SOME MULTIPLE GAUSSIAN HYPERGEOMETRIC GENERALIZATIONS OF BUSCHMAN-SRIVASTAVA THEOREM

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Some generalizations of Bailey's theorem involving the product of two Kummer functions ${}_1F_1$ are obtained by using Watson's theorem and Srivastava's identities. Its special cases yield various new transformations and reduction formulae involving Pathan's quadruple hypergeometric functions $F_p^{(4)}$, Srivastava's triple and quadruple hypergeometric functions $F^{(3)}$, $F^{(4)}$, Lauricella's quadruple hypergeometric function $F_A^{(4)}$, Exton's multiple hypergeometric functions $X_{E:G;H}^{A:B;D}$, K_{10}, K_{13}, X_8, ${}^{(k)}H_2^{(n)}$, ${}^{(k)}H_4^{(n)}$, Erdélyi's multiple hypergeometric function $H_{n,k}$, Khan and Pathan's triple hypergeometric function $H_4^{(P)}$, Kampé de Fériet's double hypergeometric function $F_{E:G;H}^{A:B;D}$, Appell's double hypergeometric function of the second kind F_2, and the Srivastava-Daoust function $F_{D:E^{(1)};E^{(2)};\dots;E^{(n)}}^{A:B^{(1)};B^{(2)};\dots;B^{(n)}}$. Some known results of Buschman, Srivastava, and Bailey are obtained.

1. Introduction

In what follows, for the sake of brevity, (a_A) denotes the sequence of A parameters given by $a_1,a_2,a_3,\dots,a_A$ in the contracted notation. Denominator parameters are neither zero nor negative integers and the Pochhammer symbol $(a)_n$ is defined by

$$(a)_n=\frac{\Gamma(a+n)}{\Gamma(a)}=\begin{cases}1 & \text{if } n=0,\\ a(a+1)\cdots(a+n-1) & \text{if } n=1,2,3,\dots,\end{cases} \tag{1.1}$$

where the notation Γ is used for the gamma function.

We will use the following power series form of multiple hypergeometric function [13, 14]:

$$F_{D:E^{(1)};\dots;E^{(n)}}^{A:B^{(1)};\dots;B^{(n)}}\left(\begin{matrix}[(a_A):\theta^{(1)},\dots,\theta^{(n)}]:\left[\left(b_{B^{(1)}}^{(1)}\right):\Phi^{(1)}\right];\dots;\left[\left(b_{B^{(n)}}^{(n)}\right):\Phi^{(n)}\right]; \\ [(d_D):\Psi^{(1)},\dots,\Psi^{(n)}]:\left[\left(e_{E^{(1)}}^{(1)}\right):\delta^{(1)}\right];\dots;\left[\left(e_{E^{(n)}}^{(n)}\right):\delta^{(n)}\right];\end{matrix}\; z_1,\dots,z_n\right)$$
$$=\sum_{m_1,\dots,m_n=0}^{\infty}\Xi(m_1,\dots,m_n)\frac{z_1^{m_1}}{(m_1)!},\dots,\frac{z_n^{m_n}}{(m_n)!}, \tag{1.2}$$

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where, for convenience,

$$\Xi(m_1,\dots,m_n) = \frac{\prod_{j=1}^{A}(a_j)_{m_1\theta_j^{(1)}+\cdots+m_n\theta_j^{(n)}}\prod_{j=1}^{B^{(1)}}\left(b_j^{(1)}\right)_{m_1\Phi_j^{(1)}}\cdots\prod_{j=1}^{B^{(n)}}\left(b_j^{(n)}\right)_{m_n\Phi_j^{(n)}}}{\prod_{j=1}^{D}(d_j)_{m_1\Psi_j^{(1)}+\cdots+m_n\Psi_j^{(n)}}\prod_{j=1}^{E^{(1)}}\left(e_j^{(1)}\right)_{m_1\delta_j^{(1)}}\cdots\prod_{j=1}^{E^{(n)}}\left(e_j^{(n)}\right)_{m_n\delta_j^{(n)}}}, \quad (1.3)$$

the coefficients $\theta_j^{(k)}$, $j=1,2,\dots,A$; $\Phi_j^{(k)}$, $j=1,2,\dots,B^{(k)}$; $\Psi_j^{(k)}$, $j=1,2,\dots,D$; $\delta_j^{(k)}$, $j=1,2,\dots,E^{(k)}$; for all $k\in\{1,2,\dots,n\}$, are zero and real constants (positive and negative) [13, equations (5), (6), (7), (8), (9), (19), (20), (21), pages 270–272] and $(b_{B^{(k)}}^{(k)})$ abbreviates the array of $B^{(k)}$ parameters $b_j^{(k)}$, $j=1,2,\dots,B^{(k)}$; for all $k\in\{1,2,\dots,n\}$, with similar interpretations for others. Here F_2, $F_{E:G;H}^{A:B;D}$, $X_{E:G;H}^{A:B;D}$, $F_A^{(3)}$, $F^{(3)}$, $F_A^{(4)}$, K_{10}, K_{13}, ${}^{(k)}H_4^{(n)}$, ${}^{(k)}H_2^{(n)}$, X_8, $H_{n,k}$, $H_4^{(P)}$, $F^{(4)}$, $F_P^{(4)}$, and $F_{D:E^{(1)};E^{(2)};\dots;E^{(n)}}^{A:B^{(1)};B^{(2)};\dots;B^{(n)}}$ are Appell's double hypergeometric function (see [3, (2), page 73]; see also [9, (139), page 265], [13, (3), page 23]), Kampé de Fériet's double hypergeometric function (see [15, (26), page 423]; see also [13, (28), page 27]), Exton's double hypergeometric function [6, (1.2), page 137], Lauricella's triple hypergeometric function [5, (2.1.1), page 41], Srivastava's triple hypergeometric function [10, page 428], Lauricella's quadruple hypergeometric function [13, (1), page 33], Exton's multiple hypergeometric functions (see [5, (3.3.10), (3.3.13), page 79], [13, (41), page 40], [5, (3.5.3), page 97], [13, (190), page 324]), Erdélyi's multiple hypergeometric function [13, (19), page 36], triple hypergeometric function of Khan-Pathan [7, (1.1), page 85], Srivastava's quadruple hypergeometric function [11, (1.2), pages 35–36], Pathan's quadruple hypergeometric function [8, (1.2), page 172], and Srivastava-Daoust's multiple hypergeometric function [13, (21), page 37], respectively.

The present note is devoted to the investigation of general multiple series identities which extend and generalize theorems of Buschman, Srivastava, and Bailey. The theorem given in Section 2 will be seen to be extremely useful, in that most properties of hypergeometric series carry over naturally and simply for these identities and provide connections with various classes of well-known hypergeometric functions and even new representations for special cases of these functions. Some applications of this theorem are given in Section 3. Clearly, the same procedure could have been utilized to extend many more results on hypergeometric functions. But, instead, we deduce fifteen special cases in Section 4.

2. General multiple series identities

Motivated by the works of Buschman, Srivastava, and Bailey, we will establish the following theorem for multiple series which is more generalized than multiple Gaussian hypergeometric functions $F^{(3)}$, $F^{(4)}$, and $F_P^{(4)}$.

Theorem 2.1. *Let $S_r(\alpha i+\beta j+\gamma k+\delta p)$, $r=1,2,\dots,7$; $S_r(0)\neq 0$, be arbitrary complex-valued functions, let independent coefficients α, β, γ, δ, θ_t, $t=1,2,3,\dots,12$, be arbitrary real constants, let x, y, z be complex variables, let c, f be arbitrary independent complex parameters (where $2f\neq 0,-1,\pm2,\pm3,\pm4,\dots$), and let any values of numerator and denominator parameters and variables x, y, z leading to the results which do not make sense be tacitly* *excluded, then*

$$\sum_{i,j,k,p=0}^{\infty} S_1(\theta_1 i+\theta_2 j+\theta_1 k+\theta_3 p)S_2(\theta_4 i+\theta_5 j+\theta_4 k)S_3(\theta_6 i+\theta_6 k+\theta_7 p)$$
$$\times S_4(\theta_8 j+\theta_9 p)S_5(\theta_{10} i+\theta_{10} k)S_6(\theta_{11} j)S_7(\theta_{12} p)\frac{(-1)^k(c)_i(f)_k x^j y^{i+k} z^p}{(2c)_i(2f)_k i!j!k!p!}$$

$$= \sum_{i,j,p=0}^{\infty} S_1(2\theta_1 i+\theta_2 j+\theta_3 p)S_2(2\theta_4 i+\theta_5 j)S_3(2\theta_6 i+\theta_7 p)S_4(\theta_8 j+\theta_9 p)$$
$$\times S_5(2\theta_{10} i)S_6(\theta_{11} j)S_7(\theta_{12} p)\frac{((c+f)/2)_i((1+c+f)/2)_i x^j (y^2/4)^i z^p}{(c+f)_i(c+1/2)_i(f+1/2)_i i!j!p!} \tag{2.1}$$

$$= \sum_{i,j,p=0}^{\infty}\sum_{u=0}^{1} S_1(2\theta_1 i+2\theta_2 j+\theta_2 u+\theta_3 p)S_2(2\theta_4 i+2\theta_5 j+\theta_5 u)S_3(2\theta_6 i+\theta_7 p)$$
$$\times S_4(2\theta_8 j+\theta_2 u+\theta_9 p)S_5(2\theta_{10} i)S_6(2\theta_{11} j+\theta_{11} u)S_7(\theta_{12} p) \tag{2.2}$$
$$\times\frac{x^u((c+f)/2)_i((1+c+f)/2)_i(x^2/4)^j(y^2/4)^i z^p}{u!(c+f)_i(c+1/2)_i(f+1/2)_i i!((1+u)/2)_j((2+u)/2)_j p!}$$

$$= \sum_{i,j,p=0}^{\infty}\sum_{u,w=0}^{1} S_1(2\theta_1 i+2\theta_2 j+\theta_2 u+2\theta_3 p+\theta_3 w)S_2(2\theta_4 i+2\theta_5 j+\theta_5 u)$$
$$\times S_3(2\theta_6 i+2\theta_7 p+\theta_7 w)S_4(2\theta_8 j+\theta_8 u+2\theta_9 p+\theta_9 w)S_5(2\theta_{10} i)$$
$$\times S_6(2\theta_{11} j+\theta_{11} u)S_7(2\theta_{12} p+\theta_{12} w)$$
$$\times\frac{x^u z^w((c+f)/2)_i((1+c+f)/2)_i(x^2/4)^j(y^2/4)^i(z^2/4)^p}{u!w!(c+f)_i(c+1/2)_i(f+1/2)_i i!((1+u)/2)_j((2+u)/2)_j((1+w)/2)_p((2+w)/2)_p}, \tag{2.3}$$

provided that each multiple series involved converges absolutely.

Proof of Theorem 2.1. Let L denote the left-hand side of (2.1). Then using the series identities [8, Lemma 10(1), page 56] (i.e., replacing i by $i-k$)

$$\sum_{i,j,k,p=0}^{\infty} A(i,j,k,p) = \sum_{i,j,p=0}^{\infty}\sum_{k=0}^{i} A(i-k,j,k,p), \tag{2.4}$$

we may write

$$L = \sum_{i,j,p=0}^{\infty} S_1(\theta_1 i+\theta_2 j+\theta_3 p)S_2(\theta_4 i+\theta_5 j)S_3(\theta_6 i+\theta_7 p)S_4(\theta_8 j+\theta_9 p)S_5(\theta_{10} i)$$
$$\times S_6(\theta_{11} j)S_7(\theta_{12} p)\frac{(c)_i x^j y^i z^p}{(2c)_i i!j!p!}\,{}_3F_2\left[\begin{matrix}-i, f, 1-2c-i; \\ 1-c-i, 2f;\end{matrix}\quad 1\right]. \tag{2.5}$$

Using Watson's summation theorem [12, (26), page 95]

$$
{}_3F_2\left[\begin{matrix} A,B,C; \\ \dfrac{A+B+1}{2},2C; \end{matrix}\; 1\right]
= \frac{\Gamma(1/2)\Gamma(1/2+C)\Gamma((1+A+B)/2)\Gamma((1-A-B)/2+C)}{\Gamma((1+A)/2)\Gamma((1+B)/2)\Gamma((1-A)/2+C)\Gamma((1-B)/2+C)} \tag{2.6}
$$

in (2.5), we get

$$
\begin{aligned}
L = \sum_{i,j,p=0}^{\infty} & S_1(\theta_1 i+\theta_2 j+\theta_3 p)S_2(\theta_4 i+\theta_5 j)S_3(\theta_6 i+\theta_7 p)S_4(\theta_8 j+\theta_9 p)S_5(\theta_{10} i)S_6(\theta_{11} j) \\
& \times S_7(\theta_{12} p)\frac{(c)_i x^j y^i z^p}{(2c)_i i!j!p!}\frac{\Gamma(1/2)\Gamma(1/2+f)\Gamma(1-i-c)\Gamma(i+c+f)}{\Gamma(1/2-i/2)\Gamma(1-i/2-c)\Gamma(1/2+f+i/2)\Gamma(i/2+c+f)}.
\end{aligned} \tag{2.7}
$$

Now applying the well-known series identity

$$
\sum_{i=0}^{\infty} A(i) = \sum_{i=0}^{\infty} A(2i) + \sum_{i=0}^{\infty} A(2i+1) \tag{2.8}
$$

in (2.7), we have

$$
\begin{aligned}
L = \sum_{i,j,p=0}^{\infty} & S_1(2\theta_1 i+\theta_2 j+\theta_3 p)S_2(2\theta_4 i+\theta_5 j)S_3(2\theta_6 i+\theta_7 p)S_4(\theta_8 j+\theta_9 p)S_5(2\theta_{10} i) \\
& \times S_6(\theta_{11} j)S_7(\theta_{12} p)\frac{(c)_i x^j y^i z^p}{(2c)_i i!j!p!}\frac{\Gamma(1/2)\Gamma(1/2+f)\Gamma(1-2i-c)\Gamma(2i+c+f)}{\Gamma(1/2-i)\Gamma(1-i-c)\Gamma(1/2+f+i)\Gamma(i+c+f)} \\
+ \sum_{i,j,p=0}^{\infty} & S_1(2\theta_1 i+\theta_1+\theta_2 j+\theta_3 p)S_2(2\theta_4 i+\theta_4+\theta_5 j)S_3(2\theta_6 i+\theta_6+\theta_7 p) \\
& \times S_4(\theta_8 j+\theta_9 p)S_5(2\theta_{10} i+\theta_{10})S_6(\theta_{11} j)S_7(\theta_{12} p) \\
& \times \frac{(c)_i x^j y^i z^p}{(2c)_i i!j!p!}\frac{\Gamma(1/2)\Gamma(1/2+f)\Gamma(-2i-c)\Gamma(2i+c+f+1)}{\Gamma(-i)\Gamma(1/2-c-i)\Gamma(1+f+i)\Gamma(1/2+c+f+i)}.
\end{aligned} \tag{2.9}
$$

Second-power series on the right-hand side of (2.9) vanishes due to the presence of $1/\Gamma(-i) = 0$, if $i = 0,1,2,\ldots$, we may then write

$$
\begin{aligned}
L = \sum_{i,j,p=0}^{\infty} & S_1(2\theta_1 i+\theta_2 j+\theta_3 p)S_2(2\theta_4 i+\theta_5 j)S_3(2\theta_6 i+\theta_7 p)S_4(\theta_8 j+\theta_9 p)S_5(2\theta_{10} i) \\
& \times S_6(\theta_{11} j)S_7(\theta_{12} p)\frac{(c)_i x^j y^i z^p}{(2c)_i i!j!p!}\frac{\Gamma(1/2)\Gamma(1/2+f)\Gamma(1-2i-c)\Gamma(2i+c+f)}{\Gamma(1/2-i)\Gamma(1-i-c)\Gamma(1/2+f+i)\Gamma(i+c+f)},
\end{aligned} \tag{2.10}
$$

and after replacing the gamma functions by Pochhammer symbols, we get the right-hand side of (2.1).

Again, now applying Srivastava's identities (see [12, pages 194–197]; see also [14, (8), page 214, (12), page 217])

$$\begin{aligned}\sum_{j=0}^{\infty} A(j) &= \sum_{u=0}^{1}\sum_{j=0}^{\infty} A(2j+u),\\ \sum_{j,p=0}^{\infty} B(j,p) &= \sum_{u=0}^{1}\sum_{w=0}^{1}\sum_{j,p=0}^{\infty} B(2j+u,2p+w)\end{aligned} \tag{2.11}$$

in (2.1), and then replacing the gamma functions by Pochhammer symbols, we get the right-hand sides of (2.2) and (2.3), respectively. □

3. Applications of formulas (2.1), (2.2), and (2.3)

3.1. Buschman-Srivastava theorem associated with Srivastava's function $F^{(3)}$. In formulas (2.1) and (2.2), setting $\theta_1 = \theta_2 = \theta_3 = \cdots = \theta_{12} = 1$, $S_1(i+j+k+p) = S_3(i+k+p) = S_4(j+p) = S_7(p) = 1$, $S_2(j+i+k) = [(a_A)]_{j+i+k}/[(b_B)]_{j+i+k}$, $S_5(i+k) = [(d_D)]_{i+k}/[(e_E)]_{i+k}$, $S_6(j) = [(g_G)]_j/[(h_H)]_j$, and $z = 0$, we get

$$F^{(3)}\left[\begin{matrix}(a_A)::—;(d_D);—:(g_G);\ c;\ f;\\ (b_B)::—;(e_E);—:(h_H);2c;2f;\end{matrix}\ x,y,-y\right]$$

$$= X^{A:G;2D+2}_{B:H;2E+3}\left[\begin{matrix}(a_A):(g_G); & \Delta(2;c+f),\Delta[2;(d_D)];\\ (b_B):(h_H);c+f,c+\frac{1}{2}, & f+\frac{1}{2},\Delta[2;(e_E)];\end{matrix}\ x,\frac{y^2}{4^{(E-D+1)}}\right] \tag{3.1}$$

$$\begin{aligned}&= \sum_{u=0}^{1}\frac{[(a_A)]_u[(g_G)]_u\, x^u}{[(b_B)]_u[(h_H)]_t\, u!}F^{2A:2G;2D+2;}_{2B:2H+1;2E+3;}\\ &\quad\times\left[\begin{matrix}\Delta[2;(a_A)+u]: & \Delta[2;(g_G)+u];\\ \Delta[2;(b_B)+u]:\Delta^*(2;1+u), & \Delta[2;(h_H)+u];\end{matrix}\right.\\ &\qquad\left.\begin{matrix}\Delta(2;c+f),\Delta[2;(d_D)]; \\ c+f,c+\frac{1}{2},f+\frac{1}{2},\Delta[2;(e_E)];\end{matrix}\ \frac{4^{(A+G)}x^2}{4^{(1+B+H)}},\frac{4^{(A+D)}y^2}{4^{(1+B+E)}}\right],\end{aligned} \tag{3.2}$$

provided that denominator parameters are neither zero nor negative integers, for convenience, the symbol $\Delta(m;b)$ abbreviates the array of m parameters given by $b/m,(b+1)/m,(b+2)/m,\ldots,(b+m-1)/m$, where $m = 1,2,3,\ldots$ and $[(a_A)]_n$ denotes the product of a Pochhammer symbol given by $[(a_A)]_n = (a_1)_n(a_2)_n(a_3)_n,\ldots,(a_A)_n$.

The asterisk in $\Delta^*(N;j+1)$ represents the fact that the (denominator) parameter N/N is always omitted, $0 \le j \le (N-1)$, so that the set $\Delta^*(N;j+1)$ obviously contains only $(N-1)$ parameters [14, page 214].

The notation $\Delta[N;(b_B)]$ denotes the array of BN parameters [14, (8), page 47 and pages 193–194] given by $\Delta(N;b_1),\Delta(N;b_2),\ldots,\Delta(N;b_B)$; similar interpretations for others.

3.2. Buschman-Srivastava theorem associated with Srivastava's function $F^{(4)}$. In formula (2.3), setting $\theta_1 = \theta_2 = \theta_3 = \cdots = \theta_{12} = 1$, $S_2(i+j+k) = S_3(i+k+p) = 1$, $S_1(i+j+k+p) = [(a_A)]_{i+j+k+p}/[(b_B)]_{j+i+k+p}$, $S_4(j+p) = [(m_M)]_{j+p}/[(n_N)]_{j+p}$, $S_5(i+k) = [(d_D)]_{i+k}/[(e_E)]_{i+k}$, $S_6(j) = [(g_G)]_j/[(h_H)]_j$, and $S_7(p) = [(q_Q)]_p/[(r_R)]_p$, we have

$$
\begin{aligned}
&F^{(4)}\left[\begin{matrix}(a_A)::\ c;(d_D);\ (g_G);(m_M):\ \ f;(d_D);(q_Q);(m_M);\\ (b_B)::2c;\ (e_E);(h_H);\ (n_N):2f;\ (e_E);\ (r_R);\ (n_N);\end{matrix}\ y,x,-y,z\right]\\
&\quad=\sum_{u=0}^{1}\sum_{w=0}^{1}\frac{[(a_A)]_{u+w}[(m_M)]_{u+w}[(g_G)]_u[(q_Q)]_w x^u z^w}{[(b_B)]_{u+w}[(n_N)]_{u+w}[(h_H)]_u[(r_R)]_w u!w!}\\
&\qquad\times F^{(3)}\left[\begin{matrix}\Delta[2;(a_A)+u+w]::\text{—};\Delta[2;(m_M)+u+w];\ \text{—}:\\ \Delta[2;(b_B)+u+w]::\text{—};\ \Delta[2;(n_N)+u+w];\ \text{—}:\end{matrix}\right.\\
&\qquad\qquad\begin{matrix}\Delta(2;c+f),\Delta[2;(d_D)]; & \Delta[2;(g_G)+u];\\ c+f,c+\frac{1}{2},f+\frac{1}{2},\Delta[2;(e_E)];\Delta^*(2;1+u),\Delta[2;(h_H)+u];\end{matrix}\\
&\qquad\qquad\left.\begin{matrix}\Delta[2;(q_Q)+w];\\ \Delta^*(2;1+w),\Delta[2;(r_R)+w];\end{matrix}\ \frac{4^{(A+D)}y^2}{4^{(1+B+E)}},\frac{4^{(A+M+G)}x^2}{4^{(1+B+N+H)}},\frac{4^{(A+M+Q)}z^2}{4^{(1+B+N+R)}}\right],
\end{aligned}
\tag{3.3}
$$

provided that denominator parameters are neither zero nor negative integers.

3.3. Buschman-Srivastava theorem associated with Pathan's function $F_P^{(4)}$. In formula (2.3), putting $\theta_1 = \theta_2 = \theta_3 = \cdots = \theta_{12} = 1$, $S_4(j+p) = S_5(i+k) = 1$, $S_1(j+p+i+k) = [(a_A)]_{j+p+i+k}/[(b_B)]_{j+p+i+k}$, $S_2(i+k+j) = [(g_G)]_{i+k+j}/[(h_H)]_{i+k+j}$, $S_3(p+i+k) = [(d_D)]_{p+i+k}/[(e_E)]_{p+i+k}$, $S_6(j) = [(m_M)]_j/[(n_N)]_j$, and $S_7(p) = [(q_Q)]_p/[(r_R)]_p$, we have

$$
\begin{aligned}
&F_P^{(4)}\left[\begin{matrix}(a_A)::\text{—};(d_D);(g_G);\text{—}:(m_M);(q_Q);\ c;\ \ f;\\ (b_B)::\text{—};(e_E);(h_H);\text{—}:\ (n_N);\ (r_R);2c;2f;\end{matrix}\ x,z,y,-y\right]\\
&\quad=\sum_{u=0}^{1}\sum_{w=0}^{1}\frac{[(a_A)]_{u+w}[(g_G)]_u[(m_M)]_u[(d_D)]_w[(q_Q)]_w x^u z^w}{[(b_B)]_{u+w}[(h_H)]_u[(n_N)]_u[(e_E)]_w[(r_R)]_w u!w!}\\
&\qquad\times F^{(3)}\left[\begin{matrix}\Delta[2;(a_A)+u+w]::\text{—};\Delta[2;(d_D)+w];\ \Delta[2;(g_G)+u]:\\ \Delta[2;(b_B)+u+w]::\text{—};\ \Delta[2;(e_E)+w];\Delta[2;(h_H)+u]:\end{matrix}\right.\\
&\qquad\qquad\begin{matrix}\Delta[2;(m_M)+u]; & \Delta[2;(q_Q)+w];\\ \Delta^*(2;1+u),\Delta[2;(n_N)+u];\Delta^*(2;1+w),\Delta[2;(r_R)+w];\end{matrix}\\
&\qquad\qquad\left.\begin{matrix}\Delta(2;c+f);\\ c+f,c+\frac{1}{2},f+\frac{1}{2};\end{matrix}\ \frac{4^{(A+G+M)}x^2}{4^{(1+B+H+N)}},\frac{4^{(A+D+G)}z^2}{4^{(1+B+E+H)}},\frac{4^{(A+D+Q)}y^2}{4^{(1+B+E+R)}}\right],
\end{aligned}
\tag{3.4}
$$

provided that denominator parameters are neither zero nor negative integers.

4. Special cases

(i) Setting $x = E = D = G = H = 0$ in (3.1) or (3.2), we get

$$F^{A:1;1}_{B:1;1}\left[\begin{matrix}(a_A): & c; & f; \\ (b_B): & 2c; & 2f;\end{matrix}\; y,-y\right] = {}_{2A+2}F_{2B+3}\left[\begin{matrix}\Delta(2;c+f),\Delta[2;(a_A)]; \\ c+f,c+\frac{1}{2},f+\frac{1}{2},\Delta[2;(b_B)];\end{matrix}\; 4^{A-B-1}y^2\right], \tag{4.1}$$

which is known as Buschman-Srivastava theorem (see [4, page 438]; see also [11, (47), page 31]).

(ii) Setting $A = B = 0$ in (4.1), we get

$${}_1F_1\left[\begin{matrix}c; \\ 2c;\end{matrix}\; y\right]{}_1F_1\left[\begin{matrix}f; \\ 2f;\end{matrix}\; -y\right] = {}_2F_3\left[\begin{matrix}\frac{1}{2}(c+f),\frac{1}{2}(c+f+1); \\ c+f,c+\frac{1}{2},f+\frac{1}{2};\end{matrix}\; \frac{y^2}{4}\right], \tag{4.2}$$

which is a known result of Bailey (see [1, (2.11), page 246]; see also [11, (186), page 322]).

(iii) Setting $B = 0$, $A = 1$, $a_1 = a$ in (4.1), we get

$$F_2(a;c,f;2c,2f;y,-y) = {}_4F_3\left[\begin{matrix}\Delta(2;a),\Delta(2;f+c); \\ c+f,c+\frac{1}{2},f+\frac{1}{2};\end{matrix}\; y^2\right], \tag{4.3}$$

which is another result of Bailey (see [2, (4.4), page 239]; see also [11, (191), page 323]).

(iv) Setting $A = G = H = 1$, $a_1 = a$, $g_1 = g$, $h_1 = h$, $B = D = E = 0$ in (3.1), we get

$$F^{(3)}_{A}[a,g,c,f;h,c,f;y,x,-y] = X^{1:2;1}_{0:3;1}\left[\begin{matrix}a: & \Delta(2;c+f);\; g; \\ —: & c+f,c+\frac{1}{2},f+\frac{1}{2};\; h;\end{matrix}\; \frac{y^2}{4},x\right]. \tag{4.4}$$

(v) Setting $B = D = E = G = N = Q = 0$, $A = M = H = R = 1$, $a_1 = a$, $m_1 = m$, $h_1 = h$, $r_1 = r$ in (3.3), we get

$$\begin{aligned} &K_{10}[a,a,a,a;m,m,c,f;h,r,2c,2f;x,z,y,-y] \\ &= \sum_{u=0}^{1}\sum_{w=0}^{1}\frac{(a)_{u+w}(m)_{u+w}x^u y^w}{(h)_u(r)_u u!w!} \\ &\quad\times F^{(3)}\left[\begin{matrix}\Delta(2;a+u+w)::—;\Delta(2;m+u+w);—: & \Delta(2;c+f); & \overline{\quad\quad}; & \overline{\quad\quad}; \\ —::—;\;—;—: & c+f,c+\frac{1}{2},f+\frac{1}{2}; & \Delta(2;h+u),\Delta^*(2;1+u); & \Delta(2;r+u),\Delta^*(2;1+w);\end{matrix}\; y^2,x^2,z^2\right]. \end{aligned} \tag{4.5}$$

(vi) Setting $B = D = E = M = H = R = 0$, $A = N = G = Q = 1$, $a_1 = a$, $n_1 = n$, $g_1 = g$, $q_1 = q$ in (3.3), we get

$$\begin{aligned} &K_{13}[a,a,a,a;g,q,c,f;n,n,2c,2f;x,z,y,-y] \\ &\quad = \sum_{u,w=0}^{1} \frac{(a)_{u+w}(g)_u(q)_w x^u y^w}{(n)_{u+w}u!w!} \\ &\qquad \times F^{(3)}\left[\begin{matrix} \Delta(2;a+u+w)::\text{—}; & \text{—} & ;\text{—}: \\ \text{—} & ::\text{—};\Delta(2;n+u+w); & \text{—}: \end{matrix}\right. \\ &\qquad\qquad \left.\begin{matrix} \Delta(2;c+f); \ \Delta(2;g+u); \ \Delta(2;q+u); \\ c+f,c+\frac{1}{2},f+\frac{1}{2};\Delta^*(2;1+u);\Delta^*(2;1+w); \end{matrix} \ y^2,x^2,z^2\right]. \end{aligned} \tag{4.6}$$

(vii) Setting $A = M = N = Q = R = 1$, $a_1 = a$, $m_1 = m$, $n_1 = n$, $q_1 = q$, $r_1 = r$, and $B = D = E = G = H = 0$ in (3.4), we get

$$\begin{aligned} &F_A^{(4)}[a;f,q,m,c;2f,r,n,2c;-y,z,x,y] \\ &\quad = \sum_{u=0}^{1}\sum_{w=0}^{1} \frac{(a)_{u+w}(m)_u(q)_w x^u z^w}{(n)_u(r)_w u!w!} \\ &\qquad \times F^{(3)}\left[\begin{matrix} \Delta(2;a+u+w)::\text{—};\text{—};\text{—}: & \Delta(2;m+u); \\ \text{—} \quad ::\text{—};\text{—};\text{—}: & \Delta^*(2;1+u),\Delta(2;n+u); \end{matrix}\right. \\ &\qquad\qquad \left.\begin{matrix} \Delta(2;q+w); & \Delta(2;c+f); \\ \Delta^*(2;1+w),\Delta(2;r+w); & c+f,c+\frac{1}{2},f+\frac{1}{2}; \end{matrix} \ x^2,z^2,y^2\right]. \end{aligned} \tag{4.7}$$

(viii) In (2.1), setting $S_1(\theta_1 i+\theta_2 j+\theta_1 k+\theta_3 p) = S_3(\theta_6 i+\theta_6 k+\theta_7 p) = S_4(\theta_8 j+\theta_9 p) = S_5(\theta_{10} i+\theta_{10} k) = S_7(\theta_{12} p) = 1$ and $z = 0$, we get

$$\begin{aligned} &\sum_{i,j,k=0}^{\infty} S_2(\theta_4 i+\theta_5 j+\theta_4 k)S_6(\theta_{11} j)\frac{(c)_i(f)_k x^j y^i(-y)^k}{(2c)_i(2f)_k i!j!k!} \\ &\quad = \sum_{i,j=0}^{\infty} S_2(2\theta_4 i+\theta_5 j)S_6(\theta_{11} j)\frac{((c+f)/2)_i((1+c+f)/2)_i x^j (y^2/4)^i}{(c+f)_i(c+1/2)_i(f+1/2)_i i!j!}. \end{aligned} \tag{4.8}$$

(ix) In (4.8), setting $\theta_4 = \theta_{11} = 1$, $\theta_5 = 2$, $S_2(2j+i+k) = (a)_{2j+i+k}$, $S_6(j) = 1/(b)_j$, we get

$$\begin{aligned} &X_8[a,c,f;b,2c,2f;x,y,-y] \\ &\quad = F_{0:3;1}^{2:2;0}\left[\begin{matrix} \Delta(2;a): & \frac{c+f}{2},\frac{c+f+1}{2}; & \text{—}; \\ \text{—} & :c+f,c+\frac{1}{2},f+\frac{1}{2}; & b; \end{matrix} \ y^2,4x\right]. \end{aligned} \tag{4.9}$$

(x) In (4.8), setting $\theta_4 = \theta_{11} = 1$, $\theta_5 = -1$, $S_2(i+k-j) = (a)_{i+k-j}$, $S_6(j) = (b)_j(g)_j$, we get

$$H_{3,2}[a,c,f,b,g;2c,2f;y,-y,x] = F^{1:2;2}_{0:3;0}\left(\begin{array}{lll} [a:2,-1]: & \left[\frac{c+f}{2}:1\right],\left[\frac{c+f+1}{2}:1\right];[b:1],[g:1]; & \\ \quad — & :[c+f:1],\left[c+\frac{1}{2}:1\right],\left[f+\frac{1}{2}:1\right]; \quad — \quad ; & \frac{y^2}{4},x \end{array}\right). \tag{4.10}$$

(xi) In (2.1), setting $S_2(\theta_4 i+\theta_5 j+\theta_4 k) = S_3(\theta_6 i+\theta_6 k+\theta_7 p) = S_4(\theta_8 j+\theta_9 p) = S_5(\theta_{10} i + \theta_{10} k) = 1$, we get

$$\begin{aligned} &\sum_{i,j,k,p=0}^{\infty} S_1(\theta_1 i+\theta_2 j+\theta_1 k+\theta_3 p)S_6(\theta_{11} j)S_7(\theta_{12} p)\frac{(c)_i(f)_k x^j y^i(-y)^k z^p}{(2c)_i(2f)_k i!j!k!p!} \\ &\quad = \sum_{i,j=0}^{\infty} S_1(2\theta_1 i+\theta_1 t+\theta_2 j+\theta_3 p)S_6(\theta_{11} j)S_7(\theta_{12} p) \\ &\qquad \times\frac{((c+f)/2)_i((1+c+f)/2)_i x^j\left(y^2/4\right)^i z^p}{(c+f)_i(c+1/2)_i(f+1/2)_i i!j!p!}. \end{aligned} \tag{4.11}$$

(xii) In (4.11), setting $\theta_1 = \theta_3 = \theta_{11} = \theta_{12} = 1$, $\theta_2 = 2$, $S_1(2j+i+k+p) = (a)_{2j+i+k+p}$, $S_6(j) = 1/(b)_j$, $S_7(p) = (d)_p/(h)_p$, we get

$$H_4^{(4)}[a,c,f,d;b,2c,2f,h;x,y,-y,z] = F^{1:2;0;1}_{0:3;1;1}\left(\begin{array}{lll} [a:2,2,1]: & \left[\frac{c+f}{2}:1\right],\left[\frac{c+f+1}{2}:1\right]; \ — \ ;[d:1]; & \\ \quad — & :[c+f:1],\left[c+\frac{1}{2}:1\right],\left[f+\frac{1}{2}:1\right];[b:1];[h:1]; & \frac{y^2}{4},x,z \end{array}\right). \tag{4.12}$$

(xiii) In (4.11), setting $\theta_1 = \theta_{11} = \theta_{12} = 1$, $\theta_2 = \theta_3 = 2$, $S_1(2j+2p+i+k) = (a)_{2j+2p+i+k}$, $S_6(j) = 1/(b)_j$, $S_7(p) = 1/(d)_p$, we get

$$H_4^{(4)}[a,c,f;b,d,2c,2f;x,z,y,-y] = F^{(3)}\left[\begin{array}{ll} \Delta(2;a)::—;—;—:—;—; \quad \frac{c+f}{2},\frac{c+f+1}{2}; & \\ \quad — \quad ::—;—;—: \quad b; \quad d;c+f,c+\frac{1}{2},f+\frac{1}{2}; & 4x,4z,y^2 \end{array}\right]. \tag{4.13}$$

(xiv) In (4.11), setting $\theta_1 = \theta_{11} = \theta_{12} = 1$, $\theta_2 = \theta_3 = -1$, $S_1(i+k-j-p) = (a)_{i+k-j-p}$, $S_6(j) = (b)_j(d)_j$, $S_7(p) = (g)_p(h)_p$, we get

$$\begin{aligned} &H_{4,2}[a,c,f,b,g,d,h;2c,2f;y,-y,x,z] \\ &= F^{1:2;2;2}_{0:0;3;0}\left(\begin{matrix} [a:-1,2,-1]:[g:1],[h:1]; \\ \text{—} \quad : \quad \text{—} \quad ; \\ \left[\frac{c+f}{2}:1\right],\left[\frac{c+f+1}{2}:1\right];[b:1],[d:1]; \\ [c+f:1],\left[c+\frac{1}{2}:1\right],\left[f+\frac{1}{2}:1\right]; \quad \text{—} \quad ; \end{matrix} \; z,\frac{y^2}{4},x\right). \end{aligned} \tag{4.14}$$

(xv) In (4.11), setting $\theta_1 = \theta_3 = \theta_{11} = \theta_{12} = 1$, $\theta_2 = -1$, $S_1(i+k+p-j) = (a)_{i+k+p-j}$, $S_6(j) = (g)_j(h)_j$, $S_7(p) = (b)_p/(d)_p$, we get

$$\begin{aligned} &H_{4,3}[a,c,f,b,g,h;2c,2f,d;y,-y,z,x] \\ &= F^{1:2;2;1}_{0:0;3;1}\left(\begin{matrix} [a:-1,2,1]:[g:1],[h:1]; \\ \text{—} \quad : \quad \text{—} \quad ; \\ \left[\frac{c+f}{2}:1\right],\left[\frac{c+f+1}{2}:1\right];[b:1]; \\ [c+f:1],\left[c+\frac{1}{2}:1\right],\left[f+\frac{1}{2}:1\right];[d:1]; \end{matrix} \; x,\frac{y^2}{4},z\right). \end{aligned} \tag{4.15}$$

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References

[1] W. N. Bailey, *Products of generalized hypergeometric series*, Proc. London Math. Soc. (2) **28** (1928), 242–254.

[2] ———, *On the sum of a terminating* ${}_3F_2(1)$, Quart. J. Math. Oxford Ser. (2) **4** (1953), 237–240.

[3] ———, *Generalized Hypergeometric Series*, Cambridge Tracts in Mathematics and Mathematical Physics, no. 32, Stechert-Hafner, New York, 1964.

[4] R. G. Buschman and H. M. Srivastava, *Series identities and reducibility of Kampé de Fériet functions*, Math. Proc. Cambridge Philos. Soc. **91** (1982), no. 3, 435–440.

[5] H. Exton, *Multiple Hypergeometric Functions and Applications*, Ellis Horwood, Chichester; Halsted Press (John Wiley & Sons), New York, 1976.

[6] ———, *Reducible double hypergeometric functions and associated integrals*, An. Fac. Ciênc. Univ. Porto **63** (1982), no. 1–4, 137–143.

[7] B. Khan and M. A. Pathan, *On certain hypergeometric function of three variables. I*, Soochow J. Math. **7** (1981), 85–91.

[8] M. A. Pathan, *On a transformation of a general hypergeometric series of four variables*, Nederl. Akad. Wetensch. Indag. Math. **41** (1979), no. 2, 171–175.

[9] E. D. Rainville, *Special Functions*, The Macmillan, New York, 1960, reprinted by Chelsea Publishing, New York, 1971.

[10] H. M. Srivastava, *Generalized Neumann expansions involving hypergeometric functions*, Proc. Cambridge Philos. Soc. **63** (1967), 425–429.

[11] ———, *A formal extension of certain generating functions. II*, Glasnik Mat. Ser. III **6** (1971), no. 1, 35–44.

[12] ———, *A note on certain identities involving generalized hypergeometric series*, Nederl. Akad. Wetensch. Indag. Math. **41** (1979), no. 2, 191–201.

[13] H. M. Srivastava and P. W. Karlsson, *Multiple Gaussian Hypergeometric Series*, Mathematics and Its Applications, Ellis Horwood, Chichester; Halsted Press (John Wiley & Sons), New York, 1985.

[14] H. M. Srivastava and H. L. Manocha, *A Treatise on Generating Functions*, Mathematics and Its Applications, Ellis Horwood, Chichester; Halsted Press (John Wiley & Sons), New York, 1984.

[15] H. M. Srivastava and R. Panda, *An integral representation for the product of two Jacobi polynomials*, J. London Math. Soc. (2) **12** (1976), no. 4, 419–425.

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