ON THE MAXIMUM MODULUS OF A POLYNOMIAL AND ITS DERIVATIVES

K. K. DEWAN AND ABDULLAH MIR

Received 18 February 2005 and in revised form 5 April 2005

Let $f(z)$ be an arbitrary entire function and $M(f,r) = \max_{|z|=r} |f(z)|$. For a polynomial $P(z)$ of degree n, having no zeros in $|z| < k$, $k \geq 1$, Bidkham and Dewan (1992) proved $\max_{|z|=r} |P'(z)| \leq (n(r+k)^{n-1}/(1+k)^n) \max_{|z|=1} |P(z)|$ for $1 \leq r \leq k$. In this paper, we generalize as well as improve upon the above inequality.

1. Introduction and statement of results

Let $P(z)$ be a polynomial of degree n and $M(P,r) = \max_{|z|=r} |P(z)|$, then according to Bernstein's inequality

$$\max_{|z|=1} \left| P'(z) \right| \leq n \max_{|z|=1} \left| P(z) \right|. \tag{1.1}$$

The result is best possible and equality in (1.1) is obtained for $P(z) = \alpha z^n$, $\alpha \neq 0$.

If we restrict ourselves to the class of polynomials not vanishing in $|z| < 1$, then Erdös conjectured and Lax [4] proved

$$\max_{|z|=1} \left| P'(z) \right| \leq \frac{n}{2} \max_{|z|=1} \left| P(z) \right|. \tag{1.2}$$

Inequality (1.2) is best possible and the extremal polynomial is $P(z) = \alpha + \beta z^n$ with $|\alpha| = |\beta|$.

As an extension of (1.2), Malik [5] proved the following.

Theorem 1.1. *If $P(z)$ is a polynomial of degree n which does not vanish in $|z| < k$, $k \geq 1$, then*

$$\max_{|z|=1} \left| P'(z) \right| \leq \frac{n}{1+k} \max_{|z|=1} \left| P(z) \right|. \tag{1.3}$$

The result is best possible and equality holds for $P(z) = (z+k)^n$.

Further, as a generalization of (1.3), Bidkham and Dewan [1] proved the following theorem.

International Journal of Mathematics and Mathematical Sciences 2005:16 (2005) 2641–2645
DOI: 10.1155/IJMMS.2005.2641

Theorem 1.2. *If $P(z)=\sum_{v=0}^{n} a_v z^v$ is a polynomial of degree n having no zeros in $|z|<k$, $k\geq 1$, then for $1\leq\rho\leq k$,*

$$\max_{|z|=\rho}|P'(z)|\leq\frac{n(\rho+k)^{n-1}}{(1+k)^n}\max_{|z|=1}|P(z)|. \tag{1.4}$$

The result is best possible and equality in (1.4) holds for $P(z)=(z+k)^n$.

In this paper, we obtain the following result which is a generalization as well as an improvement of Theorem 1.2.

Theorem 1.3. *If $P(z)=\sum_{v=0}^{n} a_v z^v$ is a polynomial of degree n having no zeros in $|z|<k$, $k\geq 1$, then for $0\leq r\leq\rho\leq k$,*

$$\max_{|z|=\rho}|P'(z)| \leq\frac{n(\rho+k)^{n-1}}{(k+r)^n}\left\{1-\frac{k(k-\rho)(n|a_0|-k|a_1|)n}{(k^2+\rho^2)n|a_0|+2k^2\rho|a_1|}\left(\frac{\rho-r}{k+\rho}\right)\left(\frac{k+r}{k+\rho}\right)^{n-1}\right\}\times M(P,r). \tag{1.5}$$

Remark 1.4. Since it is well known that if $P(z)=\sum_{v=0}^{n} a_v z^v$, $P(z)\neq 0$ in $|z|<k$, $k\geq 1$, then $|a_1|/|a_0|\leq n/k$, the above theorem with $r=1$ gives a bound that is much better than obtainable from Theorem 1.2.

If we assume $P'(0)=0$ in the above theorem, we get the following result.

Corollary 1.5. *If $P(z)=\sum_{v=0}^{n} a_v z^v$ is a polynomial of degree n having no zeros in $|z|<k$, $k\geq 1$ and $P'(0)=0$, then for $0\leq r\leq\rho\leq k$,*

$$\max_{|z|=\rho}|P'(z)|\leq\frac{n(\rho+k)^{n-1}}{(k+r)^n}\left\{1-\frac{k(k-\rho)(\rho-r)n}{(k^2+\rho^2)(k+\rho)}\left(\frac{k+r}{k+\rho}\right)^{n-1}\right\}M(P,r). \tag{1.6}$$

2. Lemmas

We require the following lemmas for the proof of the theorem. The first lemma is due to Govil et al. [2].

Lemma 2.1. *If $P(z)=\sum_{v=0}^{n} a_v z^v$ is a polynomial of degree n having all its zeros in $|z|\geq k\geq 1$, then*

$$\max_{|z|=1}|P'(z)|\leq n\frac{n|a_0|+k^2|a_1|}{(1+k^2)n|a_0|+2k^2|a_1|}\max_{|z|=1}|P(z)|. \tag{2.1}$$

Lemma 2.2. *If $P(z)=\sum_{v=0}^{n} a_v z^v$ is a polynomial of degree n having no zeros in $|z|<k$, $k>0$, then for $0\leq r\leq\rho\leq k$,*

$$M(P,r)\geq\left(\frac{r+k}{\rho+k}\right)^n M(P,\rho). \tag{2.2}$$

There is equality in (2.2) for $P(z)=(z+k)^n$.

The above lemma is due to Jain [3].

Lemma 2.3. *If* $P(z) = \sum_{v=0}^{n} a_v z^v$ *is a polynomial of degree* n *having no zeros in* $|z| < k$, $k \geq 1$, *then for* $0 \leq r \leq \rho \leq k$,

$$M(P,r) \geq \left(\frac{k+r}{k+\rho}\right)^n \left\{1 - \frac{k(k-\rho)(n|a_0| - k|a_1|)n}{(k^2+\rho^2)n|a_0| + 2k^2\rho|a_1|}\left(\frac{\rho-r}{k+\rho}\right)\left(\frac{k+r}{k+\rho}\right)^{n-1}\right\}^{-1} \times M(P,\rho). \tag{2.3}$$

Proof. Since $P(z)$ has no zeros in $|z| < k$, $k \geq 1$, therefore, the polynomial $T(z) = P(tz)$ where $0 \leq t \leq k$ has no zeros in $|z| < k/t$, where $k/t \geq 1$. Using Lemma 2.1 with the polynomial $T(z)$, we get

$$\max_{|z|=1} |T'(z)| \leq n\left\{\frac{n|a_0| + k^2/t^2|ta_1|}{(1+k^2/t^2)n|a_0| + 2(k^2/t^2)|ta_1|}\right\}\max_{|z|=1}|T(z)|, \tag{2.4}$$

which implies

$$\max_{|z|=t} |P'(z)| \leq n\left\{\frac{n|a_0|t + k^2|a_1|}{(t^2+k^2)n|a_0| + 2k^2t|a_1|}\right\}\max_{|z|=t}|P(z)|. \tag{2.5}$$

Now for $0 \leq r \leq \rho \leq k$ and $0 \leq \theta < 2\pi$, we have

$$\begin{aligned}|P(\rho e^{i\theta}) - P(re^{i\theta})| &\leq \int_r^\rho |P'(te^{i\theta})|\,dt \\ &\leq \int_r^\rho n\left\{\frac{n|a_0|t + k^2|a_1|}{(t^2+k^2)n|a_0| + 2k^2t|a_1|}\right\}\max_{|z|=t}|P(z)|\,dt \quad (\text{by } (2.5)),\end{aligned} \tag{2.6}$$

which implies on using inequality (2.2) of Lemma 2.2,

$$\begin{aligned}|P(\rho e^{i\theta}) - P(re^{i\theta})| &\leq \int_r^\rho n\left\{\frac{n|a_0|t + k^2|a_1|}{(t^2+k^2)n|a_0| + 2k^2t|a_1|}\right\}\left(\frac{k+t}{k+r}\right)^n M(P,r)\,dt \\ &\leq \frac{nM(P,r)}{(k+r)^n}\int_r^\rho \left\{\frac{n|a_0|t + k^2|a_1|}{(t^2+k^2)n|a_0| + 2k^2t|a_1|}\right\}(k+t)^n\,dt,\end{aligned} \tag{2.7}$$

which gives, for $0 \le r \le \rho \le k$,

$$
\begin{aligned}
&M(P,\rho)\\
&\le \left[1+\frac{n}{(k+r)^n}\int_r^\rho \left\{\frac{n|a_0|t+k^2|a_1|}{(t^2+k^2)n|a_0|+2k^2t|a_1|}\right\}(k+t)^n dt\right]M(P,r)\\
&\le \left[1+\frac{n(k+\rho)}{(k+r)^n}\left\{\frac{n|a_0|\rho+k^2|a_1|}{(\rho^2+k^2)n|a_0|+2k^2\rho|a_1|}\right\}\int_r^\rho (k+t)^{n-1} dt\right]M(P,r)\\
&= \left[1-\left\{\frac{(k+\rho)(n|a_0|\rho+k^2|a_1|)}{(\rho^2+k^2)n|a_0|+2k^2\rho|a_1|}\right\}+\left\{\frac{(k+\rho)(n|a_0|\rho+k^2|a_1|)}{(\rho^2+k^2)n|a_0|+2k^2\rho|a_1|}\right\}\left(\frac{k+\rho}{k+r}\right)^n\right]M(P,r)\\
&= \left[\frac{k(k-\rho)(n|a_0|-k|a_1|)}{(\rho^2+k^2)n|a_0|+2k^2\rho|a_1|}+\left\{1-\frac{k(k-\rho)(n|a_0|-k|a_1|)}{(\rho^2+k^2)n|a_0|+2k^2\rho|a_1|}\right\}\left(\frac{k+\rho}{k+r}\right)^n\right]M(P,r)\\
&= \left(\frac{k+\rho}{k+r}\right)^n\left[1-\frac{k(k-\rho)(n|a_0|-k|a_1|)}{(k^2+\rho^2)n|a_0|+2k^2\rho|a_1|}\left\{1-\left(\frac{k+r}{k+\rho}\right)^n\right\}\right]M(P,r)\\
&= \left(\frac{k+\rho}{k+r}\right)^n\left[1-\frac{k(k-\rho)(n|a_0|-k|a_1|)}{(\rho^2+k^2)n|a_0|+2k^2\rho|a_1|}\times\frac{\rho-r}{(k+\rho)\{1-((k+r)/(k+\rho))\}}\right.\\
&\qquad\qquad \left.\times\left\{1-\left(\frac{k+r}{k+\rho}\right)^n\right\}\right]M(P,r)\\
&\le \left(\frac{k+\rho}{k+r}\right)^n\left[1-\frac{k(k-\rho)(n|a_0|-k|a_1|)n}{(\rho^2+k^2)n|a_0|+2k^2\rho|a_1|}\left(\frac{\rho-r}{k+\rho}\right)\left(\frac{k+r}{k+\rho}\right)^{n-1}\right]M(P,r),
\end{aligned}
\tag{2.8}
$$

from which inequality (2.3) follows. □

3. Proof of theorem

Since the polynomial $P(z)=\sum_{\nu=0}^n a_\nu z^\nu$ has no zero in $|z|<k$, where $k\ge 1$, therefore, it follows that $F(z)=P(\rho z)$ has no zeros in $|z|<k/\rho$ where $k/\rho\ge 1$. Applying inequality (1.3) to the polynomial $F(z)$, we get

$$
\max_{|z|=1}|F'(z)| \le \frac{n}{1+k/\rho}\max_{|z|=1}|F(z)|, \tag{3.1}
$$

which gives

$$
\max_{|z|=1}|P'(z)| \le \frac{n}{\rho+k}\max_{|z|=\rho}|F(z)|. \tag{3.2}
$$

Now if $0\le r\le\rho\le k$, then applying inequality (2.3) of Lemma 2.3 to (3.2), it follows that

$$
\begin{aligned}
\max_{|z|=\rho}|P'(z)| \le{}& \frac{n(k+\rho)^{n-1}}{(k+r)^n}\left[1-\frac{k(k-\rho)(n|a_0|-k|a_1|)n}{(k^2+\rho^2)n|a_0|+2k^2\rho|a_1|}\left(\frac{\rho-r}{k+\rho}\right)\left(\frac{k+r}{k+\rho}\right)^{n-1}\right]\\
&\times\max_{|z|=r}|P(z)|,
\end{aligned}
\tag{3.3}
$$

which is (1.5) and the theorem is proved.

Acknowledgment

The authors are grateful to the referee for valuable suggestions.

References

[1] M. Bidkham and K. K. Dewan, *Inequalities for a polynomial and its derivative*, J. Math. Anal. Appl. **166** (1992), no. 2, 319–324.
[2] N. K. Govil, Q. I. Rahman, and G. Schmeisser, *On the derivative of a polynomial*, Illinois J. Math. **23** (1979), no. 2, 319–329.
[3] V. K. Jain, *Converse of an extremal problem in polynomials. II*, J. Indian Math. Soc. (N.S.) **60** (1994), no. 1-4, 41–47.
[4] P. D. Lax, *Proof of a conjecture of P. Erdös on the derivative of a polynomial*, Bull. Amer. Math. Soc. **50** (1944), 509–513.
[5] M. A. Malik, *On the derivative of a polynomial*, J. London Math. Soc. (2) **1** (1969), 57–60.

K. K. Dewan: Department of Mathematics, Faculty of Natural Sciences, Jamia Millia Islamia (Central University), New Delhi-110025, India
E-mail address: kkdewan123@yahoo.co.in

Abdullah Mir: Department of Mathematics, Faculty of Natural Sciences, Jamia Millia Islamia (Central University), New Delhi-110025, India
E-mail address: abdullah_mir@rediffmail.com