REGULARITIES AND SUBSPECTRA FOR COMMUTATIVE BANACH ALGEBRAS

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We introduce regularities in commutative Banach algebras in such a way that each regularity defines a joint spectrum on the algebra that satisfies the spectral mapping formula.

1. Introduction

Let B be a complex commutative Banach algebra with unit element denoted by e. The space of linear continuous functionals on B is denoted by B'.

We call regularity in B every nontrivial open subset $R \subset B$ which satisfies the following conditions:

$$ab \in R \quad \text{iff } a \in R, b \in R, \tag{1.1}$$

$$R = R^{\#}, \quad \text{where } R^{\#} = \{b \in B \mid \forall \varphi \in B' \ \varphi(b) = 0 \Longrightarrow 0 \in \varphi(R)\}. \tag{1.2}$$

The set $G(B)$ of invertible elements of B is the main example of a regularity. As was proved in [4], the set of elements of B which are not topological zero divisors is also a regularity.

In the present paper, we investigate a construction of joint spectra in B by means of regularities in B.

Let $\sigma(a) = \{\mu \in \mathbb{C} \mid a - \mu e \notin G(B)\}$ be the ordinary spectrum in B.

Recall that according to the terminology introduced by Żelazko [6], a subspectrum τ in B is a mapping which associates to every k-tuple $(a_1,\dots,a_k) \in B^k$ a nonempty compact set $\tau(a_1,\dots,a_k)$ such that

(a) $\tau(a_1,\dots,a_k) \subset \prod_{i=1}^k \sigma(a_i)$,

(b) $\tau(p(a_1,\dots,a_k)) = p(\tau(a_1,\dots,a_k))$ for every polynomial mapping $p = (p_1,\dots,p_m): \mathbb{C}^k \to \mathbb{C}^m$.

In Theorem 2.1, we prove that an arbitrary subspectrum τ in B defines a regularity R_τ by the formula

$$R_\tau = \{a \in B \mid 0 \notin \tau(a)\}. \tag{1.3}$$

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Lemma 2.3 used in the proof of this theorem permits us to obtain an elementary proof of a theorem belonging to Żelazko which provides the complete description of all subspectra in B.

Let $M(B)$ be the space of multiplicative functionals on B as usually identified with the space of maximal ideals in B. $M(B)$ endowed with the Gelfand topology is a compact space. For $a \in B$, $\varphi \in M(B)$, we denote by $\hat{a}(\varphi) = \varphi(a)$ the Gelfand transform of a.

Theorem of Żelazko [6] states that for every subspectrum τ in B, there is a unique compact subset $K \subset M(B)$ such that

$$\tau(a_1,\dots,a_k) = \{(\varphi(a_1),\dots,\varphi(a_k)) \mid \varphi \in K\}, \tag{1.4}$$

for $(a_1,\dots,a_k) \in B^k$.

Our proof emphasizes the role played by the spectral mapping formula (b) while the original elegant proof in [6] involves more advanced methods.

The principal result of the paper is Theorem 4.1 which states that for an arbitrary regularity R the formula

$$\sigma_R = (a_1,\dots,a_k) = \{(\lambda_1,\dots,\lambda_k) \in \mathbb{C}^k \mid I_B(a_1-\lambda_1,\dots,a_k-\lambda_k) \cap R = \varnothing\} \tag{1.5}$$

defines a subspectrum in B. By $I_B(a_1-\lambda_1,\dots,a_k-\lambda_k)$ the ideal generated in B by the elements $a_1-\lambda_1,\dots,a_k-\lambda_k$ is denoted.

It follows that, given an arbitrary subspectrum τ, we can construct the regularity R_τ and then the subspectrum σ_{R_τ}. Both subspectra τ and σ_{R_τ}, according to Żelazko theorem, are uniquely determined by compact subsets of $M(B)$, say K and K_1, respectively.

We show that

$$K_1 = \widetilde{K} = \{\varphi \in M(B) \mid \forall a \in B\ \varphi(a) = 0 \implies 0 \in \hat{a}(K)\}. \tag{1.6}$$

The idea of describing spectra of single elements in a (noncommutative) Banach algebra by means of regularities appears in [1] by Kordula and Müller (see also [2]). The present paper is concerned with the case of a commutative Banach algebra and characterizes those regularities and corresponding spectra which admit an extension to a subspectrum.

2. Regularity corresponding to a subspectrum

Let τ be a subspectrum in a commutative unital Banach algebra B and let $R_\tau = \{a \in B \mid 0 \notin \tau(a)\}$.

For the completness of the paper, we include the elementary proof of the basic fact in the following theorem.

Theorem 2.1. *R_τ is a regularity.*

Proof. By the property (a) of subspectra, we have $\varnothing \neq \tau(a) \subset \sigma(a)$ for an arbitrary $a \in B$. In particular, $\varnothing \neq \tau(0) \subset \sigma(0) = \{0\}$. Hence $\tau(0) = \{0\}$ and $0 \notin R_\tau$.

For $|\mu| > \|a\|$, the element $a-\mu$ is invertible. So $0 \notin \sigma(a-\mu)$ and $0 \notin \tau(a-\mu)$ neither. The set R_τ is not empty and not equal to B.

The particular case of the spectral mapping formula (b) is the addition formula

$$\tau(a+b) = \{\lambda+\mu \mid (\lambda,\mu) \in \tau(a,b)\}, \tag{2.1}$$

corresponding to the polynomial $p(x,y) = x+y$.

On the other hand, by (a), we have

$$\tau(a,b) \subset \sigma(a)\times\sigma(b) \subset \sigma(a)\times D(0,\|b\|). \tag{2.2}$$

If $0 \notin \tau(a)$ and $\|b\| < \min\{|\lambda| \mid \lambda \in \tau(a)\}$, then $0 \notin \tau(a+b)$. The set R_τ is open.

We apply the spectral mapping formula in the case of $p(x,y) = xy$. We obtain

$$\tau(ab) = \{\lambda\mu \mid (\lambda,\mu) \in \tau(a,b)\}. \tag{2.3}$$

Immediately, we conclude that $0 \notin \tau(ab)$ if and only if $0 \notin \tau(a)$ and $0 \notin \tau(b)$.

The set R_τ has property (1.1).

The proof of property (1.2) is based on the following two lemmas.

Lemma 2.2. (1) If $(\mu_1,\dots,\mu_k) \in \tau(a_1,\dots,a_k)$ and $b_1,\dots,b_m \in B$, then there exist $\lambda_1,\dots,\lambda_m \in \mathbb{C}$ such that

$$(\mu_1,\dots,\mu_k,\lambda_1,\dots,\lambda_m) \in \tau(a_1,\dots,a_k,b_1,\dots,b_m). \tag{2.4}$$

(2) If $(0,\dots,0) \in \tau(a_1,\dots,a_k)$ and $b_1^i,\dots,b_k^i \in B$, $1 \le i \le m$, then

$$(0,\dots,0) \in \tau\left(\sum_{j=1}^{k} a_j b_j^1,\dots,\sum_{j=1}^{k} a_j b_j^m\right). \tag{2.5}$$

Proof. (1) The spectral mapping property (b) applied to the polynomial $p(x_1,\dots,x_k,y_1,\dots,y_m) = (x_1,\dots,x_k)$ gives us the first formula.

(2) We can find in $\tau(a_1,\dots,a_k,b_1^1,\dots,b_k^1,\dots,b_1^m,\dots,b_k^m)$ an element of the form $(0,\dots,0,\lambda_1^1,\dots,\lambda_k^1,\dots,\lambda_1^m,\dots,\lambda_k^m)$ using the first part of the lemma. If we apply the spectral mapping property to the polynomial mapping

$$p(x_1,\dots,x_k,y_1^1,\dots,y_k^1,\dots,y_1^m,\dots,y_k^m) = \left(\sum_{j=1}^{k} x_j y_j^1,\dots,\sum_{j=1}^{k} x_j y_j^m\right), \tag{2.6}$$

we obtain the desired property. □

Lemma 2.3. *Let $(0,\dots,0) \in \tau(a_1,\dots,a_k)$ for some $a_1,\dots,a_k \in B$. Then there exists a maximal ideal $J \in M(B)$ such that $I_B(a_1,\dots,a_k) \subset J$ and $(0,\dots,0) \in \tau(b_1,\dots,b_m)$ for arbitrary $b_1,\dots,b_m \in J$.*

Proof. If $b_1,\dots,b_m \in I_0 = I_B(a_1,\dots,a_k)$, then $(0,\dots,0) \in \tau(b_1,\dots,b_m)$ by Lemma 2.2(2). Denote by $\mathcal{J}$ the family of all ideals I in B which contain I_0 and have the property that $(0,\dots,0) \in \tau(b_1,\dots,b_m)$ for arbitrary $b_1,\dots,b_m \in I$. For every linearly ordered subfamily I_α, $\alpha \in S$ of $\mathcal{J}$, the set $\bigcup_{\alpha\in S} I_\alpha \in \mathcal{J}$. So by Kuratowski-Zorn lemma, the family $\mathcal{J}$ contains a maximal element J. It remains to prove that $J \in M(B)$. Suppose that J is not maximal.

There exists $c \in B$ such that $c+\lambda \notin J$ for all $\lambda \in \mathbb{C}$.

However, by Lemma 2.2(1), for arbitrary $c_1,\dots,c_k \in J$, the set

$$\delta(c_1,\dots,c_k) = \{\lambda \in \mathbb{C} \mid 0 \in \tau(c_1,\dots,c_k,c-\lambda)\} \tag{2.7}$$

is nonempty. It is a compact set as an intersection of the compact set $\tau(c_1,\dots,c_k,c)$ with a line.

By the spectral mapping property again,

$$\delta(c_1,\dots,c_k,b_1,\dots,b_m) \subset \delta(c_1,\dots,c_k) \cap \delta(b_1,\dots,b_m). \tag{2.8}$$

The family of compact sets $\delta(c_1,\dots,c_k)$ has the finite intersection property, so there exists $\lambda_0 \in \mathbb{C}$ which belongs to $\delta(c_1,\dots,c_k)$ for every $(c_1,\dots,c_k) \in J^k$.

By Lemma 2.2(2), the ideal generated by J and $c-\lambda_0$ also belongs to $\mathcal{J}$, which is a contradiction. Lemma 2.3 is proved. □

We return to the proof of Theorem 2.1.

Take $a \notin R_\tau$. In order to prove that $R_\tau^\# = R_\tau$, we must find a functional $\phi \in B'$ such that $\phi(a) = 0$ and $0 \notin \phi(R_\tau)$. By definition $0 \in \tau(a)$ and by Lemma 2.2(2), $(0,\dots,0) \in \tau(b_1,\dots,b_m)$ for all $b_1,\dots,b_m \in I_B(a)$. Lemma 2.3 says that in particular, a belongs to some $J \in M(B)$ that does not intersect R_τ. J being a maximal ideal, it is equal to the kernel of a linear (multiplicative) functional. The proof follows. □

Since the way from Lemma 2.3 to Żelazko theorem is short, we include the complete proof of this important theorem.

Theorem 2.4 [6]. *For every subspectrum τ on a commutative algebra B, there exists a unique compact set $K \subset M(B)$ such that*

$$\tau(a_1,\dots,a_k) = \{(\varphi(a_1),\dots,\varphi(a_k)) \mid \varphi \in K\}. \tag{2.9}$$

Proof. We define K as the set of those multiplicative functionals φ on B for which

$$(0,\dots,0) \in \tau(b_1,\dots,b_m) \quad \text{for arbitrary } b_1,\dots,b_m \in \ker\varphi. \tag{2.10}$$

If $(\mu_1,\dots,\mu_k) \in \tau(a_1,\dots,a_k)$, then $(0,\dots,0) \in \tau(a_1-\mu_1,\dots,a_k-\mu_k)$ and by Lemma 2.3, the ideal generated by $a_1-\mu_1,\dots,a_k-\mu_k$ is contained in the kernel of a multiplicative functional φ such that condition (2.10) is satisfied.

This proves that K is nonempty and

$$\tau(a_1,\dots,a_k) \subset \{(\varphi(a_1),\dots,\varphi(a_k)) \mid \varphi \in K\}. \tag{2.11}$$

Now suppose that $\varphi \in K$ and $a_1,\dots,a_k \in B$. Obviously, $a_1-\varphi(a_1),\dots,a_k-\varphi(a_k) \in \ker\varphi$ and $(0,\dots,0) \in \tau(a_1-\varphi(a_1),\dots,a_k-\varphi(a_k))$ that implies that $(\varphi(a_1),\dots,\varphi(a_k)) \in \tau(a_1,\dots,a_k)$.

It remains to prove that K is compact. Let $\phi \notin K$. There exist $b_1,\dots,b_m \in \ker\phi$ such that $(\phi(b_1),\dots,\phi(b_m)) \notin \tau(b_1,\dots,b_m)$. By the definition of the Gelfand topology and the compactness of $\tau(b_1,\dots,b_m)$, the property $(\psi(b_1),\dots,\psi(b_m)) \notin \tau(b_1,\dots,b_m)$ holds for ψ in some neighborhood of ϕ. The set K^c is open and K is compact. □

3. $\mathcal{F}$-rationally convex sets and regularities

Let X be a topological Hausdorff space and $\mathcal{F}$ a family of continuous functions on X. For an arbitrary set $C \subset X$, we define the $\mathcal{F}$-rationally convex hull of C as follows:

$$\widetilde{C} = \{x \in X \mid \forall f \in \mathcal{F} \ f(x) = 0 \Longrightarrow 0 \in f(C)\}. \tag{3.1}$$

The term $\mathcal{F}$-rationally convex hull is justified at least when C is compact and $\mathcal{F}$ is a vector space that contains constant functions.

The case is being $x \in \widetilde{C}$ if and only if

$$\left|\frac{f}{g}\right|(x) \leq \sup_{y \in C}\left|\frac{f}{g}\right|(y) \tag{3.2}$$

for every $f, g \in \mathcal{F}$ with $0 \notin g(C)$.

A subset $C \subset X$ is $\mathcal{F}$-rationally convex if $\widetilde{C} = C$.

The hull $R^{\#}$ that appears in the definition of a regularity is just the B'-rationally convex hull of a set $R \subset B$. Condition (1.2) means that every regularity is B'-rationally convex.

We observe some basic properties of regularities.

Proposition 3.1. *Let $\varnothing \neq R \subset B$.*

(1) *If $R \subset B$ satisfies (1.1), then it contains the set $G(B)$ of all invertible elements in B,*

(2) *if R is a regularity, then*

$$R^c = \bigcup_{I \in M(B), I \cap R = \varnothing} \{I\}. \tag{3.3}$$

Proof. (1) Let $b \in R$. Then $b = be \in R$. By condition(1.1), $e \in R$. If $a \in G(B)$, then $aa^{-1} = e \in R$ and again by (1.1), we obtain that $a \in R$.

(2) By the definition, $\bigcup_{I \in M(B), I \cap R = \varnothing}\{I\} \subset R^c$.

Let $a \notin R$. By condition (1.2), there exists $\phi \in B'$ such that $\phi(a) = 0$ and $0 \notin \phi(R)$. In particular, $(\ker\phi) \cap G(B) = \varnothing$. By Gleason-Kahane-Żelazko theorem, $\phi \in M(B)$ (see [5, page 81]) and

$$a \in \ker\phi \subset \bigcup_{I \in M(B), I \cap R = \varnothing} \{I\}. \tag{3.4}$$

□

Proposition 3.2. *A nontrivial open subset $R \subset B$ is a regularity if and only if $G(B) \subset R$ and $R^{\#} = R$.*

Proof. We show that the right-hand side condition implies the property (1.1). By condition (1.2) and Gleason-Kahane-Żelazko theorem, $ab \notin R$ if and only if $\varphi(ab) = 0$ for some $\varphi \in M(B)$ with $\ker\varphi \cap R = \varnothing$. This holds if and only if $\varphi(a) = 0$ or $\varphi(b) = 0$. The proof follows. □

In general, condition (1.1) does not imply (1.2). The simplest counterexample is the set $Q = B \setminus \{0\}$, where B is an integral domain.

We us observe the following hereditary property.

Proposition 3.3. *Let R be a regularity in B. Let A be a commutative unital Banach algebra and $\phi : A \to B$ a continuous homomorphism of algebras. Then $Q = \phi^{-1}(R)$ is a regularity in A.*

Proof. The set Q is obviously open in A. Moreover,

$$G(A) \subset \phi^{-1}(G(B)) \subset \phi^{-1}(R) = Q. \tag{3.5}$$

By Proposition 3.2, it is sufficent to prove that $Q^{\#} = Q$. To this end, given $a \notin Q$, we must find $\varphi \in A'$ such that $\varphi(a) = 0$ and $\ker\varphi \cap Q = \emptyset$. Since $\phi(a) \notin R$, there exists $\psi \in B'$ such that $\psi(\phi(a)) = 0$ and $\ker\psi \cap R = \emptyset$. Hence, $\varphi = \psi \circ \phi$ has the desired properties. □

We denote by $\widehat{B}$ the set of all Gelfand transforms of elements of B.

Theorem 3.4. *Let R be a regularity in B and let*

$$K = \{\varphi \in M(B) \mid 0 \notin \varphi(R)\} = \{\varphi \in M(B) \mid \ker\varphi \cap R = \emptyset\}. \tag{3.6}$$

Then K is a nonempty, compact, $\widehat{B}$-rationally convex set.

Proof. As we know by Proposition 3.1(2), R^c is a union of a nonempty family of maximal ideals of B which are precisely kernels of each $\varphi \in K$. Hence K is nonempty.

If $\varphi \in K^c$, then $\hat{a}(\varphi) = 0$ for some $a \in R$. If at the same time $\varphi \in \widehat{K}$, we obtain $0 \in \hat{a}(K)$. Hence, $\varphi_0(a) = 0$ for some $\varphi_0 \in K$. This contradics the definition of K, and so $\widehat{K} \setminus K = \emptyset$.

Take again $\varphi \in K^c$ and $a \in R$ such that $\varphi(a) = 0$. Since R is open, there exists $\delta > 0$ such that $\|a - b\| < \delta$ implies that $b \in R$. The set $V = \{\psi \in M(B) \mid |\hat{a}(\psi)| < \delta\}$ is a neighborhood of φ in $M(B)$. For $\psi \in V$, we have that $a - \psi(a) \in R$ and $\psi(a - \psi(a)) = 0$. It follows that $V \subset K^c$. So K^c is open, K is closed, and hence compact. □

4. Subspectrum associated to a regularity

Let R be a regularity in B. For $(a_1,\dots,a_k) \in B^k$, denote

$$\sigma_R(a_1,\dots,a_k) = \{(\lambda_1,\dots,\lambda_k) \in \mathbb{C}^k \mid I_B(a_1 - \lambda_1,\dots,a_k - \lambda_k) \cap R = \emptyset\}. \tag{4.1}$$

Theorem 4.1. *For an arbitrary regularity R in a commutative unital Banach algebra, σ_R is a subspectrum. If $K = \{\varphi \in M(B) \mid 0 \notin \varphi(R)\}$, then*

$$\sigma_R(a_1,\dots,a_k) = \{(\varphi(a_1),\dots,\varphi(a_k)) \mid \varphi \in K\}. \tag{4.2}$$

Proof. The condition (a) defining subspectrum is obviously satisfied because $G(B) \subset R$.

We introduce the operator $T : B \to C(K)$ by the formula

$$T(a) = \hat{a}\,|_K . \tag{4.3}$$

The operator T is a continuous homomorphism of algebras and its image A is a unital subalgebra of $C(K)$. If $a \in R$, then $T(a)$ nowhere vanishes on K, hence it is invertible in $C(K)$. Conversely, if $a \notin R$, then by the property $R^\# = R$ and Gleason-Kahane-Żelazko theorem, there exists $\varphi \in K$ such that $\varphi(a) = 0$. So $\hat{a}$ vanishes at $\varphi \in K$ and $T(a)$ is not invertible in $C(K)$. It follows that $T(R) = G(C(K)) \cap A$.

Theorem 3.1 in [3] states that the mapping

$$\tau(f_1,\dots,f_k) = \{(\lambda_1,\dots,\lambda_k) \in \mathbb{C}^k \mid I_A(f_1 - \lambda_1,\dots,f_k - \lambda_k) \cap G(C(K)) = \varnothing\} \tag{4.4}$$

is a subspectrum on A. We extend T on A^k in a natural way: $T(a_1,\dots,a_k) = (T(a_1),\dots, T(a_k))$.

Notice that

$$\sigma_R(a_1,\dots,a_k) = \tau(T(a_1),\dots,T(a_k)) = \tau(T(a_1,\dots,a_k)). \tag{4.5}$$

Then for an arbitrary polynomial mapping $p : \mathbb{C}^k \to \mathbb{C}^m$, we have

$$\begin{aligned} p(\sigma_R(a_1,\dots,a_k)) &= p(\tau(T(a_1),\dots,T(a_k))) = \tau(p(T(a_1),\dots,T(a_k))) \\ &= \tau(T(p(a_1,\dots,a_k))) = \sigma_R(p(a_1,\dots,a_k)). \end{aligned} \tag{4.6}$$

Thus the spectral mapping formula (b) holds for σ_R.

For every $\varphi \in K$ and $a_1,\dots,a_k \in B$, we have

$$I_B(a_1 - \varphi(a_1),\dots,a_k - \varphi(a_k)) \subset \ker\varphi. \tag{4.7}$$

The kernel of φ does not intersect R, so $(\varphi(a_1),\dots,\varphi(a_k)) \in \sigma_R(a_1,\dots,a_k)$.

Now suppose that $(\mu_1,\dots,\mu_k) \in \sigma_R(a_1,\dots,a_k)$, which implies that $(0,\dots,0) \in \sigma_R(a_1 - \mu_1,\dots,a_k - \mu_k)$. By Lemma 2.3, we know that the ideal $I_B(a_1 - \mu_1,\dots,a_k - \mu_k)$ is contained in the kernel of some $\varphi \in M(B)$ and $0 \in \sigma_R(b)$ for all $b \in \ker\varphi$. It follows that $\varphi \in K$ and $(\mu_1,\dots,\mu_k) = (\varphi(a_1),\dots,\varphi(a_k))$. □

The set K is exactly the compact set which describes the subspectrum σ_R in the sense of Żelazko theorem (Theorem 2.4).

In Section 3, we have studied the regularity associated with a given subspectrum. According to the definition, the regularity associated with σ_R is the set $R_1 = \{a \in B \mid 0 \notin \sigma_R(a)\}$. Obviously, $R \subset R_1$. If $a \in R_1$, then $I_B(a) \cap R \neq \varnothing$. There exists $b \in B$ such that $ab \in R$. Hence $a \in R$ by property (1.1). We conclude that $R_1 = R$.

It is well known that different subspectra can lead to the same set of regular elements. Let τ be the approximate point spectrum. The corresponding regularity R_τ is the set of all elements of B which are not topological zero divisors while the set K_τ defining τ via formula (2.9) is the set of maximal ideals which consists of joint topological zero divisors.

The spectrum σ_{R_τ} was studied in [4] and it corresponds to K equal to the set of all maximal ideals consisting of topological zero divisors, which in general differs from K_τ.

If $K \subset M(B)$ is compact and τ is the subspectrum defined by formula (2.9), then the regularity R_τ can be described as

$$\{a \in B \mid 0 \notin \hat{a}(K)\}. \tag{4.8}$$

Proposition 4.2. *Let K_1, $K_2 \subset M(B)$ and let*

$$R_i = \{a \in B \mid 0 \notin \hat{a}(K_i)\}, \tag{4.9}$$

$i = 1,2$. *Then $R_1 = R_2$ if and only if $\widetilde{K}_1 = \widetilde{K}_2$.*

Proof. Suppose that $R_1 = R_2$. It means that for $a \in B$, the Gelfand transform $\hat{a}$ vanishes on K_1 if and only if it vanishes on K_2. If $\hat{a}(\varphi) = 0$, then $\hat{a}(K_1)$ contains zero if and only if $\hat{a}(K_2)$ does. Hence $\widetilde{K}_1 = \widetilde{K}_2$.

Now suppose that $\widetilde{K}_1 = \widetilde{K}_2$ and that $a \notin R_1$. It follows that $\hat{a}(\varphi) = 0$ for some $\varphi \in K_1 \subset \widetilde{K}_2$. We obtain $0 \in \hat{a}(K_2)$. So $a \notin R_2$. This shows that $R_1^c \subset R_2^c$, and $R_2 \subset R_1$. Similarly, we can prove the opposite. Then $R_1 = R_2$. □

For a given regularity R in B, the subspectrum σ_R is the largest subspectrum having R as the corresponding regularity.

Proposition 4.3. *Let R be a regularity and let τ be a subspectrum such that $R_\tau = R$. Then for every k-tuple $(a_1,\dots,a_k) \in B^k$,*

$$\tau(a_1,\dots,a_k) \subset \sigma_R(a_1,\dots,a_k). \tag{4.10}$$

Proof. If R is a regularity, then according to Theorem 4.1,

$$\sigma_R(a_1,\dots,a_k) = \{(\varphi(a_1),\dots,\varphi(a_k)) \mid \varphi \in K\}, \tag{4.11}$$

where $K = \widetilde{K}$ as Theorem 3.4 asserts.

If τ is a subspectrum of the form

$$\tau(a_1,\dots,a_k) = \{(\varphi(a_1),\dots,\varphi(a_k)) \mid \varphi \in K_1\} \tag{4.12}$$

and $R_\tau = R$, then $\widetilde{K}_1 = \widetilde{K} = K$ by Proposition 4.2. In particular, $K_1 \subset K$ and

$$\tau(a_1,\dots,a_k) \subset \sigma_R(a_1,\dots,a_k). \tag{4.13}$$

□

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