ANOTHER SIMPLE PROOF OF THE QUINTUPLE PRODUCT IDENTITY

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We give a simple proof of the well-known quintuple product identity. The strategy of our proof is similar to a proof of Jacobi (ascribed to him by Enneper) for the triple product identity.

1. Introduction

The well-known quintuple product identity can be stated as follows. For $z \neq 0$ and $|q| < 1$,

$$\begin{aligned} f(z,q) :&= \prod_{n=0}^{\infty}(1-q^{2n+2})(1-zq^{2n+1})\left(1-\frac{1}{z}q^{2n+1}\right)(1-z^2q^{4n})\left(1-\frac{1}{z^2}q^{4n+4}\right) \\ &= \sum_{n=-\infty}^{\infty} q^{3n^2+n}\left(z^{3n}q^{-3n}-z^{-3n-1}q^{3n+1}\right). \end{aligned} \tag{1.1}$$

The quintuple identity has a long history and, as Berndt [5] points out, it is difficult to assign priority to it. It seems that a proof of the identity was first published in H. A. Schwartz's book in 1893 [19]. Watson gave a proof in 1929 in his work on the Rogers-Ramanujan continued fractions [20]. Since then, various proofs have appeared. To name a few, Carlitz and Subbarao gave a simple proof in [8]; Andrews [2] gave a proof involving basic hypergeometric functions; Blecksmith, Brillhart, and Gerst [7] pointed out that the quintuple identity is a special case of their theorem; and Evans [11] gave a short and elegant proof by using complex function theory. For updated history up to the late 80s and early 90s, see Hirschhorn [15] (in which the author also gave a beautiful generalization of the quintuple identity) and Berndt [5] (in which the author also gave a proof that ties the quintuple identity to the larger framework of the work of Ramanujan on q-series and theta functions; see also [1]). Since the early 90s, several authors gave different new proofs of the quintuple identity; see [6, 13, 12, 17]. See also Cooper's papers [9, 10] for the connections between the quintuple product identity and Macdonald identities [18]. Quite recently, Kongsiriwong and Liu [16] gave an interesting proof that makes use of the cube root of unity.

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Our proof below is similar to the proof of the triple product identity by Jacobi (ascribed to him by Enneper; see the book by Hardy and Wright [14]). First, we set $f(z,q) = \sum a_n z^n$. Then, by considering the symmetry of $f(z,q)$ as an infinite product, we relate all a_n to a single coefficient a_0. All we need to do is to evaluate a_0. This is achieved by comparing $f(i,q)$ and $f(-q^4,q^4)$.

2. Proof of the identity

The first step of our proof is pretty standard, for example, see [16] or [4]. Set

$$f(z,q) = \sum_{n=-\infty}^{\infty} a_n z^n. \tag{2.1}$$

From the definition of $f(z,q)$, one can show that

$$f(z,q) = qz^3 f(zq^2,q), \qquad f(z,q) = -z^2 f\left(\frac{1}{z},q\right). \tag{2.2}$$

The first equality implies that for each n,

$$a_{3n} = a_0 q^{3n^2-2n}, \qquad a_{3n+1} = a_1 q^{3n^2}, \qquad a_{3n+2} = a_2 q^{3n^2+2n}, \tag{2.3}$$

whereas the second equality implies that $a_2 = -a_0$ and $a_1 = 0$. By putting all these together, we have

$$f(z,q) = a_0(q) \sum_{n=-\infty}^{\infty} q^{3n^2+n}\left(z^{3n}q^{-3n} - z^{-3n-1}q^{3n+1}\right). \tag{2.4}$$

Comparing (2.4) to (1.1) shows that all we need to do is to prove that $a_0(q) = 1$. Note that $a_0(0) = 1$.

We can also write (2.4) in the following forms (which will be useful later):

$$f(z,q) = a_0(q) \sum_{n=-\infty}^{\infty} q^{3n^2-2n}\left(z^{3n} - \frac{1}{z^{3n-2}}\right) \tag{2.5a}$$

$$= a_0(q) \sum_{n=-\infty}^{\infty} q^{3n^2+n}\left(\left(\frac{z}{q}\right)^{3n} - \left(\frac{q}{z}\right)^{3n+1}\right). \tag{2.5b}$$

To obtain (2.5a), we let $n \to n-1$ in the *second* sum on the right-hand side of (2.4). Equation (2.5b) is simply another way of writing (2.4).

By putting $z = i$ in (2.5a), we have, on the one hand,

$$f(i,q) = a_0(q) \sum_{n=-\infty}^{\infty} q^{3n^2-2n}\left(i^{3n} - \frac{1}{i^{3n-2}}\right) = 2a_0(q) \sum_{n=-\infty}^{\infty} q^{12n^2-4n}(-1)^n. \tag{2.6}$$

Note that, in the second equality, we have used the fact that

$$i^{3n} - \frac{1}{i^{3n-2}} = 2\cos\frac{3n}{2}\pi = \begin{cases} 0, & \text{if } n \text{ is odd}, \\ 2(-1)^{n/2}, & \text{if } n \text{ is even}. \end{cases} \tag{2.7}$$

On the other hand, let us evaluate $f(i,q)$ as an infinite product:

$$
\begin{aligned}
f(i,q) &= 2\prod_{n=1}^{\infty}(1-q^{2n})(1-iq^{2n-1})(1+iq^{2n-1})(1+q^{4n})^2 \\
&= 2\prod_{n=1}^{\infty}(1-q^{2n})(1+q^{4n-2})(1+q^{4n})^2 \\
&= 2\prod_{n=1}^{\infty}(1-q^{2n})(1+q^{2n})(1+q^{4n}) \\
&= 2\prod_{n=1}^{\infty}(1-q^{4n})(1+q^{4n}) \\
&= 2\prod_{n=1}^{\infty}(1-q^{8n}).
\end{aligned}
\tag{2.8}
$$

Note that we have used the fact that $\prod(1+q^{4n-2})(1+q^{4n}) = \prod(1+q^{2n})$ to derive the third equality.

By putting (2.6) and (2.8) together, we arrive at

$$
\prod_{n=1}^{\infty}(1-q^{8n}) = a_0(q)\sum_{n=-\infty}^{\infty} q^{12n^2-4n}(-1)^n. \tag{2.9}
$$

Note that, at this stage, if we appeal to Euler's pentagonal number theorem (with q replaced by q^8) [4], we have

$$
\prod_{n=1}^{\infty}(1-q^{8n}) = \sum_{n=-\infty}^{\infty} q^{12n^2-4n}(-1)^n. \tag{2.10}
$$

Compared with (2.9), we see that $a_0(q) = 1$. Alternatively, we can find $a_0(q)$ by evaluating $f(z,q)$ in a different way.

Precisely, let us evaluate $f(-q^4,q^4)$. By (2.5b), we have

$$
\begin{aligned}
f(-q^4,q^4) &= a_0(q^4)\sum_{n=-\infty}^{\infty} q^{12n^2+4n}\left((-1)^{3n}-(-1)^{3n+1}\right) \\
&= 2a_0(q^4)\sum_{n=-\infty}^{\infty} q^{12n^2+4n}(-1)^n \\
&= 2a_0(q^4)\sum_{n=-\infty}^{\infty} q^{12n^2-4n}(-1)^n.
\end{aligned}
\tag{2.11}
$$

For the second equality, we have used the fact that $(-1)^{3n}-(-1)^{3n+1} = 2(-1)^n$. For the last equality, we let $n \to -n$ in the second line.

Again, evaluating $f(-q^4,q^4)$ as an infinite product gives

$$
f(-q^4,q^4) = 2\prod_{n=1}^{\infty}(1-q^{8n})\left(\prod_{k=1}^{\infty}(1+q^{8k})(1-q^{16k-8})\right)^2 = 2\prod_{n=1}^{\infty}(1-q^{8n}). \tag{2.12}
$$

The second equality is obtained by direct computation, similar to the derivation of (2.8). Alternatively, it follows from an identity due to Euler (e.g., see [3, page 60]) that

$$\prod_{k=1}^{\infty}(1+q^{8k})(1-q^{16k-8}) = 1. \tag{2.13}$$

By putting together (2.11) and (2.12), we have

$$\prod_{n=1}^{\infty}(1-q^{8n}) = a_0(q^4)\sum_{n=-\infty}^{\infty} q^{12n^2-4n}(-1)^n. \tag{2.14}$$

Finally, by comparing (2.9) and (2.14), we conclude that $a_0(q) = a_0(q^4)$. This implies that

$$a_0(q) = a_0(q^4) = a_0(q^{16}) = \cdots = a_0(q^{4^k}) = \cdots = a_0(0) = 1 \tag{2.15}$$

and (1.1) is proven.

We remark that the evaluation of $f(-q^4,q^4)$ above also gives a simple proof of Euler's pentagonal number theorem.

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References

[1] C. Adiga, B. C. Berndt, S. Bhargava, and G. N. Watson, *Chapter* 16 *of Ramanujan's second notebook: theta-functions and q-series*, Mem. Amer. Math. Soc. **53** (1985), no. 315, v+85.

[2] G. E. Andrews, *Applications of basic hypergeometric functions*, SIAM Rev. **16** (1974), no. 4, 441–484.

[3] ______, *q-Series: Their Development and Application in Analysis, Number Theory, Combinatorics, Physics, and Computer Algebra*, CBMS Regional Conference Series in Mathematics, vol. 66, American Mathematical Society, Rhode Island, 1986.

[4] G. E. Andrews, R. Askey, and R. Roy, *Special Functions*, Encyclopedia of Mathematics and Its Applications, vol. 71, Cambridge University Press, Cambridge, 1999.

[5] B. C. Berndt, *Ramanujan's Notebooks. Part III*, Springer, New York, 1991.

[6] S. Bhargava, *A simple proof of the quintuple product identity*, J. Indian Math. Soc. (N.S.) **61** (1995), no. 3-4, 226–228.

[7] R. Blecksmith, J. Brillhart, and I. Gerst, *New proofs for two infinite product identities*, Rocky Mountain J. Math. **22** (1992), no. 3, 819–823.

[8] L. Carlitz and M. V. Subbarao, *A simple proof of the quintuple product identity*, Proc. Amer. Math. Soc. **32** (1972), no. 1, 42–44.

[9] S. Cooper, *The Macdonald identities for G_2 and some extensions*, New Zealand J. Math. **26** (1997), no. 2, 161–182.

[10] ______, *Multiplying Macdonald identities*, Special Functions and Differential Equations (Madras, 1997), Allied Publishers, New Delhi, 1998, pp. 73–82.

[11] R. J. Evans, *Theta function identities*, J. Math. Anal. Appl. **147** (1990), no. 1, 97–121.

[12] D. Foata and G.-N. Han, *Jacobi and Watson identities combinatorially revisited*, Sém. Lothar. Combin. **42** (1999), no. B42o, 12 pp.

[13] ———, *The triple, quintuple and septuple product identities revisited*, Sém. Lothar. Combin. **42** (1999), Article B42o, pp 12.

[14] G. H. Hardy and E. M. Wright, *An Introduction to the Theory of Numbers*, 5th ed., The Clarendon Press, Oxford University Press, New York, 1979.

[15] M. D. Hirschhorn, *A generalisation of the quintuple product identity*, J. Austral. Math. Soc. Ser. A **44** (1988), no. 1, 42–45.

[16] S. Kongsiriwong and Z.-G. Liu, *Uniform proofs of q-series-product identities*, Results Math. **44** (2003), no. 3-4, 312–339.

[17] V. E. Leininger and S. C. Milne, *Some new infinite families of η-function identities*, Methods Appl. Anal. **6** (1999), no. 2, 225–248.

[18] I. G. Macdonald, *Affine root systems and Dedekind's η-function*, Invent. Math. **15** (1972), 91–143.

[19] H. A. Schwarz, *Formeln und Lehrsätze zum Gebrauche der Elliptischen Funktionen. Nach Vorlesungen und Aufzeichnungen des Herrn Prof. K. Weierstrass*, Zweite Ausgabe, Erste Abteilung, Springer, Berlin, 1893.

[20] G. N. Watson, *Theorems stated by Ramanujan (VII): Theorems on continued fractions*, J. London Math. Soc. **4** (1929), 39–48.

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