous, then $\{x_n\}$ converges strongly to a common fixed point of S and T.*

Proof. Assume that $T : C \to C$ is completely continuous. Since $\{x_n\}$ is a bounded sequence and T is completely continuous, therefore $\{Tx_n\}$ must have a convergent subsequence $\{Tx_{n_i}\}$. Hence by (3.28), $\{x_n\}$ must have a subsequence $\{x_{n_i}\}$ such that $x_{n_i} \to q$ (say) in C as $n_i \to \infty$. Now continuity of S and T gives that $Sx_{n_i} \to Sq$ and $Tx_{n_i} \to Tq$ as $n_i \to \infty$. Then, again by (3.28), $\|Sq - q\| = 0 = \|Tq - q\|$. This yields that $q \in F(S)\cap F(T)$ so that $\{x_{n_i}\}$ converges strongly to q in $F(S)\cap F(T)$. As proved in

Lemma 3.2, $\lim_{n\to\infty}\|x_n - p\|$ exists for all $p \in F(S)\cap F(T)$; therefore, $\{x_n\}$ must itself converge to $q \in F(S)\cap F(T)$. Hence the proof. □

Remark 3.6. If we put $T = I$, $v_n = 0$ in the above theorem, then [2, Theorem 1] of Huang is obtained. When we take $S = T$ in the above theorem, then [2, Theorem 2] of Huang follows except when $b_n = 0$. Since a self-mapping with compact domain is completely continuous, therefore [3, Theorem 2] of Khan and Takahashi can also be obtained by putting $u_n = v_n = 0$. It is also worth mentioning that the results presented in this paper are for two mappings while the results in Huang [2] are for one mapping only. Meanwhile, calculations in this paper are made much simpler as compared to Huang [2].

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