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ON SOME PROPERTIES OF BANACH OPERATORS. II

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Using the notion of a Banach operator, we have obtained a decompositional property of a Hilbert space, and the equality of two invertible bounded linear multiplicative operators on a normed algebra with identity.

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1. Introduction. This paper is a continuation of our earlier work [7] on Banach operators. We recall that if X is a normed space and $\alpha : X \to X$ is a mapping, then following [4], α is said to be a *Banach operator* if there exists a constant k such that $0 \le k < 1$ and $\|\alpha^2(x) - \alpha(x)\| \le k\|\alpha(x) - x\|$ for all $x \in X$. Banach operators are generalizations of contraction maps and play an important role in the fixed point theory; their consideration goes back to Cheney and Goldstein [2] in the study of proximity maps on convex sets (see [4] and the references therein).

In [7], we established some decompositional properties of a normed space using Banach operators. We showed that if α is a linear Banach operator on a normed space X, then $N(\alpha-1) = N((\alpha-1)^2)$, $N(\alpha-1) \cap R(\alpha-1) = (0)$ and in case X is finite dimensional, we get the decomposition $X = N(\alpha-1) \oplus R(\alpha-1)$, where $N(\alpha-1)$ and $R(\alpha-1)$ denote the null space and the range space of $(\alpha-1)$, respectively, and 1 denotes the identity operator on X. In [7, Proposition 2.3], we proved a decompositional property of a general bounded linear operator on a Hilbert space, namely, if α is a bounded linear operator on a Hilbert space H such that α and α^* have common fixed points, then $N(\alpha-1) + R(\alpha-1)$ is dense in H.

In this paper, also we prove some properties of Banach operators on a Hilbert space. We show (Proposition 2.1) that if α is a bounded linear Banach operator on a Hilbert space H such that the sets of fixed points of α and α^* are the same, then H admits a decomposition $H = N(\alpha-1) \oplus M$, where $M = \overline{R(\alpha-1)}$, ($\overline{R(\alpha-1)}$ denotes the closure of $R(\alpha-1)$). It follows as a corollary of Proposition 2.1 that α commutes with both orthogonal projections onto $N(\alpha-1)$ and onto M.

As in [7], we also study the operator equation $\alpha + c\alpha^{-1} = \beta + c\beta^{-1}$ for a pair of invertible bounded linear multiplicative Banach operators α and β on a normed algebra X with identity, where c is an appropriate real or complex number. We prove the following result (Proposition 2.3): assume that $\alpha(x) + c\alpha^{-1}(x) = \beta(x) + c\beta^{-1}(x)$ for all $x \in X$, where c is a real or complex number such that $|c| \ge 1$, $\|\alpha\|^2 \le |c|/2$, $\|\beta\|^2 \le |c|/2$. If β is inner, then $\alpha = \beta$. We briefly recall that this operator equation has been extensively studied for automorphisms on von Neumann algebras. We refer to [1, 5, 6] for more details about this operator equation.

2. The results

Proposition 2.1. *Let α be a bounded linear Banach operator on a Hilbert space H such that the sets of fixed points of α and α^* are the same. Then the following hold:*

(i) $N(\alpha-1)\perp R(\alpha-1)$,

(ii) $H = N(\alpha-1)\oplus M$, *where* $M = \overline{R(\alpha-1)}$.

Proof. To prove (i), let $x \in N(\alpha-1)$ and $y \in R(\alpha-1)$. Then $\alpha(x) = x$ and $y = \alpha(z)-z$ for some $z \in H$. Therefore, $\alpha^*(x) = x$ and hence

$$\langle x,y\rangle = \langle x,\alpha(z)-z\rangle = \langle x,\alpha(z)\rangle - \langle x,z\rangle = \langle \alpha^*(x),z\rangle - \langle x,z\rangle = \langle x,z\rangle - \langle x,z\rangle = 0. \tag{2.1}$$

Thus $N(\alpha-1)\perp R(\alpha-1)$.

To prove (ii), it is enough to show that $N(\alpha-1) = M^\perp$. By (i) and the continuity of α, $N(\alpha-1)\perp M$. So, $N(\alpha-1)\subseteq M^\perp$. Conversely, assume that $z \in M^\perp$. Then $\langle z,y\rangle = 0$ for all $y \in M$; in particular, $\langle z,(\alpha-1)x\rangle = 0$ for all $x \in H$ because $R(\alpha-1)\subseteq M$. Thus $\langle z,\alpha(x)\rangle = \langle z,x\rangle$ for all $x \in H$. So, $\langle \alpha^*(z),x\rangle = \langle z,x\rangle$ for all $x \in H$. This shows that $\langle \alpha^*(z)-z,x\rangle = 0$ for all $x \in H$. Therefore, $\alpha^*(z)-z = 0$ or $\alpha^*(z) = z$, that is, z is a fixed point of α^* and hence by assumption, $\alpha(z) = z$, that is, $z \in N(\alpha-1)$. So, $M^\perp \subseteq N(\alpha-1)$. Thus $N(\alpha-1) = M^\perp$ and hence $H = N(\alpha-1)\oplus M$. □

Corollary 2.2. *Let α be a bounded linear Banach operator on a Hilbert space H such that the sets of fixed points of α and α^* are the same. Then α commutes with both orthogonal projections, onto $N(\alpha-1)$ and onto M.*

Proof. Since $R(\alpha-1)$ is α-invariant, so is M. Also, $M^\perp = N(\alpha-1)$ is α-invariant. Thus M reduces α and hence α commutes with both orthogonal projections, onto $N(\alpha-1)$ and onto M [3]. □

It easily follows that the orthogonal projection P onto $N(\alpha-1)$ is the largest orthogonal projection such that $\alpha P = P$.

We conclude this paper with a result about an operator equation similar to the one considered in [7].

Proposition 2.3. *Let α,β be invertible bounded linear multiplicative Banach operators on a normed algebra X with identity such that $\alpha(x)+c\alpha^{-1}(x) = \beta(x)+c\beta^{-1}(x)$ for all $x \in X$, where c is a real or complex number with $|c| \ge 1$, $\|\alpha\|^2 \le |c|/2$, $\|\beta\|^2 \le |c|/2$. If β is inner, then $\alpha = \beta$.*

Proof. It follows from [7, Proposition 3.2] that α and β commute. Therefore,

$$\begin{aligned}(\alpha\beta-c)(\beta^{-1}-\alpha^{-1})(x) &= \alpha(x)-\alpha\beta\alpha^{-1}(x)-c\beta^{-1}(x)+c\alpha^{-1}(x)\\ &= \alpha(x)-\beta\alpha(\alpha^{-1}(x))-c\beta^{-1}(x)+c\alpha^{-1}(x)\\ &= \alpha(x)-\beta(x)-c\beta^{-1}(x)+c\alpha^{-1}(x)\\ &= (\alpha(x)+c\alpha^{-1}(x))-(\beta(x)+c\beta^{-1}(x)) = 0.\end{aligned} \tag{2.2}$$

Put $(\beta^{-1}-\alpha^{-1})(x) = y$. Then we obtain $(\alpha\beta - c)(y) = 0$, that is, $\alpha\beta(y) = cy$. Therefore, by assumption, we get $|c|\|y\| = \|cy\| = \|\alpha\beta(y)\| \le \|\alpha\|\|\beta\|\|y\| \le (|c|/2)\|y\|$, that is, $|c|\|y\| \le (|c|/2)\|y\|$. This implies that $\|y\| = 0$ and hence $(\beta^{-1}-\alpha^{-1})(x) = 0$ for all $x \in X$, that is, $\beta^{-1}(x) = \alpha^{-1}(x)$ for all $x \in X$. Since α is onto, therefore replacing x by $\alpha(x)$, we get $\beta^{-1}(\alpha(x)) = x$ or $\alpha(x) = \beta(x)$ for all $x \in X$. □

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