BEST APPROXIMATIONS FOR THE LAGUERRE-TYPE WEIERSTRASS TRANSFORM ON $[0,\infty[\times\mathbb{R}$

E. JEBBARI AND F. SOLTANI

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We use reproducing kernel Hilbert spaces to give the best approximation for Laguerre-type Weierstrass transform. Estimates of extremal functions are also discussed.

1. Introduction

We consider the partial differential operators D_1 and D_2 defined on $\mathbb{K} := [0,\infty[\times\mathbb{R}$, by

$$\begin{aligned} D_1 &:= \frac{\partial}{\partial t}, \\ D_2 &:= \frac{\partial^2}{\partial x^2} + \frac{2\alpha+1}{x}\frac{\partial}{\partial x} + x^2\frac{\partial^2}{\partial t^2}, \quad \alpha > 0. \end{aligned} \tag{1.1}$$

For $\alpha = n-1$, $n \in \mathbb{N}\backslash\{0\}$, the operator D_2 is the radial part of the sub-Laplacian on the Heisenberg group $\mathbb{H}^n$ (see [2, 4]).

These operators have gained considerable interest in various fields of mathematics (see [1, 4]). They give rise to generalizations of many two-variable analytic structures like the Laguerre-Fourier transform $\mathscr{F}_L$, the Laguerre-convolution product, the dispersion and the Gaussian distributions (see [1, 2, 4]).

In this paper, we consider the Laguerre-type Weierstrass transform L_r associated with D_1 and D_2:

$$L_r f(x,t) := \int_{\mathbb{K}} E_r\big[(x,t),(y,s)\big] f(y,-s)\,\mathrm{d}m_\alpha(y,s), \tag{1.2}$$

where E_r, $r > 0$, is the generalized heat kernel given by Definition 2.8 later on and m_α is the measure defined on $\mathbb{K}$ by

$$\mathrm{d}m_\alpha(y,s) := \frac{1}{\pi\Gamma(\alpha+1)} y^{2\alpha+1}\,dy\,ds. \tag{1.3}$$

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This integral transform which generalizes the standard Weierstrass transform (see [3, 5, 6]) solves the generalized heat equation

$$\Delta_L u[(x,t),r] := (D_1^2 - D_2^2)u[(x,t),r] = \frac{\partial}{\partial r}u[(x,t),r] \tag{1.4}$$

on $\mathbb{K}\times]0,\infty[$ with the initial condition $u[(x,t),0] = f(x,t)$ on $\mathbb{K}$, (see Proposition 2.11).

Let $L^2_\alpha(\mathbb{K})$ be the space of square integrable functions on $\mathbb{K}$ with respect to the measure m_α and let $\langle\cdot,\cdot\rangle_{2,m_\alpha}$ be its inner product. For $\nu \in \mathbb{R}$, we consider the space $H^\nu_\alpha(\mathbb{K})$ of functions f in $L^2_\alpha(\mathbb{K})$, such that the function $[1+\lambda^2(1+m^2)]^{\nu/2}\mathcal{F}_L(f)$ is square integrable on $\Gamma = \mathbb{R}\times\mathbb{N}$ with respect to some measure γ_α, defined later in Section 2.

The space $H^\nu_\alpha(\mathbb{K})$ is a Hilbert space with the inner product

$$\langle f,g\rangle_{H^\nu_\alpha} := \int_\Gamma [1+\lambda^2(1+m^2)]^\nu \mathcal{F}_L(f)(\lambda,m)\overline{\mathcal{F}_L(g)(\lambda,m)}d\gamma_\alpha(\lambda,m). \tag{1.5}$$

For $\mu > 0$, by introducing the inner product

$$\langle f,g\rangle_\mu = \langle f,g\rangle_{H^\nu_\alpha} + \langle L_r f, L_r g\rangle_{2,m_\alpha}, \tag{1.6}$$

we construct the Hilbert space $H_\mu(\mathbb{K})$ comprising elements of $H^\nu_\alpha(\mathbb{K})$. Next, we exhibit explicit reproducing kernels for $H^\nu_\alpha(\mathbb{K})$ and $H_\mu(\mathbb{K})$. After that, we provide an explicit solution of the following problem. Given a function g in $L^2_\alpha(\mathbb{K})$. Let $\nu > (\alpha+2)/2$ and $\mu > 0$, we prove that the infimum of $\{\mu\|f\|^2_{H^\nu_\alpha} + \|g - L_r f\|^2_{2,m_\alpha},\ f \in H^\nu_\alpha(\mathbb{K})\}$ is attained at some function denoted by $f^*_{\mu,g}$, which is unique, called the extremal function. We also establish the estimate of the extremal function $f^*_{\mu,g}$, that is,

$$||f^*_{\mu,g} - f||^2_{H^\nu_\alpha} \longrightarrow 0 \quad \text{as } \mu \longrightarrow 0, \tag{1.7}$$

when $f \in H^\nu_\alpha(\mathbb{K})$ and $g = L_r f$.

In the classical case [3, 6], the authors obtain analogous results by using the theory of reproducing kernels from the ideas of best approximations. Also the authors illustrated their numerical experiments by using computers.

2. The reproducing kernels

We begin this section by recalling some results about harmonic analysis associated with the differential operators D_1 and D_2. Next we exhibit the reproducing kernels of some Hilbert spaces associated to these operators.

Notations 2.1. We denote the following.

(i) $\mathbb{K} := [0,\infty[\times\mathbb{R}$ and $\Gamma := \mathbb{R}\times\mathbb{N}$.

(ii) $L^p_\alpha(\mathbb{K})$, $p \in [1,\infty]$, is the space of measurable functions f on $\mathbb{K}$, such that

$$\|f\|_{p,m_\alpha} := \left[\int_{\mathbb{K}} |f(x,t)|^p \,\mathrm{d}m_\alpha(x,t)\right]^{1/p} < \infty, \quad p \in [1,\infty[,$$
$$\|f\|_{\infty,m_\alpha} := \operatorname{ess}\sup_{(x,t)\in\mathbb{K}} |f(x,t)| < \infty, \tag{2.1}$$

where m_α is the measure given by (1.3).

(iii) $L_\alpha^p(\Gamma)$, $p \in [1,\infty]$, is the space of measurable functions g on Γ, such that

$$\|g\|_{p,\gamma_\alpha} := \left[\int_\Gamma |g(\lambda,m)|^p d\gamma_\alpha(\lambda,m)\right]^{1/p} < \infty, \quad p \in [1,\infty[,$$
$$\|g\|_{\infty,\gamma_\alpha} := \operatorname{ess}\sup_{(\lambda,m)\in\Gamma} |g(\lambda,m)| < \infty, \tag{2.2}$$

where γ_α is the positive measure defined on Γ by

$$\int_\Gamma g(\lambda,m)d\gamma_\alpha(\lambda,m) = \sum_{m=0}^{\infty} L_m^{(\alpha)}(0)\int_{\mathbb{R}} g(\lambda,m)|\lambda|^{\alpha+1}d\lambda. \tag{2.3}$$

Here $L_m^{(\alpha)}$ is the Laguerre polynomial of degree m and order α.

Proposition 2.2 (see [4, page 135]). (i) *The system*

$$\begin{gathered} D_1 u = i\lambda u, \\ D_2 u = -2|\lambda|(2m+\alpha+1)u, \\ u(0,0) = 1, \qquad \frac{\partial u}{\partial x}(0,t) = 0, \quad \forall t \in \mathbb{R}, \end{gathered} \tag{2.4}$$

admits a unique solution $\varphi_{\lambda,m}(x,t)$, $(\lambda,m) \in \Gamma$, *given by*

$$\varphi_{\lambda,m}(x,t) = \frac{L_m^{(\alpha)}(|\lambda|x^2)}{L_m^{(\alpha)}(0)}\exp\left(i\lambda t - |\lambda|\frac{x^2}{2}\right), \quad (x,t) \in \mathbb{K}. \tag{2.5}$$

(ii) *For all* $(\lambda,m) \in \Gamma$,

$$\sup_{(x,t)\in\mathbb{K}} |\varphi_{\lambda,m}(x,t)| = 1. \tag{2.6}$$

The function $\varphi_{\lambda,m}$ *gives rise to an integral transform, called the Fourier-Laguerre transform on* $\mathbb{K}$, *which is studied in* [2, 4].

Definition 2.3. The Fourier-Laguerre transform $\mathcal{F}_L$ is defined on $L_\alpha^1(\mathbb{K})$ by

$$\mathcal{F}_L(f)(\lambda,m) := \int_{\mathbb{K}} \varphi_{-\lambda,m}(x,t)f(x,t)\,\mathrm{dm}_\alpha(x,t), \quad (\lambda,m) \in \Gamma. \tag{2.7}$$

From Proposition 2.2(ii), the integral makes sense.

The Fourier-Laguerre transform satisfies the following properties [2, 4].

Theorem 2.4. (i) *Plancherel theorem. The Fourier-Laguerre transform* $\mathcal{F}_L$ *can be extended to an isometric isomorphism from* $L_\alpha^2(\mathbb{K})$ *onto* $L_\alpha^2(\Gamma)$, *denoted also by* $\mathcal{F}_L$. *In particular,*

$$\|\mathcal{F}_L(f)\|_{2,\gamma_\alpha} = \|f\|_{2,m_\alpha}, \quad f \in L_\alpha^2(\mathbb{K}). \tag{2.8}$$

(ii) *Inversion formula. Let* f *be in* $L_\alpha^1(\mathbb{K})$ *such that* $\mathcal{F}_L(f)$ *belongs to* $L_\alpha^1(\Gamma)$, *then*

$$f(x,t) = \int_\Gamma \mathcal{F}_L(f)(\lambda,m)\varphi_{\lambda,m}(x,t)d\gamma_\alpha(\lambda,m), \quad \text{a.e. } (x,t) \in \mathbb{K}. \tag{2.9}$$

Notations 2.5. We denote the following.

(i) $\mathscr{W}(\mathbb{K}) = \{f \in L^1_\alpha(\mathbb{K})/\mathscr{F}_L(f) \in L^1_\alpha(\Gamma)\}$.

(ii) $H^\nu_\alpha(\mathbb{K})$, $\nu \in \mathbb{R}$, is the space

$$H^\nu_\alpha(\mathbb{K}) := \left\{f \in L^2_\alpha(\mathbb{K})/\left[1+\lambda^2(1+m^2)\right]^{\nu/2}\mathscr{F}_L(f) \in L^2_\alpha(\Gamma)\right\}. \tag{2.10}$$

The space $H^\nu_\alpha(\mathbb{K})$ provided with the inner product

$$\langle f,g\rangle_{H^\nu_\alpha} := \int_\Gamma \left[1+\lambda^2(1+m^2)\right]^\nu \mathscr{F}_L(f)(\lambda,m)\overline{\mathscr{F}_L(g)(\lambda,m)}\,d\gamma_\alpha(\lambda,m) \tag{2.11}$$

and the norm $\|f\|^2_{H^\nu_\alpha} = \langle f,f\rangle_{H^\nu_\alpha}$ is a Hilbert space.

Proposition 2.6. *For $\nu > (\alpha+2)/2$, the Hilbert space $H^\nu_\alpha(\mathbb{K})$ admits the reproducing kernel*

$$\mathscr{K}_\alpha[(x,t),(y,s)] = \int_\Gamma \frac{\varphi_{\lambda,m}(x,t)\varphi_{-\lambda,m}(y,s)}{\left[1+\lambda^2(1+m^2)\right]^\nu}\,d\gamma_\alpha(\lambda,m), \tag{2.12}$$

that is,

(i) *for every $(y,s) \in \mathbb{K}$, the function $(x,t) \to \mathscr{K}_\alpha[(x,t),(y,s)] \in H^\nu_\alpha(\mathbb{K})$;*

(ii) *for every $f \in H^\nu_\alpha(\mathbb{K})$ and $(y,s) \in \mathbb{K}$,*

$$\langle f,\mathscr{K}_\alpha[\cdot,(y,s)]\rangle_{H^\nu_\alpha} = f(y,s). \tag{2.13}$$

Proof. (i) Let $(y,s) \in \mathbb{K}$. Since from Proposition 2.2(ii), the function

$$(\lambda,m) \longrightarrow \frac{\varphi_{-\lambda,m}(y,s)}{\left[1+\lambda^2(1+m^2)\right]^\nu} \tag{2.14}$$

belongs to $L^2_\alpha(\Gamma)$ for $\nu > (\alpha+2)/2$, then from Theorem 2.4(i), there exists a function in $L^2_\alpha(\mathbb{K})$, which we denote by $\mathscr{K}_\alpha[\cdot,(y,s)]$, such that

$$\mathscr{F}_L(\mathscr{K}_\alpha[\cdot,(y,s)])(\lambda,m) = \frac{\varphi_{-\lambda,m}(y,s)}{\left[1+\lambda^2(1+m^2)\right]^\nu}. \tag{2.15}$$

Let $\Gamma_N := [-N,N] \times \{0,1,\dots,N\}$. Then we have

$$\mathscr{K}_\alpha[\cdot,(y,s)] = \lim_{N\to\infty}\int_{\Gamma_N} \frac{\varphi_{\lambda,m}(\cdot)\varphi_{-\lambda,m}(y,s)}{\left[1+\lambda^2(1+m^2)\right]^\nu}\,d\gamma_\alpha(\lambda,m), \tag{2.16}$$

in the $L^2_\alpha(\mathbb{K})$ sense.

So there exists a subsequence $(N_p)_{p\in\mathbb{N}}$, such that

$$\mathscr{K}_\alpha[(x,t),(y,s)] = \lim_{p\to\infty}\int_{\Gamma_{N_p}} \frac{\varphi_{\lambda,m}(x,t)\varphi_{-\lambda,m}(y,s)}{\left[1+\lambda^2(1+m^2)\right]^\nu}\,d\gamma_\alpha(\lambda,m), \quad \text{a.e. } (x,t) \in \mathbb{K}. \tag{2.17}$$

Let

$$g_{N_p}(\lambda,m) := \frac{\varphi_{\lambda,m}(x,t)\varphi_{-\lambda,m}(y,s)}{\left[1+\lambda^2(1+m^2)\right]^{\nu}}\mathbf{1}_{\Gamma_{N_p}}, \quad (\lambda,m)\in\Gamma. \tag{2.18}$$

Since

$$\lim_{p\to\infty} g_{N_p}(\lambda,m) = \frac{\varphi_{\lambda,m}(x,t)\varphi_{-\lambda,m}(y,s)}{\left[1+\lambda^2(1+m^2)\right]^{\nu}}\mathbf{1}_{\Gamma} \tag{2.19}$$

and from Proposition 2.2(ii),

$$\left|g_{N_p}(\lambda,m)\right| \le \frac{1}{\left[1+\lambda^2(1+m^2)\right]^{\nu}}. \tag{2.20}$$

Then from the dominated convergence theorem, $\mathcal{K}_\alpha[(x,t),(y,s)]$ is given by

$$\mathcal{K}_\alpha[(x,t),(y,s)] = \int_\Gamma \frac{\varphi_{\lambda,m}(x,t)\varphi_{-\lambda,m}(y,s)}{\left[1+\lambda^2(1+m^2)\right]^{\nu}}d\gamma_\alpha(\lambda,m). \tag{2.21}$$

(ii) Let $f\in\mathcal{W}(\mathbb{K})\cap H_\alpha^\nu(\mathbb{K})$ and $(y,s)\in\mathbb{K}$. From (2.11) and (2.15) and Theorem 2.4(ii), we have

$$\langle f,\mathcal{K}_\alpha[\cdot,(y,s)]\rangle_{H_\alpha^\nu} = \int_\Gamma \mathcal{F}_L(f)(\lambda,m)\varphi_{\lambda,m}(y,s)d\gamma_\alpha(\lambda,m) = f(y,s). \tag{2.22}$$

The assertion (ii) follows by the density of $\mathcal{W}(\mathbb{K})$ in $L_\alpha^2(\mathbb{K})$. □

Definition 2.7. Let $\alpha\ge 0$.

(i) Define the Laguerre translation operators $T_{(x,t)}^\alpha$, $(x,t)\in\mathbb{K}$, for $f\in L_\alpha^1(\mathbb{K})$, by the following.

(a) If $\alpha=0$,

$$T_{(x,t)}^\alpha(f)(y,s) := \frac{1}{2\pi}\int_0^{2\pi} f(\Delta_1(x,y,\theta),s+t+xy\sin\theta)\,d\theta. \tag{2.23}$$

(b) If $\alpha>0$,

$$T_{(x,t)}^\alpha(f)(y,s) := \frac{\alpha}{\pi}\int_0^{2\pi}\int_0^1 f(\Delta_r(x,y,\theta),s+t+xyr\sin\theta)r(1-r^2)^{\alpha-1}dr\,d\theta. \tag{2.24}$$

Here and in what follows, $\Delta_r(x,y,\theta) = (x^2+y^2+2xyr\cos\theta)^{1/2}$.

(ii) The Laguerre convolution product $*_L$ of two functions $f,g\in L_\alpha^1(\mathbb{K})$ is defined by

$$f*_L g(x,t) := \int_{\mathbb{K}} T_{(x,t)}^\alpha(f)(y,s)g(y,-s)\,\mathrm{dm}_\alpha(y,s), \quad (x,t)\in\mathbb{K}. \tag{2.25}$$

Definition 2.8. Let $r > 0$. Define

$$\mathscr{E}_r(x,t) := \int_\Gamma \exp\big(-r\lambda^2[1+4(2m+\alpha+1)^2]\big)\varphi_{\lambda,m}(x,t)\,d\gamma_\alpha(\lambda,m). \tag{2.26}$$

The generalized heat kernel E_r is given by

$$E_r[(x,t),(y,s)] := T^\alpha_{(x,t)}\mathscr{E}_r(y,s); \quad (x,t),(y,s) \in \mathbb{K}. \tag{2.27}$$

Proposition 2.9. *Let $(x,t),(y,s) \in \mathbb{K}$, and $r > 0$. Then, the following exist.*

(i) *The function $\mathscr{E}_r$ solves the generalized heat equation*

$$\Delta_L \mathscr{E}_r = \frac{\partial}{\partial r}\mathscr{E}_r, \tag{2.28}$$

where Δ_L is the operator given by (1.4).

(ii) $\mathscr{F}_L(E_r[(x,t),\cdot])(\lambda,m) = \exp(-r\lambda^2[1+4(2m+\alpha+1)^2])\varphi_{\lambda,m}(x,t)$.

(iii) $\int_\mathbb{K} E_r[(x,t),(y,s)]\,\mathrm{dm}_\alpha(x,t) = 1$.

(iv) *For fixed $(y,s) \in \mathbb{K}$, the function $u[(x,t),r] := E_r[(x,t),(y,s)]$ solves the generalized heat equation:*

$$\Delta_L u[(x,t),r] = \frac{\partial}{\partial r}u[(x,t),r]. \tag{2.29}$$

Proof. The assertion (i) follows from Definition 2.8 and Proposition 2.2(i) by applying derivation theorem under the integral sign.

(ii), (iii), and (iv) will be easily proved. □

Definition 2.10. The Laguerre-type Weierstrass transform is the integral operator given for $f \in L^2_\alpha(\mathbb{K})$ by

$$L_r f(x,t) := \mathscr{E}_r *_L f(x,t) = \int_\mathbb{K} E_r[(x,t),(y,s)]f(y,-s)\,\mathrm{dm}_\alpha(y,s). \tag{2.30}$$

Proposition 2.11. (i) *The integral transform L_r, $r > 0$, solves the generalized heat equation*

$$\Delta_L u[(x,t),r] = \frac{\partial}{\partial r}u[(x,t),r], \tag{2.31}$$

on $\mathbb{K}\times]0,\infty[$ with the initial condition $u[(x,t),0] = f(x,t)$ on $\mathbb{K}$.

(ii) *The integral transform L_r, $r > 0$, is a bounded linear operator from $H^\nu_\alpha(\mathbb{K})$, $\nu > (\alpha+2)/2$, into $L^2_\alpha(\mathbb{K})$, and*

$$\|L_r f\|_{2,m_\alpha} \le c_\alpha(r)\|f\|_{H^\nu_\alpha}, \tag{2.32}$$

where

$$c_\alpha(r) := \int_\Gamma \exp\big(-r\lambda^2[1+4(2m+\alpha+1)^2]\big)\,d\gamma_\alpha(\lambda,m). \tag{2.33}$$

Proof. (i) This assertion follows from Definition 2.10 and Proposition 2.9(iv).

(ii) Let $f \in H^{\nu}_{\alpha}(\mathbb{K})$. Applying Hölder's inequality, we get

$$||L_r f||_{2,m_\alpha} \leq ||E_r[(x,t),\cdot]||_{\infty,m_\alpha} \|f\|_{2,m_\alpha}. \tag{2.34}$$

From Theorem 2.4(ii) and Proposition 2.2(ii), we obtain

$$||E_r[(x,t),\cdot]||_{\infty,m_\alpha} \leq \int_\Gamma \exp\big(-r\lambda^2[1+4(2m+\alpha+1)^2]\big)\,d\gamma_\alpha(\lambda,m) := c_\alpha(r). \tag{2.35}$$

On the other hand, from Theorem 2.4(i), we see that $\|f\|_{2,m_\alpha} \leq \|f\|_{H^{\nu}_{\alpha}}$. which proves (ii). □

Definition 2.12. Let $\mu > 0$. Define the Hilbert space $H_\mu(\mathbb{K}) = H_{\mu,\nu}(\mathbb{K})$ with the norm square

$$\|f\|^2_{H_\mu} := \mu\|f\|^2_{H^{\nu}_{\alpha}} + ||L_r f||^2_{2,m_\alpha}. \tag{2.36}$$

Proposition 2.13. *For $\nu > (\alpha+2)/2$, the Hilbert space $H_\mu(\mathbb{K})$ admits the following reproducing kernel:*

$$K_\mu[(x,t),(y,s)] = \int_\Gamma \frac{\varphi_{\lambda,m}(x,t)\varphi_{-\lambda,m}(y,s)\,d\gamma_\alpha(\lambda,m)}{\mu[1+\lambda^2(1+m^2)]^{\nu} + \exp\big(-2r\lambda^2[1+4(2m+\alpha+1)^2]\big)}. \tag{2.37}$$

Proof. (i) Let $(y,s) \in \mathbb{K}$. In the same way as in the proof of Proposition 2.6(i), we can prove that the function $(x,t) \to K_\mu[(x,t),(y,s)]$ belongs to $L^2_\alpha(\mathbb{K})$ and we have

$$\mathcal{F}_L(K_\mu[\cdot,(y,s)])(\lambda,m) = \frac{\varphi_{-\lambda,m}(y,s)}{\mu[1+\lambda^2(1+m^2)]^{\nu} + \exp\big(-2r\lambda^2[1+4(2m+\alpha+1)^2]\big)}. \tag{2.38}$$

On the other hand, since for $(\lambda,m) \in \Gamma$,

$$\begin{aligned}&\mathcal{F}_L(L_r(K_\mu[\cdot,(y,s)]))(\lambda,m)\\ &\quad= \exp\big(-r\lambda^2[1+4(2m+\alpha+1)^2]\big)\mathcal{F}_L(L_r(K_\mu[\cdot,(y,s)]))(\lambda,m),\end{aligned} \tag{2.39}$$

then from Theorem 2.4(i), we obtain

$$\begin{aligned}&||L_r(K_\mu[\cdot,(y,s)])||^2_{2,m_\alpha}\\ &\quad= \int_\Gamma \exp\big(-2r\lambda^2[1+4(2m+\alpha+1)^2]\big)\,|\mathcal{F}_L(K_\mu[\cdot,(y,s)])(\lambda,m)|^2 d\gamma_\alpha(\lambda,m)\\ &\quad\leq \frac{1}{\mu^2}\int_\Gamma \frac{\exp\big(-2r\lambda^2[1+4(2m+\alpha+1)^2]\big)}{[1+\lambda^2(1+m^2)]^{\nu}}\,d\gamma_\alpha(\lambda,m) < \infty.\end{aligned} \tag{2.40}$$

Therefore, we conclude that $\|K_\mu[\cdot,(y,s)]\|^2_{H_\mu} < \infty$.

(ii) Let $f \in \mathcal{W} \cap H_\mu(\mathbb{K})$ and $(y,s) \in \mathbb{K}$. Then

$$\langle f, K_\mu[\cdot,(y,s)]\rangle_{H_\alpha^\nu} = \mu I_1 + I_2, \tag{2.41}$$

where

$$I_1 = \langle f, K_\mu[\cdot,(y,s)]\rangle_{H_\alpha^\nu}, \qquad I_2 = \langle L_r f, L_r(K_\mu[\cdot,(y,s)])\rangle_{2,m_\alpha}. \tag{2.42}$$

From (2.38), we have

$$I_1 = \int_\Gamma \frac{[1+\lambda^2(1+m^2)]^\nu \mathcal{F}_L(f)(\lambda,m)\varphi_{\lambda,m}(y,s)d\gamma_\alpha(\lambda,m)}{\mu[1+\lambda^2(1+m^2)]^\nu + \exp(-2r\lambda^2[1+4(2m+\alpha+1)^2])}. \tag{2.43}$$

From (2.39), (2.38), and Theorem 2.4(i), we have

$$I_2 = \int_\Gamma \frac{\exp(-2r\lambda^2[1+4(2m+\alpha+1)^2])\mathcal{F}_L(f)(\lambda,m)\varphi_{\lambda,m}(y,s)d\gamma_\alpha(\lambda,m)}{\mu[1+\lambda^2(1+m^2)]^\nu + \exp(-2r\lambda^2[1+4(2m+\alpha+1)^2])}. \tag{2.44}$$

The relation (2.41) and Theorem 2.4(ii) imply that

$$\langle f, K_\mu[\cdot,(y,s)]\rangle_{H_\alpha^\nu} = f(y,s). \tag{2.45}$$

The assertion (ii) follows also from the density of $\mathcal{W}(\mathbb{K})$ in $L_\alpha^2(\mathbb{K})$. □

3. Extremal function for Laguerre-type Weierstrass transform

In this section, we prove for a given function $g \in L_\alpha^2(\mathbb{K})$ that the infimum of $\{\mu\|f\|_{H_\alpha^\nu}^2 + \|g - L_r f\|_{2,m_\alpha}^2,\ f \in H_\alpha^\nu(\mathbb{K})\}$ is attained at some function denoted by $f_{\mu,g}^*$, which is unique, called the extremal function. We start with the following fundamental theorem (see [3, 6, 7]).

Theorem 3.1. *Let H_K be a Hilbert space admitting the reproducing kernel $K(p,q)$ on a set E and H a Hilbert space. Let $L: H_K \to H$ be a bounded linear operator on H_K into H. For $\mu > 0$, introduce the inner product in H_K and call it H_{K_μ} as*

$$\langle f_1, f_2\rangle_{H_{K_\mu}} = \mu\langle f_1, f_2\rangle_{H_K} + \langle Lf_1, Lf_2\rangle_H. \tag{3.1}$$

Then, the following hold.

(i) *H_{K_μ} is the Hilbert space with the reproducing kernel $K_\mu(p,q)$ on E and satisfying the equation*

$$K(\cdot,q) = (\mu I + L^* L)K_\mu(\cdot,q), \tag{3.2}$$

where L^* is the adjoint operator of $L: H_K \to H$.

(ii) *For any $\mu > 0$ and for any $g \in H$, the infimum*

$$\inf_{f \in H_K}\{\mu\|f\|_{H_K}^2 + \|Lf - g\|_H^2\} \tag{3.3}$$

is attained by a unique function $f^*_{\mu,g} \in H_K$ *and this extremal function is given by*

$$f^*_{\mu,g}(p) = \langle g, LK_\mu(\cdot,p)\rangle_H. \tag{3.4}$$

The main result of this paragraph can be stated now.

THEOREM 3.2. *Let* $\nu > (\alpha+2)/2$. *For any* $g \in L^2_\alpha(\mathbb{K})$ *and for any* $\mu > 0$, *the infimum*

$$\inf_{f\in H^\nu_\alpha}\left\{\mu\|f\|^2_{H^\nu_\alpha} + \|g - L_r f\|^2_{2,m_\alpha}\right\} \tag{3.5}$$

is attained by a unique function $f^*_{\mu,g} = f^*_{\mu,\nu,g}$ *and this extremal function is given by*

$$f^*_{\mu,g}(x,t) = \int_{\mathbb{K}} g(y,s)Q_\mu[(x,t),(y,s)]\,\mathrm{d}m_\alpha(y,s), \tag{3.6}$$

where

$$\begin{aligned} Q_\mu[(x,t),(y,s)] &= Q_{\mu,\nu}[(x,t),(y,s)] \\ &= \int_\Gamma \frac{\exp\left(-r\lambda^2[1+4(2m+\alpha+1)^2]\right)\varphi_{\lambda,m}(x,t)\varphi_{-\lambda,m}(y,s)}{\mu[1+\lambda^2(1+m^2)]^\nu + \exp\left(-2r\lambda^2[1+4(2m+\alpha+1)^2]\right)}d\gamma_\alpha(\lambda,m). \end{aligned} \tag{3.7}$$

Proof. By Proposition 2.13 and Theorem 3.1(ii), the infimum given by (3.5) is attained by a unique function $f^*_{\mu,g}$, and from (3.4), the extremal function $f^*_{\mu,g}$ is represented by

$$f^*_{\mu,g}(y,s) = \langle g, L_r(K_\mu[\cdot,(y,s)])\rangle_{2,m_\alpha}, \quad (y,s)\in\mathbb{K}, \tag{3.8}$$

where K_μ is the kernel given by Proposition 2.13.

Since for $(x,t)\in\mathbb{K}$,

$$L_r f(x,t) = \int_\Gamma \exp\left(-r\lambda^2[1+4(2m+\alpha+1)^2]\right)\mathcal{F}_L(f)(\lambda,m)\varphi_{\lambda,m}(x,t)d\gamma_\alpha(\lambda,m), \tag{3.9}$$

and by (2.38), we obtain

$$\begin{aligned} &L_r(K_\mu[\cdot,(y,s)])(x,t) \\ &\quad = \int_\Gamma \frac{\exp\left(-r\lambda^2[1+4(2m+\alpha+1)^2]\right)\varphi_{\lambda,m}(x,t)\varphi_{-\lambda,m}(y,s)}{\mu[1+\lambda^2(1+m^2)]^\nu + \exp\left(-2r\lambda^2[1+4(2m+\alpha+1)^2]\right)}d\gamma_\alpha(\lambda,m) \\ &\quad = Q_\mu[(x,t),(y,s)]. \end{aligned} \tag{3.10}$$

This gives (3.6). □

COROLLARY 3.3. *Let* $\nu > (\alpha+2)/2$. *The extremal function* $f^*_{\mu,g}$ *in (3.6) can be estimated as follows:*

$$\|f^*_{\mu,g}\|^2_{2,m_\alpha} \le \frac{M_\alpha}{4\mu N_\alpha}\int_{\mathbb{K}} e^{(y^2+s^2)}|g(y,s)|^2\,\mathrm{d}m_\alpha(y,s), \tag{3.11}$$

where

$$M_\alpha = \int_{\mathbb{K}} e^{-(y^2+s^2)}\,\mathrm{dm}_\alpha(y,s), \qquad N_\alpha = \left(\int_\Gamma \frac{d\gamma_\alpha(\lambda,m)}{[1+\lambda^2(1+m^2)]^\nu}\right)^{-1}. \tag{3.12}$$

Proof. Applying Hölder's inequality to relation (3.6), we obtain

$$|f^*_{\mu,g}(x,t)|^2 \le M_\alpha \int_{\mathbb{K}} e^{(y^2+s^2)}|g(y,s)|^2|Q_\mu[(x,t),(y,s)]|^2\,\mathrm{dm}_\alpha(y,s). \tag{3.13}$$

Thus, and from Fubini-Tonnelli theorem, we get

$$\|f^*_{\mu,g}\|^2_{2,m_\alpha} \le M_\alpha \int_{\mathbb{K}} e^{(y^2+s^2)}|g(y,s)|^2\|Q_\mu[\cdot,(y,s)]\|^2_{2,m_\alpha}\,\mathrm{dm}_\alpha(y,s). \tag{3.14}$$

On the other hand from Theorem 2.4(i), we have

$$\|Q_\mu[\cdot,(y,s)]\|^2_{2,m_\alpha} = \int_\Gamma |\mathcal{F}_L(Q_\mu[\cdot,(y,s)])(\lambda,m)|^2 d\gamma_\alpha(\lambda,m). \tag{3.15}$$

But for $(\lambda,m)\in\Gamma$, we have

$$\mathcal{F}_L(Q_\mu[\cdot,(y,s)])(\lambda,m) = \frac{\exp(r\lambda^2[1+4(2m+\alpha+1)^2])\varphi_{-\lambda,m}(y,s)}{1+\mu[1+\lambda^2(1+m^2)]^\nu\exp(2r\lambda^2[1+4(2m+\alpha+1)^2])}. \tag{3.16}$$

Then the inequality $(x+y)^2 \ge 4xy$ yields

$$\|Q_\mu[\cdot,(y,s)]\|^2_{2,m_\alpha} \le \frac{1}{4\mu}\int_\Gamma \frac{1}{[1+\lambda^2(1+m^2)]^\nu}d\gamma_\alpha(\lambda,m). \tag{3.17}$$

From this inequality and (3.14), we deduce the result. □

Corollary 3.4. *Let $\nu > (\alpha+2)/2$, $\delta > 0$ and $g, g_\delta \in L^2_\alpha(\mathbb{K})$ such that*

$$\|g-g_\delta\|_{2,m_\alpha} \le \delta. \tag{3.18}$$

Then,

$$\|f^*_{\mu,g} - f^*_{\mu,g_\delta}\|_{H^\nu_\alpha} \le \frac{\delta}{2\sqrt{\mu}}. \tag{3.19}$$

Proof. From (3.6) and Fubini's theorem, we have for $(\lambda,m)\in\Gamma$,

$$\mathcal{F}_L\left(f^*_{\mu,g}\right)(\lambda,m) = \frac{\exp(r\lambda^2[1+4(2m+\alpha+1)^2])\mathcal{F}_L(g)(\lambda,m)}{1+\mu[1+\lambda^2(1+m^2)]^\nu\exp(2r\lambda^2[1+4(2m+\alpha+1)^2])}. \tag{3.20}$$

Hence

$$\mathscr{F}_L\left(f^*_{\mu,g} - f^*_{\mu,g_\delta}\right)(\lambda,m) = \frac{\exp\left(r\lambda^2[1+4(2m+\alpha+1)^2]\right)\mathscr{F}_L(g-g_\delta)(\lambda,m)}{1+\mu[1+\lambda^2(1+m^2)]^\nu \exp\left(2r\lambda^2[1+4(2m+\alpha+1)^2]\right)}. \tag{3.21}$$

Using the inequality $(x+y)^2 \geq 4xy$, we obtain

$$[1+\lambda^2(1+m^2)]^\nu \left|\mathscr{F}_L\left(f^*_{\mu,g} - f^*_{\mu,g_\delta}\right)(\lambda,m)\right|^2 \leq \frac{1}{4\mu}\left|\mathscr{F}_L(g-g_\delta)(\lambda,m)\right|^2. \tag{3.22}$$

Thus, and from Theorem 2.4(i), we obtain

$$\left\|f^*_{\mu,g} - f^*_{\mu,g_\delta}\right\|^2_{H^\nu_\alpha} \leq \frac{1}{4\mu}\left\|\mathscr{F}_L(g-g_\delta)\right\|^2_{2,\gamma_\alpha} \leq \frac{1}{4\mu}\left\|g-g_\delta\right\|^2_{2,m_\alpha}, \tag{3.23}$$

which gives the desired result. □

Corollary 3.5. *Let $\nu > (\alpha+2)/2$, $f \in H^\nu_\alpha(\mathbb{K})$, and $g = L_r f$. Then*

$$\left\|f^*_{\mu,g} - f\right\|^2_{H^\nu_\alpha} \longrightarrow 0 \quad \text{as } \mu \longrightarrow 0. \tag{3.24}$$

Proof. From (3.20), we have

$$\begin{gathered}\mathscr{F}_L(f)(\lambda,m) = \exp\left(r\lambda^2[1+4(2m+\alpha+1)^2]\right)\mathscr{F}_L(g)(\lambda,m),\\ \mathscr{F}_L\left(f^*_{\mu,g}\right)(\lambda,m) = \frac{\exp\left(r\lambda^2[1+4(2m+\alpha+1)^2]\right)\mathscr{F}_L(g)(\lambda,m)}{1+\mu[1+\lambda^2(1+m^2)]^\nu \exp\left(2r\lambda^2[1+4(2m+\alpha+1)^2]\right)}.\end{gathered} \tag{3.25}$$

Thus

$$\mathscr{F}_L\left(f^*_{\mu,g} - f\right)(\lambda,m) = -\frac{\mu[1+\lambda^2(1+m^2)]^\nu \exp\left(2r\lambda^2[1+4(2m+\alpha+1)^2]\right)\mathscr{F}_L(g)(\lambda,m)}{1+\mu[1+\lambda^2(1+m^2)]^\nu \exp\left(2r\lambda^2[1+4(2m+\alpha+1)^2]\right)}. \tag{3.26}$$

Then we obtain

$$\left\|f^*_{\mu,g} - f\right\|^2_{H^\nu_\alpha} = \int_\Gamma h_{\mu,r,\nu}(\lambda,m)\left|\mathscr{F}_L(g)(\lambda,m)\right|^2 d\gamma_\alpha(\lambda), \tag{3.27}$$

with

$$h_{\mu,r,\nu}(\lambda,m) = \frac{\mu^2[1+\lambda^2(1+m^2)]^{3\nu}\exp\left(4r\lambda^2[1+4(2m+\alpha+1)^2]\right)}{\left(1+\mu[1+\lambda^2(1+m^2)]^\nu\right)^2 \exp\left(4r\lambda^2[1+4(2m+\alpha+1)^2]\right)}. \tag{3.28}$$

Since

$$\begin{gathered}\lim_{\mu\to 0} h_{\mu,r,\nu}(\lambda,m) = 0,\\ \left|h_{\mu,r,\nu}(\lambda,m)\right| \leq [1+\lambda^2(1+m^2)]^\nu,\end{gathered} \tag{3.29}$$

we obtain the result from the dominated convergence theorem. □

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E. Jebbari: Department of Mathematics, Faculty of Sciences of Tunis, University of El-Manar Tunis, 2092 Tunis, Tunisia

E-mail address: emna.jebbari@fst.rnu.tn

F. Soltani: Department of Mathematics, Faculty of Sciences of Tunis, University of El-Manar Tunis, 2092 Tunis, Tunisia

E-mail address: fethi.soltani@fst.rnu.tn