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STABILITY OF MULTIPLIERS ON BANACH ALGEBRAS

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Suppose A is a Banach algebra without order. We show that an approximate multiplier $T : A \to A$ is an exact multiplier. We also consider an approximate multiplier T on a Banach algebra which need not be without order. If, in addition, T is approximately additive, then we prove the Hyers-Ulam-Rassias stability of T.

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1. Introduction and statement of results. It seems that the stability problem of functional equations had been first raised by Ulam (cf. [5, Chapter VI] and [6]): for what metric groups G is it true that a ε-automorphism of G is necessarily near to a strict automorphism?

An answer to the above problem has been given as follows. Suppose E_1 and E_2 are two real Banach spaces and $f : E_1 \to E_2$ is a mapping. If there exist $\delta \geq 0$ and $p \geq 0$, $p \neq 1$, such that

$$\left\|f(x+y)-f(x)-f(y)\right\| \leq \varepsilon\left(\|x\|^p+\|y\|^p\right) \tag{1.1}$$

for all $x, y \in E_1$, then there is a unique additive mapping $T : E_1 \to E_2$ such that $\|f(x) - T(x)\| \leq 2\varepsilon\|x\|^p/|2-2^p|$ for every $x \in E_1$. This result is called the Hyers-Ulam-Rassias stability of the additive Cauchy equation $g(x+y) = g(x)+g(y)$. Indeed, Hyers [2] obtained the result for $p = 0$. Then Rassias [3] generalized the above result of Hyers to the case where $0 \leq p < 1$. Gajda [1] solved the problem for $1 < p$, which was raised by Rassias. In the same paper, Gajda also gave an example that a similar result does not hold for $p = 1$. We can also find another example in [4]. If $p < 0$, then $\|x\|^p$ is meaningless for $x = 0$. In this case, if we assume that $\|0\|^p$ means ∞, then the proof given in [3] shows the existence of a mapping $T : E_1 \setminus \{0\} \to E_2$ such that $\|f(x)-T(x)\| \leq 2\varepsilon\|x\|^p/|2-2^p|$ for every $x \in E_1 \setminus \{0\}$. Moreover, if we define $T(0) = 0$, then we see that the extended mapping, denoted by the same letter T, is additive. The last inequality is valid for $x = 0$ since we assume $\|0\|^p = \infty$. Thus, the Hyers-Ulam-Rassias stability holds for $p \in \mathbb{R} \setminus \{1\}$, where $\mathbb{R}$ denotes the real number field.

Suppose A is a Banach algebra. We say that a mapping $T : A \to A$ is a multiplier if $a(Tb) = (Ta)b$ for all $a, b \in A$. Recall that a Banach algebra A is *not* without order if there exist $x_0, y_0 \in A \setminus \{0\}$ such that $x_0 A = A y_0 = \{0\}$. Therefore, A is without order if and only if for all $x \in A$, $xA = \{0\}$ implies $x = 0$, or, for all $x \in A$, $Ax = \{0\}$ implies $x = 0$. We first prove the superstability of multipliers on a Banach algebra without order; that is, each approximate multiplier is an exact multiplier.

Theorem 1.1. *Suppose A is a complex Banach algebra without order. If $T : A \to A$ is a mapping such that*

$$\|a(Tb) - (Ta)b\| \leq \varepsilon \|a\|^p \|b\|^p \quad (a, b \in A) \tag{1.2}$$

for some $\varepsilon \geq 0$ and $p \geq 0$, $p \neq 1$, then T is a multiplier.

In Theorem 1.1, we only consider the case where $p \geq 0$, $p \neq 1$. Even if $p < 0$, we can also obtain a result similar to Theorem 1.1 under an additional but natural assumption that $T(0) = 0$.

Theorem 1.2. *Suppose A is a complex Banach algebra without order and suppose $T : A \to A$ is a mapping such that $T(0) = 0$ and $\|a(Tb) - (Ta)b\| \leq \varepsilon \|a\|^p \|b\|^p$ $(a, b \in A)$ for some $\varepsilon \geq 0$ and $p < 0$, where $\|0\|^p$ means ∞. Then T is a multiplier.*

Theorem 1.1 need not be true for $p = 1$. In fact, in Remark 2.1, we give an approximate multiplier which is not an exact multiplier; however, in Remark 2.2, we see that the Hyers-Ulam-Rassias stability holds for approximate multipliers between unital commutative Banach algebras.

If A is a Banach algebra which need not be without order, then under an additional assumption, we show the Hyers-Ulam-Rassias stability of multiplier on A: if f is an approximate multiplier which is also approximately additive, then there is a multiplier near to f.

Theorem 1.3. *Suppose A is a Banach algebra, which need not be without order, and $f : A \to A$ is a mapping such that*

$$\|f(a+b) - f(a) - f(b)\| \leq \varepsilon \left(\|a\|^p + \|b\|^p\right) \quad (a, b \in A), \tag{1.3}$$

$$\|af(b) - f(a)b\| \leq \varepsilon \|a\|^p \|b\|^p \quad (a, b \in A) \tag{1.4}$$

for some $\varepsilon \geq 0$ and $p \in \mathbb{R}$. If $p \geq 0$ and $p \neq 1$, or $p < 0$ and $f(0) = 0$, then there is a multiplier $T : A \to A$ such that

$$\|f(a) - Ta\| \leq \frac{2\varepsilon}{|2 - 2^p|} \|a\|^p \quad (a \in A). \tag{1.5}$$

2. Proofs of the results

Proof of Theorem 1.1. We first show that T is homogeneous, that is, $T(\lambda a) = \lambda Ta$ for all $\lambda \in \mathbb{C}$ and $a \in A$. To do this, pick $\lambda \in \mathbb{C}$, $a \in A$ and fix $x \in A$ arbitrarily. Put $s = (1-p)/|1-p|$. For each $n \in \mathbb{N}$, it follows from (1.2) that

$$\begin{aligned} \|n^s x[T(\lambda a) - \lambda Ta]\| &\leq \|n^s x[T(\lambda a)] - [T(n^s x)](\lambda a)\| \\ &\quad + \|[T(n^s x)](\lambda a) - n^s x(\lambda Ta)\| \\ &\leq \varepsilon \|n^s x\|^p \|\lambda a\|^p + |\lambda| \varepsilon \|n^s x\|^p \|a\|^p \\ &\leq n^{sp} \varepsilon \left(|\lambda|^p + |\lambda|\right) \|x\|^p \|a\|^p, \end{aligned} \tag{2.1}$$

and hence

$$\|x[T(\lambda a)-\lambda Ta]\| \le n^{s(p-1)}\varepsilon(|\lambda|^p+|\lambda|)\|x\|^p\|a\|^p \tag{2.2}$$

for all $n\in\mathbb{N}$. Since $s(p-1)<0$, we obtain by letting $n\to\infty$ in (2.2) that $x[T(\lambda a)-\lambda Ta]=0$. Similarly to the argument above, we can also get $[T(\lambda a)-\lambda Ta]x=0$. Since A is without order, we conclude that $T(\lambda a)=\lambda Ta$, which implies the homogeneity of T.

Now we are ready to prove that T is a multiplier. Since T is homogeneous, $T(a)=n^{-s}T(n^s a)$ for all $n\in\mathbb{N}$. Recall that, by definition, $s(p-1)<0$. We thus obtain for all $a,b\in A$,

$$\begin{aligned}\|a(Tb)-(Ta)b\| &= n^{-s}\|n^s a(Tb)-T(n^s a)b\|\\ &\le n^{-s}\varepsilon\|n^s a\|^p\|b\|^p = n^{s(p-1)}\varepsilon\|a\|^p\|b\|^p\\ &\longrightarrow 0 \quad \text{as } n\longrightarrow\infty.\end{aligned} \tag{2.3}$$

Hence $a(Tb)=(Ta)b$, proving T is a multiplier. □

Proof of Theorem 1.2. Since $T(0)=0$, it suffices to show that $a(Tb)=(Ta)b$ for all $a,b\in A\setminus\{0\}$. So, fix $a,b\in A\setminus\{0\}$ arbitrarily. In this case, inequalities (2.1) and (2.2) are also valid for $p<0$. Recall that we assume $\|0\|^p=\infty$, and hence

$$x[T(\lambda a)-\lambda Ta]=0, \qquad [T(\lambda a)-\lambda Ta]x=0, \tag{2.4}$$

for $\lambda\in\mathbb{C}\setminus\{0\}$ and $x\in A\setminus\{0\}$. Note that (2.4) is also true for $x=0$. Since A is without order, we thus obtain $T(\lambda a)=\lambda Ta$ for all $\lambda\in\mathbb{C}\setminus\{0\}$. An argument similar to (2.3) shows $a(Tb)=(Ta)b$, and the proof is complete. □

Remark 2.1. A result similar to Theorem 1.1 need not be true for $p=1$, that is, there exists an approximate multiplier which is not an exact multiplier. More explicitly, to each $\varepsilon>0$ there corresponds a function $f:\mathbb{C}\to\mathbb{C}$ which is not a multiplier such that

$$|z_1 f(z_2)-f(z_1)z_2| \le \varepsilon|z_1||z_2| \tag{2.5}$$

for all $z_1,z_2\in\mathbb{C}$. Fix $\varepsilon>0$ arbitrarily. By the continuity of the function $t\mapsto e^{it}$, there corresponds a δ with $0<\delta<1$ such that $|t|<2\pi(1-\delta)$ implies $|e^{it}-1|<\varepsilon$. With this δ, we define the mapping $f:\mathbb{C}\to\mathbb{C}$ by

$$f(z)=\begin{cases}0 & z=0,\\ |z|e^{i\delta\theta} & z\in\mathbb{C}\setminus\{0\},\end{cases} \tag{2.6}$$

where $\theta\in[0,2\pi)$ denotes the argument of z. Then we see that f satisfies inequality (2.5) for all $z_1,z_2\in\mathbb{C}$. Since the case where $z_1=0$ or $z_2=0$ is trivial, we only consider $z_1,z_2\in\mathbb{C}\setminus\{0\}$. If $z_j=|z_j|e^{i\theta_j}$ for $j=1,2$, then we get

$$|z_1 f(z_2)-f(z_1)z_2| = |z_1||z_2||e^{i(1-\delta)(\theta_1-\theta_2)}-1|. \tag{2.7}$$

Note that $|\theta_1-\theta_2|<2\pi$. By the definition of δ, we obtain (2.5), which implies that f is an approximate multiplier. Moreover, f is not an exact multiplier, and hence Theorem 1.1 does not hold for $p=1$ in general.

Remark 2.2. Suppose A is a unital commutative Banach algebra. If $f : A \to A$ is a mapping such that

$$\|af(b) - f(a)b\| \le \varepsilon \|a\| \|b\| \quad (a, b \in A) \tag{2.8}$$

for some $\varepsilon \ge 0$, then there is an exact multiplier $T : A \to A$ such that

$$\|f(a) - Ta\| \le \varepsilon \|a\| \quad (a \in A). \tag{2.9}$$

Indeed, let $e \in A$ be a unit element. Taking $b = e$ in (2.8), we obtain

$$\|af(e) - f(a)\| \le \varepsilon \|a\| \quad (a \in A). \tag{2.10}$$

If we consider the mapping $T : A \to A$ defined by

$$Ta = af(e) \quad (a \in A), \tag{2.11}$$

then T is a multiplier such that $\|f(a) - Ta\| \le \varepsilon \|a\|$ for all $a \in A$.

Proof of Theorem 1.3. Suppose $p \ne 1$. By (1.3), it follows from a theorem of Rassias [3] and Gajda [1] that there exists a unique additive mapping $T : A \to A$ such that (1.5) holds. So, we need to show that $a(Tb) = (Ta)b$ for all $a, b \in A$. Since T is additive, $T(0) = 0$, and hence it is enough to consider $a, b \in A \setminus \{0\}$. Put $s = (1-p)/|1-p|$ and fix $a, b \in A \setminus \{0\}$ arbitrarily. Since T is additive, we see that $Ta = n^{-s}T(n^s a)$ for each $n \in \mathbb{N}$. Now it follows from (1.5) that

$$\|n^{-s} f(n^s b) - Tb\| \le n^{-s} \frac{2\varepsilon}{|2 - 2^p|} \|n^s b\|^p = n^{s(p-1)} \frac{2\varepsilon}{|2 - 2^p|} \|b\|^p \tag{2.12}$$

for all $n \in \mathbb{N}$, and hence

$$\|n^{-s} f(n^s b) - Tb\| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{2.13}$$

Since f is an approximate multiplier, we get

$$\begin{aligned} \|n^{-s} af(n^s b) - f(a)b\| &= n^{-s} \|af(n^s b) - f(a) n^s b\| \\ &\le n^{-s} \varepsilon \|a\|^p \|n^s b\|^p \\ &= n^{s(p-1)} \varepsilon \|a\|^p \|b\|^p \end{aligned} \tag{2.14}$$

for all $n \in \mathbb{N}$. Hence,

$$\|n^{-s} af(n^s b) - f(a)b\| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{2.15}$$

Now it follows from (2.13) and (2.15) that

$$\begin{aligned} &\|a(Tb) - (Ta)b\| \\ &\quad \le \|a\| \|Tb - n^{-s} f(n^s b)\| + \|n^{-s} af(n^s b) - f(a)b\| + \|f(a)b - (Ta)b\| \\ &\quad \longrightarrow \|f(a)b - (Ta)b\| \quad \text{as } n \longrightarrow \infty. \end{aligned} \tag{2.16}$$

By (1.5), we obtain

$$||a(Tb) - (Ta)b|| \le \frac{2\varepsilon}{|2 - 2^p|} \|a\|^p \|b\|. \tag{2.17}$$

An argument similar to (2.3) implies $\|a(Tb) - (Ta)b\| = 0$, proving T is a multiplier. □

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