IDEMPOTENT-SEPARATING EXTENSIONS OF REGULAR SEMIGROUPS

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Received 1 July 2004 and in revised form 10 March 2005

For a regular biordered set E, the notion of E-diagram and the associated regular semigroup was introduced in our previous paper (1995). Given a regular biordered set E, an E-diagram in a category C is a collection of objects, indexed by the elements of E and morphisms of C satisfying certain compatibility conditions. With such an E-diagram A we associate a regular semigroup $\mathrm{Reg}_E(\mathbf{A})$ having E as its biordered set of idempotents. This regular semigroup is analogous to automorphism group of a group. This paper provides an application of $\mathrm{Reg}_E(\mathbf{A})$ to the idempotent-separating extensions of regular semigroups. We introduced the concept of crossed pair and used it to describe all extensions of a regular semigroup S by a group E-diagram A. In this paper, the necessary and sufficient condition for the existence of an extension of S by A is provided. Also we study cohomology and obstruction theories and find a relationship with extension theory for regular semigroups.

1. Introduction

If $\pi : T \to S$ is an idempotent-separating surjective homomorphism of regular semigroups, then the kernel of π defines a group $E(S)$-diagram $\mathbf{A} : \underline{C}(E(S)) \to \mathbf{GR}$ that factors through $\mathbf{D}(B(E(S)))$ and π induces an idempotent-separating homomorphism $\Psi : S \to (\mathrm{Reg}_{E(S)}(\mathbf{A}))/\mathrm{Inn}_{E(S)}(\mathbf{A})$ ((T,π) is called an *extension of S by the group $E(S)$-diagram* $\mathbf{A}$ with abstract kernel Ψ). In this paper, we discuss the following extension problem for regular semigroups.

Given $\Psi : S \to (\mathrm{Reg}_{E(S)}(\mathbf{A}))/\mathrm{Inn}_{E(S)}(\mathbf{A})$, *find all extensions of S by A with abstract kernel* Ψ. Of course, given Ψ, is it possible that no extension of S by A with abstract kernel Ψ can exist. In this connection, an *obstruction theory* is developed for finding extensions of S by A which induce the given Ψ.

In Section 1, we introduce the concept of a *crossed pair* and use it to describe all extensions of S by a group $E(S)$-diagram $\mathbf{A}$. In Section 2, we associate with each $\Psi : S \to \mathrm{Reg}_{E(S)}(\mathbf{A})/\mathrm{Inn}_{E(S)}(\mathbf{A})$ a three-dimensional cohomology class in the Leech cohomology of S^I. We show in Theorem 3.6 that the vanishing of this cohomology class is necessary and sufficient condition for the existence of an extension of S by $\mathbf{A}$ with abstract kernel Ψ.

International Journal of Mathematics and Mathematical Sciences 2005:18 (2005) 2945–2975
DOI: 10.1155/IJMMS.2005.2945

We further show that if Ψ has an extension, then the set of all equivalence classes of extensions of S by $\mathbf{A}$ with abstract kernel Ψ is in bijective correspondence with the set of all elements of certain second cohomology group.

Before proceeding further, let us recall some known definitions and results.

For any regular semigroup S, we denote by $E(S)$ the set of idempotents of S and by $V(x)$ the set of inverses of an element $x \in S$. Thus $V(x) = \{x' \in S : xx'x = x,\ x'xx' = x'\}$. A pair of elements (x,x') such that $x' \in V(x)$ is called a *regular pair* in S.

A homomorphism $\theta : T \to S$ of regular semigroups is called *idempotent-separating* if θ is one-to-one on the idempotents of T. A *congruence* ρ is called idempotent-separating if the associated projection homomorphism is idempotent-separating. Let ρ be an idempotent-separating congruence on S. Then $\rho \subseteq \mathbf{H}$. For each $e \in E(S)$, let $\mathbf{K}_e = \rho(e) = \{x \in S : x\rho e\}$. Then $\mathbf{K}_e$ is a subgroup of the maximal subgroup $\mathbf{H}_e$ of S. The family $\mathbf{K} = \{\mathbf{K}_e : e \in E(S)\}$, where $(\mathbf{K}_S)_e = \{a \in \mathbf{H}_e : af = fa$ for each idempotent $f \leq e\}$.

Definition 1.1 [7]. Let S be a regular semigroup. For each $e \in E(S)$, let $\mathbf{K}_e$ be a subgroup of $\mathbf{H}_e$. Then $\mathbf{K} = \{\mathbf{K}_e : e \in E(S)\}$ is called a *group kernel normal system* of S if it satisfies

(i) $af = fa$ for all $a \in \mathbf{K}_e$ and for all $f \in E(S)$ such that $f \leq e$,

(ii) $x'\mathbf{K}_{xx'}x \subseteq \mathbf{K}_{x'x}$ for each regular pair (x,x') of S.

Proposition 1.2 [3]. *Let S be a regular semigroup. Let $\mathbf{K} = \{\mathbf{K}_e : e \in E(S)\}$ be a group kernel normal system of S. Define*

$$\begin{aligned}\rho_K = \{(x,y) \in S \times S : &\ \text{for some } x' \in V(x) \text{ and } y' \in V(y),\\ &\ xx' = yy',\ x'x = y'y, \text{ and } y'x \in \mathbf{K}_{x'x}\}.\end{aligned} \tag{1.1}$$

Then ρ_K is an idempotent-separating congruence on S whose kernel is the group kernel normal system $\mathbf{K}$ of S. Conversely, if ρ is an idempotent-separating congruence on S, then the kernel $\mathbf{K}$ of ρ is a group kernel normal system of S and $\rho_K = \rho$.

Let us recall some results from [10, 13].

Let E be a regular biordered set. We write $\omega^r = \{(e,f) : fe = e\}$, $\omega^l = \{(e,f) : ef = e\}$ and $\mathbf{R} = \omega^r \cap (\omega^r)^{-1}$, $\mathbf{L} = \omega^l \cap (\omega^l)^{-1}$, $\omega = \omega^r \cap \omega^l$.

Definition 1.3. Let E be a regular biordered set and $\mathbf{G}(\mathbf{E})$ the ordered groupoid of E-chains of E [13]. The category $\underline{C}(E)$ has objects as the elements of E and a morphism from e to f is a pair (e,c), where $c = c(e_0,\dots,e_n) \in \mathbf{G}(\mathbf{E})$ such that $e \geq e_0$ and $e_n = f$.

If $(e,c) : e \to f$, $(f,c') : f \to g$ are two morphisms, with $c = c(e_0,\dots,e_n)$, $c' = c(f_0,\dots,f_m)$, then the composite is given by $(e,c)(f,c') = (e,(c * f_0)c')$, where $(c * f_0)c'$ is the composite of $(c * f_0)$ and c' in $\mathbf{G}(\mathbf{E})$. The identity morphism at e is $(e,c(e))$ and the associativity of the composition follows from the transitivity property of the ordered groupoid $\mathbf{G}(\mathbf{E})$ [13, Proposition 3.3].

Throughout this paper, S will denote a regular semigroup with biordered set of idempotents $E = E(S)$. $B(E)$ the universal regular idempotent generated semigroup on E [13]. $\mathbf{C}(S)$ denotes the category with the set of idempotents $E(S)$ as its objects and morphism from an object e to an object f is a triple $(e,x,x') : e \to f$, where (x,x') is a regular pair such that $e \geq xx'$ and $x'x = f$. Composition of morphisms is defined by $(e,x,x')(x'x,y,y') = (e,xy,y'x')$. Define an equivalence relation $\sim$ on the morphisms of $\mathbf{C}(S)$ as follows: if (e,x,x'), $(e,y,y') : e \to f$ are two morphisms, then $(e,x,x') \sim (e,y,y')$ if and only if there exist idempotents $e_0,e_1,\dots,e_n \in \omega(e)$ with $(e_{i-1},e_i) \in \mathbf{R} \cup \mathbf{L}$, $i = 1$ to n, such that $e_0 = xx'$, $e_n = yy'$, and $(y,y') = (e_n e_{n-1} \cdots e_0 x, x' e_0 e_1 \cdots e_n)$. Then $\mathbf{D}(S) = \mathbf{C}(S)/\sim$ is the quotient category of $\mathbf{C}(\mathbf{S})$. If we view the underlying groupoid of the inductive groupoid $\mathbf{G}(S)$ of S as a subcategory of $\mathbf{C}(S)$ via the embedding $(x,x') \to (xx',x,x')$, then the evaluation map $\varepsilon_S : c(e_0,e_1,\dots,e_n) \to (e_0e_1\cdots e_n, e_ne_{n-1}\cdots e_0) : \mathbf{G}(E) \to \mathbf{G}(S)$ extends to a functor $\bar{\varepsilon}_S : \underline{C}(E) \to \mathbf{C}(S)$ such that $\bar{\varepsilon}_S(e,c) \to (e,\bar{\varepsilon}_S(c))$ for every morphism (e,c) of $\underline{C}(E)$. In particular, by taking $S = B(E)$ we obtain a functor $\bar{\varepsilon}_{B(E)} : \underline{C}(E) \to \mathbf{C}(B(E))$. By [13, Theorem 6.9], the inclusion $E \subseteq S$ extends uniquely to an idempotent-separating homomorphism $\delta : B(E) \to S$. If $\mathbf{C}(\delta) : (e,x,x') \to (e,x\delta,x'\delta) : \mathbf{C}(B(E)) \to \mathbf{C}(S)$ is the induced functor, then $\mathbf{C}(\delta)\bar{\varepsilon}_{B(E)} = \bar{\varepsilon}_S$.

If $\theta : S \to S'$ is a homomorphism of regular semigroups, then the maps $e \to e\theta$; $[e,x,x'] \to [e\theta,x\theta,x'\theta]$ define a functor $\mathbf{D}(\theta) : \mathbf{D}(S) \to \mathbf{D}(S')$. Let E be a biordered set. For each $e \in E$, the inclusion $\omega(e) \subseteq E$ induces a functor $\mathbf{D}(B(\omega(e))) \to \mathbf{D}(B(E))$. Let $[e,x,x']$, $[e,y,y'] : e \to f$ be two morphisms of $\mathbf{D}(B(E))$, then

$$[e,x,x'] = [e,y,y'] \quad \text{if } x = y \text{ or } x' = y'. \tag{1.2}$$

Lemma 1.4. *Let $[e,x,x'] : e \to g$ be any morphism of $\mathbf{D}(B(\omega(e)))$ with domain e. Then $[e,x,x'] = [e,g,g]$ in $\mathbf{D}(B(\omega(e)))$ and hence in $\mathbf{D}(B(E))$.*

Let S be a regular semigroup. Let ρ_S be the maximum idempotent-separating congruence on S. Then the kernel $\mathbf{K}_S$ of ρ_S defines a group-valued functor $\mathbf{K}_S : \mathbf{C}(S) \to \mathbf{GR}$, where $\mathbf{GR}$ denotes the category of groups, which associates to each object e of $\mathbf{C}(S)$ the group $(\mathbf{K}_S)_e$ and to each morphism $(e,x,x') : e \to f$ the group homomorphism $\mathbf{K}_S(e,x,x') : (\mathbf{K}_S)_e \to (\mathbf{K}_S)_e$ given by $(a)\mathbf{K}_S(e,x,x') = x'ax$.

Proposition 1.5. *If ρ is an idempotent-separating congruence on S, then $K^\rho : \mathbf{C}(S) \to \mathbf{GR}$ defined by $K^\rho(e) = \rho(e)$; $K^\rho(e,x,x') = \mathbf{K}_S(e,x,x')/K^\rho(e)$ is a subfunctor of $\mathbf{K}_S$. Conversely, if $\mathbf{K}' : \mathbf{C}(S) \to \mathbf{GR}$ is a subfunctor of $\mathbf{K}_S$, then $\mathbf{K}' = \{\mathbf{K}'_e : e \in E(S)\}$ is a group kernel normal system of S and defines, by (1.1), an idempotent-separating congruence $\rho_{K'}$ on S. Further $\rho \to \mathbf{K}^\rho$ defines a bijective correspondence between the idempotent-separating congruences on S and the subfunctors of $\mathbf{K}_S$.*

Let $\pi : T \to S$ be an idempotent-separating homomorphism from T onto S. Then $\mathbf{K}^{\pi\pi^{-1}} : \mathbf{C}(T) \to \mathbf{GR}$ factors through $\mathbf{D}(T)$. That is, there is a functor $\operatorname{Ker}\pi : \mathbf{D}(T) \to \mathbf{GR}$ such that the diagram

$$\begin{tikzcd} \mathbf{C}(T) \arrow[rr, "\mathbf{K}^{\pi\pi^{-1}}"] \arrow[dr] & & \mathbf{GR} \\ & \mathbf{D}(T) \arrow[ur, "\operatorname{Ker}\pi"'] & \end{tikzcd} \tag{1.3}$$

is commutative. Thus $(\operatorname{Ker}\pi)_e = \{a \in T : a\pi = e\pi\}$, $e \in E(T)$, and $a\operatorname{Ker}\pi(e,x,x') = x'ax$, $a \in (\operatorname{Ker}\pi)_e$.

Lemma 1.6 [9, Lemma 4.1]. *Let $\pi : T \to S$ be an idempotent-separating onto homomorphism of regular semigroups. If $t\pi = u\pi = x$, $t,u \in T$, $x \in S$, then for each $e \in E(S) \cap \mathbf{L}_x$, there exists a unique element $a \in T$ such that $u = ta$ and $a\pi = e$.*

Definition 1.7. Let $\mathbf{GR}$ be the category of groups. By an E-diagram in $\mathbf{GR}$ we mean a functor $\mathbf{A} : \underline{C}(E) \to \mathbf{GR}$ which factors through $\mathbf{C}(B(E))$. In other words, a functor $\mathbf{A} : \underline{C}(E) \to \mathbf{GR}$ is an E-diagram in $\mathbf{GR}$ if there is a (necessarily unique) functor $\hat{\mathbf{A}} : \mathbf{C}(B(E)) \to \mathbf{GR}$ such that $\mathbf{A} = \hat{\mathbf{A}}\bar{\varepsilon}_{B(E)}$.

Observe that if $\mathbf{A}$ is an E-diagram in $\mathbf{GR}$, then for any two morphisms (e,c), $(e,c') : e \to f$ in $\underline{C}(E)$, $\mathbf{A}(e,c) = \mathbf{A}(e,c')$ whenever $\bar{\varepsilon}_{B(E)}(c) = \bar{\varepsilon}_{B(E)}(c')$.

Let $\mathbf{A}$ be a contravariant E-diagram in $\mathbf{GR}$. Then there exists a contravariant functor $\hat{\mathbf{A}} : \mathbf{C}(B(E)) \to \mathbf{GR}$ such that $\mathbf{A} = \hat{\mathbf{A}}\bar{\varepsilon}_{B(E)}$. For each $e \in E$, let $\mathbf{A}^e$ denote the composite

$$\mathbf{A}^e : \underline{C}(\omega(e)) \xrightarrow{i_e} \underline{C}(E) \xrightarrow{\mathbf{A}} \mathbf{GR}. \tag{1.4}$$

Define $\mathbf{G}(\mathbf{A})$ to be the category whose objects are the elements of E. A morphism $e \to f$ is a pair of (α,ϕ) consisting of an ω-isomorphism $\alpha : \omega(e) \to \omega(f)$ and a natural isomorphism $\phi : \mathbf{A}^e \to \mathbf{A}^f \underline{C}(\alpha)$, where $\underline{C}(\alpha) : \underline{C}(\omega(e)) \to \underline{C}(\omega(f))$ is the functor defined by the ω-isomorphism α, and $\mathbf{A}^f \underline{C}(\alpha)$ is the composite

$$\mathbf{C}(\omega(e)) \xrightarrow{\mathbf{C}(\alpha)} \underline{C}(\omega(f)) \xrightarrow{\mathbf{A}^f} \mathbf{GR}. \tag{1.5}$$

Note that the natural isomorphism ϕ assigns to each object h in $\underline{C}(\omega(e))$ an isomorphism $\phi_h : \mathbf{A}_h \to \mathbf{A}_{(h)\alpha}$ such that, for any morphism $(h,c) : h \to k$ in $\underline{C}(\omega(e))$, the following diagram commutes:

$$\begin{tikzcd} \mathbf{A}_h \arrow[r, "\phi_{\mathbf{h}}"] & \mathbf{A}_{(h)\alpha} \\ \mathbf{A}_k \arrow[u, "\mathbf{A}(h{,}c)"] \arrow[r, "\phi_{\mathbf{k}}"] & \mathbf{A}_{(k)\alpha} \arrow[u, "\mathbf{A}(h\alpha{,}c\alpha)"'] \end{tikzcd} \tag{1.6}$$

The composite of two morphisms $(\alpha,\phi) : e \to f$, $(\beta,\Psi) : f \to g$ is given by $(\alpha,\phi)(\beta,\Psi) = (\alpha\beta, \phi(\underline{C}(\alpha)\Psi))$, where $\alpha\beta$ is the composite $\omega(e) \xrightarrow{\alpha} \omega(f) \xrightarrow{\beta} \omega(g)$ and the natural isomorphism $\phi(\underline{C}(\alpha)\Psi) : \mathbf{A}^e \to \mathbf{A}^g \underline{C}(\alpha\beta)$ is defined by

$$\left(\phi(\underline{C}(\alpha)\Psi)\right)_h = \phi_h \circ \Psi_{(h)\alpha} : \mathbf{A}_h \longrightarrow \mathbf{A}_{(h)\alpha\beta} \tag{1.7}$$

for all $h \in \omega(e)$. For each object e, $(1_e, \mathbf{1}_e) : e \to e$, where $1_e : \omega(e) \to \omega(e)$ is the identity ω-isomorphism and $\mathbf{1}_e : \mathbf{A}^e \to \mathbf{A}^e$ is the identity isomorphism, is the identity morphism of e. For an E-chain $c = c(e_0, e_1, \ldots, e_n) \in \mathbf{G}(E)$, define $\varepsilon : \mathbf{G}(E) \to \mathbf{G}(\mathbf{A})$ by $\varepsilon(c) = (\alpha^c, \phi^c)$, where $\alpha^c : \omega(e_0) \to \omega(e_n)$ and $\phi^c : \mathbf{A}^{e_0} \to \mathbf{A}^{e_n}\underline{C}(\alpha^c)$ are such that

$$(h)\alpha^c = (h)\tau_E(c) = \tau(e_0,e_1)\tau(e_1,e_2)\cdots\tau(e_{n-1},e_n), \tag{1.8}$$

$\tau_E : \mathbf{G}(E) \to T^*(E)$ is the evaluation map of the inductive groupoid $T^*(E)$ of ω-isomorphism of E, and ${\phi_h}^c = \mathbf{A}(h,h*c)^{-1} : \mathbf{A}_h \to {\mathbf{A}_{(h)\alpha}}^c$ for every $h \in \omega(e_0)$. By [10], it follows that $(\mathbf{G}(\mathbf{A}),\varepsilon)$ is an inductive groupoid.

Let $\mathbf{A} : \underline{C}(E) \to \mathbf{GR}$ be an E-diagram in $\mathbf{GR}$. Let $\mathrm{Reg}_E(\mathbf{A})$ be the quotient of $\mathbf{G}(\mathbf{A})$ by the equivalence relation ρ, where for any two morphisms $(\alpha,\phi) : e \to f$, $(\beta,\Psi) : f \to g$ in $\mathbf{G}(\mathbf{A})$,

$$(\alpha,\phi)\rho(\beta,\Psi) \Longleftrightarrow e\mathbf{R}g, \qquad f\mathbf{L}h, \varepsilon(c(e,g))(\beta,\Psi) = (\alpha,\phi)\varepsilon(c(f,h)). \tag{1.9}$$

Also if $[\alpha,\phi]$, $[\beta,\Psi]$ are the elements of $\mathrm{Reg}_E(\mathbf{A})$ with representatives $(\alpha,\phi) : e \to f$, $(\beta,\Psi) : f \to g$, then as in [13]

$$[\alpha,\phi][\beta,\Psi] = [(\alpha,\phi)\circ_1(\beta,\Psi)] = [((\alpha,\phi)*fl)(\alpha^c,\phi^c)(lg*(\beta,\Psi))], \tag{1.10}$$

where $l \in S(f,g)$, the sandwich set of f and g, and $c = c(fl,l,lg)$. Also note that $E(\mathrm{Reg}_E(\mathbf{A})) = [1_e,\mathbf{1}_e]$.

Lemma 1.8. *Let $([x],[y])$ be a regular pair in S such that $[x][y] = [1_e]$ and $[y][x] = [1_f]$. Then there exists $z : e \to f$ in the inductive groupoid $\mathbf{G}$ such that $[z] = [x]$ and $[z^{-1}] = [y]$.*

2. Idempotent-separating extensions of regular semigroups

Consider a regular semigroup T and an idempotent-separating homomorphism $\pi : T \to S$ of T onto S. Let $\mathrm{Ker}\,\pi : \mathbf{D}(T) \to \mathbf{GR}$ be the group-valued functor defined by the kernel of π. The inverse $i = (\pi/E(T))^{-1} : E \to E(T)$ of the biorder isomorphism $\pi/E(T) : E(T) \to E(S) = E$ extends to an idempotent-separating homomorphism $\hat{\mathbf{i}} : B(E) \to T$ by [13, Theorem 6.9] and hence induces a functor $\mathbf{D}(\hat{\mathbf{i}}) : \mathbf{D}(B(E)) \to \mathbf{D}(T)$. If $\pi_1 : \underline{C}(E) \to \mathbf{D}(B(E))$ denotes the functor

$$(e,c(e_0,\ldots,e_n)) \longrightarrow [e,e_0e_1\cdots e_n,e_ne_{n-1}\cdots e_0], \tag{2.1}$$

then the composite

$$\mathbf{A}^\pi = \mathrm{Ker}\,\pi\mathbf{D}(\hat{\mathbf{i}})\pi_1 : \underline{C}(E) \longrightarrow \mathbf{GR} \tag{2.2}$$

is a group E-diagram which factors through $\mathbf{D}(B(E))$. Thus ${\mathbf{A}_e}^\pi = \{t \in T : t\pi = e\}$ for each object e of $\underline{C}(E)$ and

$$A^\pi(e,c(e_o,\ldots,e_n)) = \mathrm{Ker}\,\pi[ei,(e_oi)\cdots(e_ni),(e_ni)\cdots(e_oi)] : \mathbf{A}_e \longrightarrow \mathbf{A}_f \tag{2.3}$$

for each morphism $(e,c(e_o,\ldots,e_n)) : e \to f$ of $\underline{C}(E)$. This observation motivates the following.

Definition 2.1. Let $\mathbf{A} : \underline{C}(E) \to \mathbf{GR}$ be a (covariant) group E-diagram that factors through $\mathbf{D}(B(E))$. An *extension of the regular semigroup S by the group E-diagram* $\mathbf{A}$ is a triple $\varepsilon_T = (T, \pi, U)$ consisting of a regular semigroup T, an idempotent-separating homorphism $\pi : T \to S$ of T onto S, and a natural isomorphism of functors $U : \mathbf{A} \to \mathbf{A}^\pi$.

Remark 2.2. Let $e \in E$ and let $\mathbf{A}^e$ be the composite $\underline{C}(\omega(e)) \xrightarrow{i_e} \underline{C}(E) \xrightarrow{\mathbf{A}} \mathbf{GR}$. For each $x \in \mathbf{A}_e$, we define a natural isomorphism $\eta^x : \mathbf{A}^e \to \mathbf{A}^e$ as follows. Given $h \in \omega(e)$, let $x_h = (x)\mathbf{A}(e,h) \in \mathbf{A}_h$, and let ${\eta_h}^x : \mathbf{A}_h \to \mathbf{A}_h$; $(a){\eta_h}^x = {x_h}^{-1} a x_h$ be the inner automorphism defined by x_h. If $m = (h, c(h_0, h_1, \dots, h_n)) : h \to k$ is a morphism of $\underline{C}(\omega(e))$, then

$$x_h \mathbf{A}(m) = (x)\mathbf{A}(e,h)\mathbf{A}(m) = (x)\mathbf{A}\big((e,h)m\big) = (x)\mathbf{A}(e,k) = x_k \tag{2.4}$$

and therefore the diagram

$$\begin{array}{ccc} \mathbf{A}_h & \xrightarrow{\eta_{\mathbf{h}}{}^{\mathbf{x}}} & \mathbf{A}_h, \\ {\scriptstyle \mathbf{A}(m)}\big\downarrow & & \big\downarrow{\scriptstyle \mathbf{A}(m)} \\ \mathbf{A}_k & \xrightarrow{\eta_{\mathbf{k}}{}^{\mathbf{x}}} & \mathbf{A}_k \end{array} \tag{2.5}$$

is commutative. Thus the map $h \to {\eta_h}^x$, $h \in \omega(e)$, defines a natural isomorphism $\eta^x : \mathbf{A}^e \to \mathbf{A}^e$. If $\mathrm{Reg}_E(\mathbf{A})$ is the regular semigroups of partial isomorphisms of the E-diagram $\mathbf{A}$, then $[1_e, \eta^x] \in \mathrm{Reg}_E(\mathbf{A})$, where $1_e : \omega(e) \to \omega(e)$ is the identity isomorphism. Clearly $\eta^x \eta^y = \eta^{xy}$ for all x, $y \in \mathbf{A}_e$ and hence the map

$$\eta : x \to [1_e, \eta^x] : \mathbf{A}_e \longrightarrow \mathrm{Reg}_E(\mathbf{A}) \tag{2.6}$$

is a homomorphism. Denote the image of $\mathbf{A}_e$ under η by $\mathrm{Inn}(\mathbf{A})_e$. Then $\mathrm{Inn}(\mathbf{A})_e$ is a subgroup of the maximal group $\mathbf{H}_{[1_e, \mathbf{1}_e]}$ of $\mathrm{Reg}_E(\mathbf{A})$. We write

$$\mathrm{Inn}_E(\mathbf{A}) = \{\mathrm{Inn}_E(\mathbf{A})\}_{e \in E}. \tag{2.7}$$

Proposition 2.3. $\mathrm{Inn}_E(\mathbf{A})$ *is a group kernel normal system in* $\mathrm{Reg}_E(\mathbf{A})$.

Proof. Let $x \in \mathbf{A}_e$ and $h \in \omega(e)$. Then $[1_e, \eta^x][1_h, \mathbf{1}_h] = [1_h, {\eta^x}_h] = [1_h, \mathbf{1}_h][1_e, \eta^x]$. Next, let (s, s') be a regular pair in $\mathrm{Reg}_E(\mathbf{A})$ such that $ss' = [1_e, \mathbf{1}_e]$ and $s's = [1_f, \mathbf{1}_f]$. Using Lemma 1.8, choose a morphism $(\alpha, \phi) : e \to f$ in $G(A)$ such that $[\alpha, \phi] = s$, $[\alpha^{-1}, \phi^{-1}] = s'$. Then, for any $x \in \mathbf{A}_e$, we have $s'(x)\eta s = [\alpha^{-1}, \phi^{-1}][1_e, \eta^x][\alpha, \phi] = [1_f, {\eta^{(x)\phi}}_e] = ((x)\phi_e)\eta \in \mathrm{Inn}(\mathbf{A})_f$. Hence, by Definition 1.1, $\mathrm{Inn}_E(\mathbf{A})$ is a group kernel normal system in $\mathrm{Reg}_E(\mathbf{A})$. □

Let $\mathrm{Reg}_E(\mathbf{A})/\mathrm{Inn}_E(\mathbf{A})$ be the quotient of $\mathrm{Reg}_E(\mathbf{A})$ by the idempotent-separating congruence determined by $\mathrm{Inn}_E(\mathbf{A})$ (see Proposition 1.2) and let

$$t : \mathrm{Reg}_E(\mathbf{A}) \longrightarrow \mathrm{Reg}_E(\mathbf{A})/\mathrm{Inn}_E(\mathbf{A}) \tag{2.8}$$

be the associated projection homomorphism.

We next define the centre of the E-diagram $\mathbf{A}$. For any $e \in E$, let

$$\begin{aligned} \mathbf{Z}(\mathbf{A})_e &= \operatorname{Ker}\{\eta : \mathbf{A}_e \longrightarrow \operatorname{Reg}_E(\mathbf{A})\} = \{a \in \mathbf{A}_e : (a)\eta = [1_e, \mathbf{1}_e]\} \\ &= \{a \in \mathbf{A}_e : (a)A(e,h) = a_h \in \mathbf{Z}(\mathbf{A}_h) \text{ for every } h \in \omega(e)\}. \end{aligned} \tag{2.9}$$

Evidently $\mathbf{Z}(\mathbf{A})_e$ is an abelian normal subgroup of $\mathbf{A}_e$. If $(e, c = c(e_0,\ldots,e_n)) : e \to f$ is a morphism in $\underline{C}(E)$, then $(\mathbf{Z}(\mathbf{A})_e)\mathbf{A}(e,c) \subseteq \mathbf{Z}(\mathbf{A})_f$. For, if $a \in \mathbf{Z}(\mathbf{A})_e$, then for any element $h \in \omega(f)$, letting $c' = c * h = c(h_0,h_1,\ldots,h_n)$, we have $(a)\mathbf{A}(e,c)\mathbf{A}(f,h) = (a)\mathbf{A}((e,c)(f,h)) = (a)\mathbf{A}(e,c*h) = (a)\mathbf{A}(e,c') = (a)\mathbf{A}(e,h_0)\mathbf{A}(h_0,c') \in \mathbf{Z}(\mathbf{A})_h$, since $(a)\mathbf{A}(e,h_0) \in \mathbf{Z}(\mathbf{A})_{h0}$ and $\mathbf{A}(h_0,c') : \mathbf{A}_{h0} \to \mathbf{A}_h$ is an isomorphism of groups. Therefore, the maps

$$e \longrightarrow \mathbf{Z}(\mathbf{A})_e; \quad (e,c) \longrightarrow \mathbf{A}(e,c) \mid \mathbf{Z}(\mathbf{A})_e : \mathbf{Z}(\mathbf{A})_e \longrightarrow \mathbf{Z}(\mathbf{A})_f \tag{2.10}$$

define a functor $\mathbf{Z}(\mathbf{A}) : \underline{C}(E) \to \mathbf{GR}$, which is a subfunctor of $\mathbf{A} : \underline{C}(E) \to \mathbf{GR}$. Since $\mathbf{Z}(\mathbf{A})_e$'s are abelian groups, we may also view $\mathbf{Z}(\mathbf{A})$ as a functor from $\underline{C}(E)$ to $\mathbf{Ab}$, the category of abelian groups.

Definition 2.4. The functor $\mathbf{Z}(\mathbf{A}) : \underline{C}(E) \to \mathbf{Ab}$ is called the *centre* of $\mathbf{A}$.

PROPOSITION 2.5. *The sequence*

$$1 \longrightarrow \mathbf{Z}(\mathbf{A}) \xrightarrow{i} \mathbf{A} \xrightarrow{\eta} \operatorname{Reg}_E(\mathbf{A}) \xrightarrow{t} \operatorname{Reg}_E(\mathbf{A})/\operatorname{Inn}_E(\mathbf{A}) \longrightarrow 1 \tag{2.11}$$

is exact in the sense that t is an idempotent-separating onto homomorphism and the sequence

$$1 \longrightarrow \mathbf{Z}(\mathbf{A})_e \xrightarrow{ie} \mathbf{A}_e \longrightarrow (\operatorname{Ker} t)_e \longrightarrow 1 \tag{2.12}$$

is an exact sequence of groups for each $e \in E$.

Since $\mathbf{A}$ factors through $\mathbf{D}(B(E))$, so is the centre $\mathbf{Z}(\mathbf{A}) : \underline{C}(E) \to \mathbf{Ab}$. Let $\operatorname{Reg}_E(Z(\mathbf{A}))$ be the regular semigroup of partial isomorphisms of $\mathbf{Z}(\mathbf{A})$. If $(\alpha,\phi) : e \to f$ is a morphism in $\mathbf{G}(\mathbf{A})$, then for each $h \in \omega(e)$, $\phi_h : \mathbf{A}_h \to \mathbf{A}_{(h)\alpha}$ induces by restriction an isomorphism $\overline{\phi}_h : Z(\mathbf{A})_h \to Z(\mathbf{A})_{(h)\alpha}$ and therefore the map $h \to \overline{\phi}_h$ defines a natural $\overline{\phi} : \mathbf{Z}(\mathbf{A})^e \to \mathbf{Z}(\mathbf{A})^f \underline{C}(\alpha)$. Thus we have an idempotent-separating homomorphism

$$u : \operatorname{Reg}_E(\mathbf{A}) \longrightarrow \operatorname{Reg}_E \mathbf{Z}((\mathbf{A})) \tag{2.13}$$

defined by $[\alpha,\phi]u = [\alpha,\overline{\phi}]$, for $[\alpha,\phi] \in \operatorname{Reg}_E(\mathbf{A})$. If $x \in \mathbf{A}_e$, then clearly $\overline{\eta}^x : \mathbf{Z}(\mathbf{A})^e \to \mathbf{Z}(\mathbf{A})^e$ is the identity natural isomorphism. Hence, u induces an idempotent-separating homomorphism

$$v : \operatorname{Reg}_E(\mathbf{A})/\operatorname{Inn}_E(\mathbf{A}) \longrightarrow \operatorname{Reg}_E \mathbf{Z}((\mathbf{A})) \tag{2.14}$$

such that $tv = u$.

Definition 2.6. Two extensions $\varepsilon_T = (T,\pi,U)$ and $\varepsilon_{T'} = (T',\pi',U')$ of S by $\mathbf{A}$ are equivalent if there exists an isomorphism $\theta : T \to T'$ of regular semigroups such that

(i) $\theta\pi' = \pi$,

(ii) for each $e \in E$, the diagram

$$\begin{tikzcd} & \mathbf{A}_e \arrow[dl, "U_e"'] \arrow[dr, "U_e{}'"] & \\ \mathbf{A}_e^{\pi} \arrow[rr, "\theta"] & & \mathbf{A}_e^{\pi} \end{tikzcd} \tag{2.15}$$

is commutative.

This defines an equivalence relation on any set of extensions of S by $\mathbf{A}$.

Given an extension $\varepsilon_T = (T,\pi,U)$ of S by $\mathbf{A}$, we usually identify $\mathbf{A}$ with $\mathbf{A}^{\pi}$ so that $U = \mathbf{1}$, the identity natural isomorphism on $\mathbf{A}$.

Let $\varepsilon_T = (T,\pi,\mathbf{1})$ be an extension of S by $\mathbf{A}$ and let $\mathrm{Reg}_E(\mathbf{A})$ be the regular semigroup of partial isomorphisms of $\mathbf{A}$. We define a map $\overline{\mu} : T \to \mathrm{Reg}_E(\mathbf{A})$ as follows. Given $x \in T$, choose $x' \in V(x)$ and let

$$(x)\overline{\mu} = [\beta(x,x'),\Psi(x,x')], \tag{2.16}$$

where the ω-isomorphism $\beta(x,x') : \omega((xx')\pi) \to \omega((x'x)\pi)$ is given by

$$(h)\beta(x,x') = (x'\pi)h(x\pi), \quad h \in \omega((xx')\pi), \tag{2.17}$$

and the natural isomorphism $\Psi(x,x') : \mathbf{A}^{(xx')\pi} \to A^{(x'x)\pi}\underline{C}(\beta(x,x'))$ sends each object h of $\underline{C}(\omega(xx')\pi)$ to the isomorphism

$$\Psi_h(x,x') : a \longrightarrow x'ax : \mathbf{A}_h \longrightarrow \mathbf{A}_{(x'\pi)h(x\pi)}. \tag{2.18}$$

The element $(x)\overline{\mu}$ is independent of the chosen $x' \in V(x)$, and $x \to (x)\overline{\mu}$ defines an idempotent-separating homomorphism $\overline{\mu} : T \to \mathrm{Reg}_E(\mathbf{A})$ such that $(e)\overline{\mu} = [1_{e\pi},\mathbf{1}_{e\pi}]$ for every $e \in E(T)$. These facts are immediate from [10, Theorem 1.6], as $\overline{\mu}$ is essentially the idempotent-separating homomorphism induced by the composite: $\mathrm{Ker}\,\pi : \mathbf{D}(T) \to \mathbf{GR}$ with the projection functor $\mathbf{C}(T) \to \mathbf{D}(T) : (e,x,x') \to [e,x,x']$. If $x \in \mathbf{A}_e$ with inverse x^{-1} in $A_{e'}$, then from (2.17) and (2.18) we obtain

$$(x)\overline{\mu} = [\beta(x,x^{-1}),\Psi(x,x^{-1})] = [1_e,\eta^x] \in (\mathrm{Inn}\,\mathbf{A})_e, \tag{2.19}$$

where $\eta : \mathbf{A}_e \to \mathrm{Reg}_E(\mathbf{A})$ is as in (2.6). Hence, $\overline{\mu}$ induces an idempotent-separating homomorphism $\Psi : S \to \mathrm{Reg}_E(\mathbf{A})/\mathrm{Inn}_E(\mathbf{A})$ completing the square

$$\begin{tikzcd} T \arrow[r, "\overline{\mu}"] \arrow[d, "\pi"'] & \mathrm{Reg}_E(\mathbf{A}) \arrow[d, "t"] \\ S \arrow[r, "\Psi"] & \mathrm{Reg}_E(\mathbf{A})/\mathrm{Inn}_E(\mathbf{A}) \end{tikzcd} \tag{2.20}$$

where t is the projection homomorphism. From (2.17) it is clear that the diagram

$$\begin{tikzcd} S \arrow[rr, "\Psi"] \arrow[dr, "\theta_S"'] & & \mathrm{Reg}_E(\mathbf{A})/\mathrm{Inn}_E(\mathbf{A}) \arrow[dl, "\theta'_A"] \\ & T_E & \end{tikzcd} \tag{2.21}$$

is commutative. Here as in [13], θ_S denotes the fundamental representation of S and θ'_A is the idempotent-separating homomorphism induced by the fundamental representation $\theta_A : \mathrm{Reg}_E(\mathbf{A}) \to T_E : [\alpha,\phi] \mapsto [\alpha]$ of $\mathrm{Reg}_E(\mathbf{A})$.

Definition 2.7. Let $\mathbf{A} : \underline{C}(E) \to \mathbf{GR}$ be a group E-diagram that factors through $\mathbf{D}(B(E))$. Let $\Psi : S \to \mathrm{Reg}_E(\mathbf{A})/\mathrm{Inn}_E(\mathbf{A})$ be an idempotent-separating homomorphism such that diagram (2.21) is commutative. Then the triple $(S,\Psi,\mathbf{A})$, or just Ψ, is called an *abstract kernel.*

The discussion preceding Definition 2.7 shows that an extension $\varepsilon_T = (T,\pi,\mathbf{1})$ of S by $\mathbf{A}$ defines an abstract kernel $\Psi : S \to \mathrm{Reg}_E(\mathbf{A})/\mathrm{Inn}_E(\mathbf{A})$ which we call *the abstract kernel of the extension* ε_T. ε_T is called *an extension of the abstract kernel* Ψ.

Remark 2.8. If $\Psi : S \to \mathrm{Reg}_E(\mathbf{A})/\mathrm{Inn}_E(\mathbf{A})$ is an abstract kernel, then the following two properties of Ψ are immediate from the commutativity of diagram (2.21):

(i) $[1_e,\mathbf{1}_e] \in (e)\Psi$ for every $e \in E$,

(ii) if $[\alpha,\phi] \in (x)\Psi$, then for some $x' \in V(x)$, there is a representative $(\alpha',\phi') : xx' \to x'x$ of $[\alpha,\phi]$ in the inductive groupoid $\mathbf{G}(\mathbf{A})$ such that $(e)\alpha' = x'ex$ for all $e \in \omega(xx')$.

Note that if (ii) holds for one $x' \in V(x)$, then it holds for all $x' \in V(x)$.

The rest of the section is devoted to a description of extensions of S by $\mathbf{A}$ which induce the given abstract kernel Ψ. We first fix some notation and develop necessary preliminaries for this purpose.

Remark 2.9. Suppose $\mathbf{A}$ is a covariant E-diagram in an arbitrary category C which factors through $\mathbf{D}(B(E))$. That is, there is a (necessarily unique) functor $\hat{\mathbf{A}} : \mathbf{D}(B(E)) \to C$ such that the diagram

$$\begin{tikzcd} \underline{C}(E) \arrow[rr, "\mathbf{A}"] \arrow[dr, "\pi_1"'] & & \subset \\ & \mathbf{D}(B(E)) \arrow[ur, "\hat{\mathbf{A}}"'] & \end{tikzcd} \tag{2.22}$$

is commutative, where $\pi_1 : (e,c(e_0,e_1,\dots,e_n)) \to [e,e_0e_1\cdots e_n,e_ne_{n-1}\cdots e_0] : \underline{C}(E) \to \mathbf{D}(B(E))$ is the composite $\underline{C}(E) \xrightarrow{\overline{\varepsilon}_{B(E)}} \mathbf{C}(B(E)) \to \mathbf{D}(B(E))$. In this case, for any idempotent-separating homomorphism $\mu : S \to \mathrm{Reg}_E(\mathbf{A})$ with $\mu\theta_S = \theta_A$, the associated covariant functor $\mathbf{A}_\mu : \mathbf{C}(S) \to C : (\mathbf{A}_\mu)_e = \mathbf{A}_e$; $e \in \mathbf{C}(S)$ and $\mathbf{A}_\mu(e,x,x') = (\phi_{xx'}(x,x'))^{-1}\mathbf{A}(e,xx')$ for each morphism $(e,x,x') : e \to f$ of $\mathbf{C}(S)$ factors through $\mathbf{D}(S)$. We denote the functor

$e \to \mathbf{A}_e$; $[e,x,x'] \to \mathbf{A}(e,xx')\phi_{xx'}(x,x') : \mathbf{D}(S) \to C$ by $\overline{\mathbf{A}}_\mu$ itself so that the diagram

$$\begin{tikzcd} \underline{C}(E) \arrow[d, "\varepsilon_S"'] \arrow[dr, "\mathbf{A}"] & \\ \mathbf{C}(S) \arrow[d] & C \\ \mathbf{D}(S) \arrow[ur, "\mathbf{A}_\mu"'] & \end{tikzcd} \tag{2.23}$$

is commutative.

If $e, f \in E$, then for any $h \in S(e,f)$, $(e, c(eh,h,hf))$ is a morphism from e to hf in $\underline{C}(E)$.

We write

$$D(h,e,f) = \big(e, c(eh,h,hf)\big). \tag{2.24}$$

If $f\omega^l e$, then we write

$$L(e,f) = \big(e, c(ef,f)\big). \tag{2.25}$$

Note that $L(e,f) = D(f,e,f)$. Also note that

$$\pi_1 D(h,e,f) = [e,ef,h] : e \longrightarrow hf, \qquad \pi_1 L(e,f) = [e,ef,f] : e \longrightarrow f, \tag{2.26}$$

where $\pi_1 : \underline{C}(E) \to \mathbf{D}(B(E))$ is as in Remark 2.9.

Lemma 2.10. *Let $\mathbf{A} : \underline{C}(E) \to \mathbf{GR}$ be an E-diagram that factors through $\mathbf{D}(B(E))$. Let $e, f, g, \dots$ denote arbitrary elements of E.*

(i) If $g\omega^l f\omega^l e$, then

$$\mathbf{A}\big(L(e,f)L(f,g)\big) = \mathbf{A}\big(L(e,g)\big). \tag{2.27}$$

(ii) If $f\omega^l e$, $h \in S(f,g)$, then $h \in S(e,hg)$ and

$$\mathbf{A}\big(L(e,f)D(h,f,g)\big) = \mathbf{A}\big(D(h,e,hg)\big). \tag{2.28}$$

If, in addition, $h \in S(e,g)$ (this happens, e.g., $f\mathbf{L}e$), then

$$\mathbf{A}\big(L(e,f)D(h,f,g)\big) = \mathbf{A}\big(D(h,e,g)\big). \tag{2.29}$$

(iii) If $h \in S(e,f)$ and $g \in S(e,k)$, with $k\omega f$, then $gf \in S(hf,k)$ and

$$\mathbf{A}\big(D(h,e,f)D(gf,hf,k)\big) = \mathbf{A}\big(D(g,e,k)\big). \tag{2.30}$$

(iv) If $h \in S(e,f)$, $g\omega^l e$, $g\omega^r f$, then $g \in S(e,gf)$, $gf\omega^l hf$, and

$$\mathbf{A}\big(D(h,e,f)L(hf,gf)\big) = \mathbf{A}\big(D(g,e,gf)\big). \tag{2.31}$$

If, in addition, $g \in S(e,f)$, then

$$\mathbf{A}\big(D(h,e,f)L(hf,gf)\big) = \mathbf{A}\big(D(g,e,f)\big). \tag{2.32}$$

(v) If $h \in S(e,f)$, $g\mathbf{R}f$, then $h \in S(e,g)$, $hf \in S(hf,g)$, and

$$\mathbf{A}\big(D(h,e,f)D(hf,hf,g)\big) = \mathbf{A}\big(D(h,e,g)\big). \tag{2.33}$$

(vi) If $f\omega^l e$, $g \in S(e,n)$, $h \in S(f,eg)$, $m \in S(f,n)$, $k \in S(h(eg),n)$, with $mn\mathbf{L}kn$, then

$$\mathbf{A}\big(D(h,f,eg)D\big(k,h(eg),n\big)\big) = \mathbf{A}\big(D(m,f,n)L(mn,kn)\big). \tag{2.34}$$

Proof. By Remark 2.9, it is sufficient to prove (i)-(ii) replacing $\mathbf{A}$ by the functor $\pi_1 : \underline{C}(E) \to \mathbf{D}(B(E))$. We frequently use (1.2) to prove the lemma.

(i) Using (1.2) we get $\pi_1(L(e,f)L(f,g)) = \pi_1(L(e,f)\pi_1 L(f,g)) = [e,ef,f][f,fg,g] = [e,(ef)(fg),g] = [e,eg,g] = \pi_1(L(e,g))$. This proves (i).

(ii) Let $f\omega^l e$ and $h \in S(f,g)$. Then clearly $h \in S(e,hg)$, and $\pi_1(L(e,f)D(h,f,g)) = [e,ef,f][f,fg,h] = [e,(ef)(fg),h] = [e,e(hg),h] = \pi_1(D(h,e,hg))$. If $h \in S(e,g)$, then $D(h,e,g) = D(h,e,hg)$. Therefore, the second statement follows from the first.

(iii) Clearly $gf \in S(hf,k)$. Now $\pi_1(D(h,e,f)D(gf,hf,k)) = [e,ef,h][hf,(hf)k,gf] = [e,(ef)((hf)k),(gf)h] = [e,ek,gh] = [e,ek,g] = \pi_1(D(g,e,k))$, since $(ef)((hf)k) = ek$, and $(gf)h = gh$.

(iv) Clearly $g \in S(e,gf)$ and $gf\omega^l hf$. By taking $k = gf$ in (iii) and observing $D(gf,hf,gf) = L(hf,gf)$, we get $\pi_1(D(h,e,f)L(hf,gf)) = \pi_1(D(g,e,gf))$. The last relation follows from this since $D(g,e,gf) = D(g,e,f)$, if $g \in S(e,f)$.

(v) Let $h \in S(e,f)$, $g\mathbf{R}f$. Then, by [13, Proposition 2.12], $S(e,f) = S(e,g)$ and so $h \in S(e,g)$. Clearly $hf \in S(hf,g)$. Further, since $(hf)h = h(fh) = h$, we get $\pi_1(D(h,e,f)D(hf,hf,g)) = [e,ef,h][hf,(hf)g,hf] = [e,(ef)((hf)g),h] = [e,eg,h] = \pi_1(D(h,e,g))$.

(vi)

$$\begin{aligned}
&\pi_1\big(D(h,f,eg)D\big(k,h(eg),n\big)\big)\\
&\quad= \big[f,f(eg),h\big]\big[h(eg),\big(h(eg)n,k\big)\big]\\
&\quad= \big[f,\big(f(eg)\big)\big(\big(h(eg)\big)n\big),kh\big]\\
&\quad= \big[f,\big(f(eg)\big)n,kh\big] \quad \text{since } (eg)h = h,\ (fh)(eg) = f(eg)\\
&\quad= \big[f,fn,kh\big] \quad \text{since } (eg)n = en, fe = f\\
&\quad= \big[f,f(nm)n,kh\big] \quad \text{since } fmn = fn, nm = m\\
&\quad= \big[f,(fn)(mn),kh\big]\\
&\quad= \big[f,(fn)(mn),(kn)m\big]\\
&\quad= \big[f,fn,m\big]\big[mn,mn,kn\big]\\
&\quad= \pi_1\big(D(m,f,n)L(mn,kn)\big).
\end{aligned} \tag{2.35}$$

The proof of the lemma is complete. □

We fix once and for all a map $*: S \to S$ such that

(i) $x^* \in V(x)$ for every $x \in S$,

(ii)

$$x^* \in \mathbf{H}_e \quad \text{if } x \in \mathbf{H}_e. \tag{2.36}$$

Suppose $\Psi : S \to \mathrm{Reg}_E(\mathbf{A})/\mathrm{Inn}_E(\mathbf{A})$ is an abstract kernel, and let $\sigma : S \to \mathrm{Reg}_E(\mathbf{A})$ be a map such that $(x)\sigma \in (x)\Psi$, for every $x \in S$. By Remark 2.9(ii) and by [13], each $(x)\sigma$ has a unique representative in $\mathbf{G}(\mathbf{A})$ with domain xx^* and range x^*x. We denote this morphism by $(\alpha(x),\phi(x)) : xx^* \to x^*x$ so that $[\alpha(x),\phi(x)] = (x)\sigma$; recall by Remark 2.9(ii), $(h)\alpha(x) = x^*hx$ for all $h \in \omega(xx^*)$. Using σ we will define a biaction of S on the disjoint union

$$\mathbf{A} = \bigcup_{x\in S} \mathbf{A}_x, \quad \text{where } \mathbf{A}_x = \mathbf{A}_{x^*x}. \tag{2.37}$$

For $x, y \in S$, define

$$a \longrightarrow x \bullet a : \mathbf{A}_y \longrightarrow \mathbf{A}_{xy}, \qquad a \longrightarrow a \bullet x : \mathbf{A}_y \longrightarrow \mathbf{A}_{yx} \tag{2.38}$$

by

$$x \bullet a = a\mathbf{A}\big(L(y^*y,(xy)^*xy)\big), \tag{2.39}$$

$$a \bullet x = a\mathbf{A}\big(D(h,y^*y,xx^*)\big)\phi_{hxx^*}(x)\mathbf{A}\big(L(x^*hx,(yx)^*yx)\big), \tag{2.40}$$

where $h \in S(y^*y,xx^*)$ and $\phi_{hxx^*}(x) : \mathbf{A}_{hxx^*} \to \mathbf{A}_{(hxx^*)\alpha(x)=x^*hx}$ is the component of $\phi(x)$ at $hxx^* \in \omega(xx^*)$. If $k \in S(y^*y,xx^*)$ is any other element, then the following diagram commutes:

$$\begin{tikzcd}
 & \mathbf{A}_{hxx^*} \arrow[r, "\phi_{hxx^*}(x)"] \arrow[dd, "\mathbf{A}(L(hxx^*,kxx^*))"] & \mathbf{A}_{x^*hx} \arrow[dr, "\mathbf{A}(L(x^*hx,(yx)^*yx))"] \arrow[dd, "\mathbf{A}(L(x^*hx,x^*kx))"'] & \\
\mathbf{A}_y = \mathbf{A}_{y^*y} \arrow[ur, "\mathbf{A}(D(h,y^*y,xx^*))"] \arrow[dr, "\mathbf{A}(D(k,y^*y,xx^*))"'] & & & \mathbf{A}_{(yx)^*yx} = \mathbf{A}_{yx} \\
 & \mathbf{A}_{kxx^*} \arrow[r, "\phi_{kxx^*}(x)"] & \mathbf{A}_{x^*kx} \arrow[ur, "\mathbf{A}(L(x^*kx,(yx)^*yx))"'] &
\end{tikzcd} \tag{2.41}$$

The first triangle is commutative by Lemma 2.10(iv) and the last triangle is commutative by Lemma 2.10(i), since $(yx)^*yx\omega^l x^*kx\omega^l x^*hx$. Finally the commutativity of the rectangle follows from the naturality of $\phi(x)$. Hence, $a \bullet x$ does not depend on the choice of h. Clearly $a \to a \bullet x$ and $a \to x \bullet a$ are homomorphisms of groups.

The following lemma explains to what extent the biaction of S on $\mathbf{A}$ depends on σ.

Lemma 2.11. *Suppose $\sigma,\sigma' : S \to \mathrm{Reg}_E(\mathbf{A})$ are maps, with $(x)\sigma,(x)\sigma' \in (x)\Psi$, and let $\bullet$ and $\spadesuit$ be the corresponding biactions of S on $\mathbf{A}$. Then*

(i)

$$x\spadesuit a = x\bullet a \quad x\in S,\ a\in \mathbf{A}_y, \tag{2.42}$$

(ii) *there exists a map $\beta : S \to A$, with $(x)\beta \in A_x$, such that*

$$\begin{gathered}(x)\sigma' = (x)\sigma\big((x)\beta\big)\eta,\\ a\spadesuit x = (a\bullet x)\big(y\bullet (x)\beta\big)\eta, \quad x\in S,\ a\in \mathbf{A}_y,\end{gathered} \tag{2.43}$$

where $\eta : \mathbf{A}_{yx}(=\mathbf{A}_{(yx)^*yx}) \to \mathrm{Reg}_E(\mathbf{A})$ is as in (2.6).

Proof. (i) is clear, since $x\spadesuit a = a\mathbf{A}(L(y^*y,(xy)^*xy)) = x\bullet a$.

(ii) Since $(x)\sigma$, $(x)\sigma'$ belong to the same class $(x)\Psi$, by Proposition 2.5, there must be elements $(x)\beta \in \mathbf{A}_x$ such that $(x)\sigma' = (x)\sigma((x)\beta)\eta$. Let $(\alpha(x),\phi(x)) : xx^* \to x^*x$ and $(\overline{\alpha(x)},\overline{\phi(x)}) : xx^* \to x^*x$ be unique representatives of $(x)\sigma$ and $(x)\sigma'$ with domain xx^* and range x^*x.

$$\begin{aligned}&\Longrightarrow \big[\overline{\alpha(x)},\overline{\phi(x)}\big] = \big[\alpha(x),\phi(x)\big]\big[1_{x^*x},\eta^{(x)\beta}\big]\\ &\Longrightarrow \overline{\phi_e}(x) = \big\{\phi(x)\big(\underline{C}(\alpha(x))\eta^{(x)\beta}\big)\big\}_e = \phi_e(x)\eta^{(x)\beta}{}_{x^*ex},\end{aligned} \tag{2.44}$$

for every $e\in\omega(xx^*)$. Hence, for $h\in S(y^*y,xx^*)$,

$$\begin{aligned}a\spadesuit x &= a\mathbf{A}\big(D(h,y^*y,xx^*)\big)\overline{\phi_{hxx^*}}(x)\mathbf{A}\big(L(x^*hx,(yx)^*yx)\big)\\ &= \big\{\big\{a\mathbf{A}\big(D(h,y^*y,xx^*)\big)\phi_{hxx^*}(x)\big\}\eta^{(x)\beta}{}_{x^*hx}\big\}\mathbf{A}\big(L(x^*hx,(yx)^*yx)\big)\\ &= \big[(x)\beta\mathbf{A}\big(L(x^*x,x^*hx)\big)\mathbf{A}\big(L(x^*hx,(yx)^*yx)\big)\big]^{-1}\\ &\quad\times\big[a\mathbf{A}\big(D(h,y^*y,xx^*)\big)\phi_{hxx^*}(x)\mathbf{A}\big(L(x^*hx,(yx)^*yx)\big)\big]\\ &\quad\times\big[(x)\beta\mathbf{A}\big(L(x^*x,x^*hx)\big)\mathbf{A}\big(L(x^*hx,(yx)^*yx)\big)\big]\\ &= \big[(x)\beta\mathbf{A}\big(L(x^*x,(yx)^*yx)\big)\big]^{-1}(a\bullet x)\big[(x)\beta\mathbf{A}\big(L(x^*x,(yx)^*yx)\big)\big]\\ &\qquad\text{by Lemma 2.10(i), since } (yx)^*yx\omega^l x^*hx\omega^l x^*x\\ &= \big(y\bullet(x)\beta\big)^{-1}(a\bullet x)\big(y\bullet(x)\beta\big)\\ &= (a\bullet x)\big(y\bullet(x)\beta\big)\eta.\end{aligned} \tag{2.45}$$

□

Definition 2.12. Let $\Psi : S \to \mathrm{Reg}_E(\mathbf{A})/\mathrm{Inn}_E(\mathbf{A})$ be an abstract kernel. Let $\sigma : S \to \mathrm{Reg}_E(\mathbf{A})$ and $p : S\times S \to \mathbf{A}$, $(x,y)p \in \mathbf{A}_{xy}$, be maps such that

(i)

$$(x)\sigma \in (x)\Psi, \tag{2.46}$$

(ii)

$$(x)\sigma(y)\sigma = (xy)\sigma((x,y)p)\eta, \tag{2.47}$$

(iii)

$$(xy,z)p((x,y)p\bullet z) = (x,yz)p(x\bullet(y,z))p, \tag{2.48}$$

where $\eta : \mathbf{A}_{xy}[= \mathbf{A}_{(xy)^*xy}] \to \mathrm{Reg}_E(\mathbf{A})$ is as in (2.6) and the biaction $\bullet$ of S on $\mathbf{A}$ is with respect to the map σ. Then the pair (σ, p) is called a *crossed pair*.

Let Ψ be an abstract kernel and let σ, p be maps satisfying Definition 2.12(i) and (ii). In the next two lemmas, we establish some of the essential properties of the biaction $\bullet$ of S on $\mathbf{A}$ induced by σ.

Lemma 2.13. *Let $\Psi : S \to \mathrm{Reg}_E(\mathbf{A})/\mathrm{Inn}_E(\mathbf{A})$ be an abstract kernel. Let (σ,p) satisfy Definition 2.12(i) and (ii) and let $\bullet$ denote the biaction of S on $\mathbf{A}$ induced by σ.*

(i) If $(\alpha(e),\phi(e)) : e \to e$ is a representative of $(e)\sigma$ with domain and range e, then $\phi_e(e)$ coincides with the inner automorphism defined by $(e,e)p$. More generally, if $e_1\omega^r e$, then for any $a \in \mathbf{A}_e = \mathbf{A}_e$, $a \bullet e_1 = (e_1,e_1)p^{-1}a\mathbf{A}(e,c(e_1e,e_1))(e_1,e_1)p$.

(ii) If $x \in S$, $e \in E(S)$, with $ex = x$, then for $a \in \mathbf{A}_x$, $e \bullet a = a$. If (σ,p) also satisfies (2.48), then for $y \in S$, $e \in E(S)$, $(e,e)p \bullet y = (e,y)p^{-1}(e,ey)p(e,y)p$.

Proof. (i) For $e \in E(S)$, $(e)\sigma(e)\sigma = (e)\sigma((e,e)p)\eta \Rightarrow (e)\sigma = ((e,e)p)\eta \Rightarrow \phi_e(e) = \eta_e{}^{(e,e)p}$, the inner automorphism defined by the element $(e,e)p$.

If $e_1\omega^r e$, then $e_1e \in S(e,e_1)$ and for $a \in \mathbf{A}_e$, by (2.40),

$$\begin{aligned} a\bullet e_1 &= a\mathbf{A}(D(e_1e,e,e_1))\phi_{e1}(e_1) = a\mathbf{A}(e,c(e_1e,e_1))\eta_{e_1}{}^{(e_1,e_1)p} \\ &= (e_1,e_1)p^{-1}a\mathbf{A}(e,c(e_1e,e_1))(e_1,e_1)p. \end{aligned} \tag{2.49}$$

(ii) Clearly $e\bullet a = a\mathbf{A}(L(x^*x,(ex)^*ex)) = a\mathbf{A}(L(x^*x,x^*x)) = a$. To prove the last assertion, let $y \in S$, $e \in E(S)$. Then, by (2.48), $(e,y)p((e,e)p\bullet y) = (e,ey)p(e\bullet(e,y))p$ or $(e,e)p\bullet y = (e,y)p^{-1}(e,ey)p(e,y)p$, since $e(ey) = ey$ implies $(e\bullet(e,y))p = (e,y)p$. □

Lemma 2.14. *Let Ψ, σ, p be as in the first paragraph of Lemma 2.13. Then*

(i)

$$x\bullet(y\bullet a) = xy\bullet a, \quad x,y \in S,\ a \in \mathbf{A}_z, \tag{2.50}$$

(ii)

$$x\bullet(b\bullet z) = (x\bullet b)\bullet z, \quad x,z \in S,\ b \in \mathbf{A}_y, \tag{2.51}$$

(iii)

$$\begin{aligned} (d\bullet y)\bullet z &= (d\bullet yz)(x\bullet(y,z)p)\eta \\ &= (x\bullet(y,z)p)^{-1}(d\bullet yz)(x\bullet(y,z)p) \quad y,z \in S,\ d \in \mathbf{A}_x. \end{aligned} \tag{2.52}$$

Proof. (i) is immediate from Lemma 2.10(i), since $(xyz)^*xyz\omega^l(yz)^*yz\omega^l z^*z$.

As before for $x \in S$, let $(\alpha(x),\phi(x)) : xx^* \to x^*x$ denote the unique representative of $(x)\sigma$ with domain xx^* and range x^*x. Let $x,y,z \in S$. Choose $h \in S(x^*x, yy^*)$, $h_2 \in S(y^*y, zz^*)$. Then $(xy)^*xy\mathbf{L}y^*hy$ and $yz(yz)^*\mathbf{R}yh_2y^*$. Let $h_1 \in S((xy)^*xy, zz^*) = S(y^*hy, zz^*)$, $h_3 \in S(x^*x, yz(yz)^*) = S(x^*x, yh_2y^*)$. Since $\phi(z) : \mathbf{A}^{zz^*} \to \mathbf{A}^{z^*z}\underline{C}(\alpha(z))$ is a natural isomorphism, the diagram

$$\begin{tikzcd}
\mathbf{A}_{h_2zz^*} \arrow[r, "\phi_{h_2zz^*}(z)"] \arrow[d, "\mathbf{A}(L(h_2zz^*,h_1zz^*))"'] & \mathbf{A}_{(h_2zz^*)\alpha(z)=z^*h_2z} \arrow[d, "\mathbf{A}(L(z^*h_2z,z^*h_1z))"] \\
\mathbf{A}_{h_1zz^*} \arrow[r, "\phi_{h_1zz^*}(z)"] & \mathbf{A}_{(h_1zz^*)\alpha(z)=z^*h_1z}
\end{tikzcd} \tag{2.53}$$

is commutative. Now, using Lemma 2.10(i) twice, we get

$$\begin{aligned}
x\bullet(b\bullet z) &= b\mathbf{A}\big(D(h_2,y^*y,zz^*)\big)\phi_{h_2zz^*}(z)\mathbf{A}\big(L(z^*h_2z,(yz)^*yz)\big) \\
&\quad\times\mathbf{A}\big(L((yz)^*yz,(xyz)^*xyz)\big) \\
&= b\mathbf{A}\big(D(h_2,y^*y,zz^*)\big)\phi_{h_2zz^*}(z)\mathbf{A}\big(L(z^*h_2z,z^*h_1z)\big)\mathbf{A}\big(L(z^*h_1z,(xyz)^*xyz)\big) \\
&= b\mathbf{A}\big[\big(D(h_2,y^*y,zz^*)L(h_2zz^*,h_1zz^*)\big)\big]\phi_{h_1zz^*}(z)\mathbf{A}\big(L(z^*h_1z,(xyz)^*xyz)\big) \\
&= b\mathbf{A}\big(D(h_1,y^*y,h_1zz^*)\big)\phi_{h_1zz^*}(z)\mathbf{A}\big(L(z^*h_1z,(xyz)^*xyz)\big) \\
&\qquad\text{by Lemma 2.10(iv), since } h_2\in S(y^*y,zz^*),\ h_1\omega^l y^*y,\ h_1\omega^r zz^* \\
&= b\mathbf{A}\big(L(y^*y,(xy)^*xy)\big)\mathbf{A}\big(D(h_1,(xy)^*xy,zz^*)\big)\phi_{h_1zz^*}(z) \\
&\quad\times\mathbf{A}\big(L(z^*h_1z,(xyz)^*xyz)\big) \\
&\qquad\text{by Lemma 2.10(ii), since } (xy)^*xy\omega^l y^*y \text{ and } h_1\in S((xy)^*xy,zz^*) \\
&= (x\bullet b)\bullet z.
\end{aligned} \tag{2.54}$$

Hence, the proof of (ii) is complete. To prove (iii), consider the diagram

$$\begin{tikzcd}
 & & & \mathbf{A}_{(xy)^*xy}=\mathbf{A}_{xy} \arrow[dr, "C_2"] & & & \\
 & \mathbf{A}_{hyy^*} \arrow[r, "C_4"] \arrow[d, "C_7"] \arrow[dr, phantom, "\mathrm{III}"] & \mathbf{A}_{y^*hy} \arrow[ur, "C_1"] \arrow[rr, "C_5"] \arrow[d, "C_8"] \arrow[r, phantom, "\mathrm{I}"] \arrow[drr, phantom, "\mathrm{IV}"] & & \mathbf{A}_{h_1zz^*} \arrow[r, "C_6"] \arrow[d, "C_9"] \arrow[dr, phantom, "\mathrm{V}"] & \mathbf{A}_{z^*h_1z} \arrow[d, "C_{10}"] \arrow[dr, "C_{11}"] & \\
\mathbf{A}=\mathbf{A}_{x^*x} \arrow[ur, "C_3"] \arrow[r, "C_{12}"] \arrow[dr, "C_{16}"'] \arrow[u, phantom, "\mathrm{II}"] & \mathbf{A}_{h_3yh_2y^*} \arrow[r, "C_{13}"] & \mathbf{A}_{y^*h_3yh_2} \arrow[rr, "C_{14}"] & & \mathbf{A}_{h_2y^*h_3h_2zz^*} \arrow[r, "C_{15}"] & \mathbf{A}_{z^*h_2y^*h_3yz} \arrow[d, "C_{18}"] \arrow[dr, phantom, "\mathrm{VIII}"] & \mathbf{A}_{(xyz)^*xyz}=\mathbf{A}_{xyz} \\
 & \mathbf{A}_{h_3yz(yz)^*} \arrow[u, "C_{17}"'] \arrow[rrrr, "C_{20}"] \arrow[ul, phantom, "\mathrm{VI}"] \arrow[urr, phantom, "\mathrm{VII}"] & & & & \mathbf{A}_{(yz)^*h_3yz} \arrow[ur, "C_{19}"'] &
\end{tikzcd} \tag{2.55}$$

where

$$
\begin{aligned}
&C_1 = \mathbf{A}(L(y^*hy,(xy)^*xy)), \qquad C_2 = \mathbf{A}(D(h_1,(xy)^*xy,zz^*)),\\
&C_3 = \mathbf{A}(D(h,x^*x,yy^*)), \qquad C_4 = \phi_{hyy^*}(y), \qquad C_5 = \mathbf{A}(D(h_1,y^*hy,zz^*)),\\
&C_6 = \phi_{h_1zz^*}(h), \qquad C_7 = \mathbf{A}(D(h_3yy^*,hyy^*,yh_2y^*)),\\
&C_8 = \mathbf{A}(D(y^*h_3y,y^*hy,y^*yh_2)), \qquad C_9 = \mathbf{A}(L(h_1zz^*,h_2y^*h_3yh_2zz^*)),\\
&C_{10} = \mathbf{A}(L(z^*h_1z,z^*h_2y^*h_3yz)), \qquad C_{11} = \mathbf{A}(L(z^*h_1z,(xyz)^*xyz)),\\
&C_{12} = \mathbf{A}(D(h_3,x^*x,yh_2y^*)), \qquad C_{13} = \phi_{h_3yh_2y^*}(y),\\
&C_{14} = \mathbf{A}(D(h_2y^*h_3yh_2,y^*h_3yh_2,zz^*)), \qquad C_{15} = \phi_{h_2y^*h_3yh_2zz^*}(z),\\
&C_{16} = \mathbf{A}(D(h_3,x^*x,yz(yz)^*)), \qquad C_{17} = \mathbf{A}(D(h_3yz(yz)^*,h_3yz(yz)^*,yh_2y^*)),\\
&C_{18} = \mathbf{A}(L(z^*h_3y^*h_3yz,(yz)^*h_3yz)), \qquad C_{19} = \mathbf{A}(L((yz)^*h_3yz,(xyz)^*xyz)),\\
&C_{20} = \phi_{h_3yz(yz)^*}(yz)\eta_{(yz)h_3yz}^{(y,z)p\mathbf{A}(L((yz)^*yz,(yz)^*h_3yz))}.
\end{aligned}
\tag{2.56}
$$

The commutativity of the diagram I follows from Lemma 2.10(ii), since $(xy)^*xy\mathbf{L}y^*hy$ and $h_1 \in S((xy)^*xy,zz^*) = S(y^*hy,zz^*)$. Since $yh_2y^*\omega yy^*$, by Lemma 2.10(iii), $h_3yy^* \in S(hyy^*,yh_2y^*)$ and the diagram II is commutative. The diagrams III and V are commutative, since $\phi(y)$ and $\phi(z)$ are natural isomorphisms. Next we show that the diagram IV is commutative. Now

$$
y^*h_3yh_2\omega^l h_2 \Longrightarrow y^*h_3yh_2\mathbf{L}h_2y^*h_3yh_2 \Longrightarrow h_2y^*h_3yh_2 \in S(y^*h_3yh_2,zz^*). \tag{2.57}
$$

Also

$$
h_3\omega^r yh_2y^* \Longrightarrow h_1zz^*h_2y^*h_3 = h_1(xy)^*xh_3 \Longrightarrow h_1zz^*\mathbf{L}h_2y^*h_3yzz^*, \tag{2.58}
$$

since $zz^*h_2 = h_2$ and $h_3yh_1zz^* = h_3x^*xyh_1zz^* = h_3yzz^*$. Take $e = y^*y$, $f = y^*hy$, $g = h_2$, $h = y^*h_3y$, $k = h_2y^*h_3yh_2$, $m = h_1$, $n = zz^*$. The commutativity of the diagram IV now follows from Lemma 2.10(vi). Since $yz(yz)^*\mathbf{R}yh_2y^*$, the commutativity of the diagram VI follows from Lemma 2.10(v). Since $z^*h_1z\mathbf{L}z^*h_2y^*h_3yz\mathbf{L}(yz)^*h_3yz\mathbf{L}(xyz)^*xyz$, the diagram VIII is commutative by Lemma 2.10(i). Finally we establish the commutativity of the diagram VII. Put $c_1 = c(y^*yh_2,h_2,h_2zz^*)$, $c_2 = c(yz(yz)^*,yh_2y^*)$, $c_3 = c(z^*h_2z,(yz)^*yz)$, $(\alpha_1,\phi_1) = yh_2y^* * (\alpha(y),\phi(y))$, $(\alpha_2,\phi_2) = (\alpha(z),\phi(z)) * z^*h_2z$. Then

$$
\begin{aligned}
&[(\alpha(yz),\phi(yz))(1_{(yz)^*yz},\eta^{(y,z)p})]\\
&\quad = [\alpha(yz),\phi(yz)][1_{(yz)^*yz},\eta^{(y,z)p}]\\
&\quad = (yz)\sigma((y,z)p)\eta\\
&\quad = (y)\sigma(z)\sigma\\
&\quad = [\alpha(y),\phi(y)][\alpha(z),\phi(z)]\\
&\quad = [((\alpha(y),\phi(y)) * y^*yh_2)\varepsilon(c_1)(h_2zz^* * (\alpha(z),\phi(z)))] \quad \text{by [10]}\\
&\quad = [(\alpha_1,\phi_1)\varepsilon(c_1)(\alpha_2,\phi_2)] \quad \text{by [13, Proposition 3.2]}.
\end{aligned}
\tag{2.59}
$$

Since $yz(yz)^*\mathbf{R}yh_2y^*$ and $z^*h_2z\mathbf{L}(yz)^*yz$, (1.9) implies that

$$\begin{aligned}(\alpha(yz),\phi(yz))&(1_{(yz)^*yz},\eta^{(y,z)p})\\ &=\varepsilon(c_2)(\alpha_1,\phi_1)\varepsilon(c_1)(\alpha_2,\phi_2)\varepsilon(c_3)\\ &=(\alpha^{c_2},\phi^{c_2})(\alpha_1,\phi_1)(\alpha^{c_1},\phi^{c_1})(\alpha_2,\phi_2)(\alpha^{c_3},\phi^{c_3}).\end{aligned} \tag{2.60}$$

Therefore, the component at e of the natural isomorphism defined by the left-hand side coincides with the component at e of the natural isomorphism defined by the right-hand side for each $e\in\omega(yz(yz)^*)$. In particular, by taking $e=h_3yz(yz)^*$ and noting that ${\phi_e}^{c_2}=\mathbf{A}(D(e,e,yh_2y^*))$ and ${\phi^{c1}}_{y^*h_3yh_2}=\mathbf{A}(D(h_2y^*h_3yh_2,y^*h_3yh_2,zz^*))$, we obtain the commutativity of the diagram VII. As the interior diagrams are commutative, the outer diagram is commutative. Hence, for $d\in\mathbf{A}_x=\mathbf{A}_{x^*x}$,

$$\begin{aligned}(d\bullet y)\bullet z&=d\mathbf{A}(D(h,x^*x,yy^*))\phi_{hyy^*}(y)\mathbf{A}(L(y^*hy,(xy)^*xy))\\ &\qquad\times\mathbf{A}(D(h_1,(xy)^*xy,zz^*))\phi_{h_1zz^*}(z)\mathbf{A}(L(z^*h_1z,(xyz)^*xyz))\\ &=d\mathbf{A}(D(h_3,x^*x,yz(yz)^*))\phi_e(yz)\mathbf{A}(L((yz)^*h_3yz,(xyz)^*xyz))\eta^{(x\bullet(y,z)p)}_{(xyz)^*xyz}\\ &=(x\bullet(y,z)p)^{-1}(d\bullet yz)(x\bullet(y,z)p).\end{aligned} \tag{2.61}$$

With these preliminaries we are now in a position to describe the extensions of S by $\mathbf{A}$ which induce the given abstract kernel Ψ. □

Theorem 2.15. *Let $\Psi:S\to\operatorname{Reg}_E(\mathbf{A})/\operatorname{Inn}_E(\mathbf{A})$ be an abstract kernel and let (σ,p) be a crossed pair. Let*

$$T_p=\{(x,a):x\in S,\ a\in\mathbf{A}_x\}. \tag{2.62}$$

Define a multiplication on T_p by

$$(x,a)(y,b)=(xy,(x,y)p(a\bullet y)(x\bullet b)). \tag{2.63}$$

Then T_p is a regular semigroup with

$$E(T_p)=\{(e,(e,e)p^{-1}):e\in E(S)\}. \tag{2.64}$$

The map $\pi_p:T_p\to S$ defined by $(x,a)\pi_p=x$, is an idempotent-separating homomorphism of T_p onto S. For each $e\in E=E(S)$, define $(U_p)_e:\mathbf{A}_e\to{\mathbf{A}_e}^{\pi p}$ by

$$(a)(U_p)_e=(e,(e,e)p^{-1}a). \tag{2.65}$$

Then $U_p:e\to(U_p)_e$ defines a natural isomorphism between $\mathbf{A}$ and $\mathbf{A}^{\pi p}$. The triple (T_p,π_pU_p) is an extension of S by $\mathbf{A}$.

Proof. For $(x,a),(y,b),(z,c)\in T_p$, by (2.48), Lemma 2.14, we easily prove that $((x,a)(y,b))(z,c)=(x,a)((y,b)(z,c))$. So the multiplication is associative. For each $e\in E(S)$, $(e,(e,e)p^{-1})(e,(e,e)p^{-1})=(e,(e,e)p((e,e)p^{-1}\bullet e)(e\bullet(e,e)p^{-1}))=(e,(e,e)p^{-1})$, since

$(e,e)p^{-1} \bullet e = e \bullet (e,e)p^{-1} = (e,e)p^{-1}$, by Lemma 2.13(ii). Hence, $(e,(e,e)p^{-1}) \in E(T_p)$. On the other hand, $(e,a)(e,a) = (e,a) \Rightarrow (ee,(e,e)p(a \bullet e)(e \bullet a)) = (e,a) \Rightarrow ee = e$ and $(e,e)p(a \bullet e)(e \bullet a) = a \Rightarrow e \in E(S)$ and $(e,e)p = a^{-1} \bullet e = (e,e)p^{-1}a^{-1}(e,e)p$ (by Lemma 2.13(i) and (ii)) $\Rightarrow e \in E(S)$ and $a = (e,e)p^{-1}$.

Hence, $E(T_p) = \{(e,(e,e)p^{-1}) : e \in E(S)\}$. To prove T_p is a regular semigroup, take any $(x,a) \in T_p$ and let y be an inverse of x in S. Put $b = y \bullet ((xy,xy)p(x,y)p(a \bullet y))^{-1}$. Then $(y,b) \in T_p$, and $x \bullet b = xy \bullet ((xy,xy)p(x,y)p(a \bullet y))^{-1}$ (by Lemma 2.14(i)) $= ((xy,xy)p(x,y)p(a \bullet y))^{-1}$ (by Lemma 2.13(ii)). Then

$$\begin{aligned} (x,a)(y,b) &= (xy,(x,y)p(a \bullet y)(x \bullet b)) \\ &= (xy,(x,y)p(a \bullet y)(a \bullet y)^{-1}(x,y)p^{-1}(xy,xy)p^{-1}) \\ &= (xy,(xy,xy)p^{-1}). \end{aligned} \tag{2.66}$$

Therefore, $(x,a)(y,b)(x,a) = (xy,(xy,xy)p^{-1})(x,a) = (xyx,(xy,x)p((xy,xy)p^{-1} \bullet x)(xy \bullet a)) = (x,a)$, since by Lemma 2.13(ii), $xy \bullet a = a$ and $(xy,xy)p \bullet x = (xy,x)p$ and $(y,b)(x,a)(y,b) = (y,b)(xy,(xy,xy)p^{-1}) = (y,(y,xy)p(b \bullet xy)(y \bullet (xy,xy)p^{-1})) = (y,b)$, since

$$\begin{aligned} b \bullet xy &= \left[y \bullet ((a^{-1} \bullet y)(x,y)p^{-1}(xy,xy)p^{-1})\right] \bullet xy \\ &= y \bullet \left[((a^{-1} \bullet y)(x,y)p^{-1}(xy,xy)p^{-1}) \bullet xy\right] \quad \text{by Lemma 2.14(ii)} \\ &= y \bullet \left[((a^{-1} \bullet y) \bullet xy)((x,y)p^{-1} \bullet xy)((xy,xy)p^{-1} \bullet xy)\right] \\ &= y \bullet ((x \bullet (y,xy)p^{-1})(a^{-1} \bullet y)(x \bullet (y,xy)p)((x,y)p^{-1} \bullet xy)(xy,xy)p^{-1}) \\ &\qquad \text{by Lemmas 2.14(iii) and 2.13(ii)} \\ &= y \bullet ((x \bullet (y,xy)p^{-1})(a^{-1} \bullet y)(x,y)p^{-1}) \quad \text{using (2.48) for the triple } x,\ y,\ xy \\ &= (yx \bullet (y,xy)p^{-1})(y \bullet (a^{-1} \bullet y)(x,y)p^{-1}) \quad \text{by Lemma 2.14(i)} \\ &= (y,xy)p^{-1}(y \bullet (a^{-1} \bullet y)(x,y)p^{-1}) \quad \text{by Lemma 2.13(ii).} \end{aligned} \tag{2.67}$$

Hence, (y,b) is an inverse of (x,a), and T_p is a regular semigroup. The map $\pi_p : T_p \to S$, $(x,a)\pi_p = x$, is clearly an idempotent-separating homomorphism from T_p onto S with $\mathbf{A}_e{}^{\pi p} = \{(e,a) : a \in \mathbf{A}_e\}$ for each $e \in E$. The map $U_e = (U_p)_e : \mathbf{A}_e \to \mathbf{A}_e{}^{\pi p}$ defined by (2.65) is clearly a bijection. By Lemma 2.13, it is clear that U_e is also a homomorphism.

We next show that the isomorphisms U_e define a natural isomorphism $U_p : \mathbf{A} \to \mathbf{A}^{\pi p}$. We must show that for each morphism $(e,c(e_0,\dots,e_n)) : e \to f$ in $\underline{C}(E)$, the diagram

$$\begin{tikzcd} \mathbf{A}_e \arrow[r, "U_e"] \arrow[d, "\mathbf{A}(e{,}c)"'] & \mathbf{A}_e{}^{\pi p} \arrow[d, "\mathbf{A}^{\pi p}(e{,}c)"] \\ \mathbf{A}_f \arrow[r, "U_f"] & \mathbf{A}_f{}^{\pi p} \end{tikzcd} \tag{2.68}$$

is commutative. Since $(e,c(e_0,\dots,e_n)) = (e,e_0)(e_0,c(e_0,e_1))\cdots(e_{n-1},c(e_{n-1},e_n))$, it is enough to prove the commutativity of the diagram for morphisms of the form $(e,c(e,f))$, with $e \geq f$ or $e(\mathbf{R}\cup\mathbf{L})f$.

Case 1 ($e \geq f$). Let $a \in \mathbf{A}_e$. Then, since

$$\begin{aligned}
\mathbf{A}^{\pi p}(e,f) &= \operatorname{Ker}\pi_p\big[(e,(e,e)p^{-1}),(f,(f,f)p^{-1}),(f,(f,f)p^{-1})\big],\\
(a)U_e\mathbf{A}^{\pi p}(e,f) &= (e,(e,e)p^{-1}a)\mathbf{A}^{\pi p}(e,f)\\
&= (f,(f,f)p^{-1})(e,(e,e)p^{-1}a)(f,(f,f)p^{-1})\\
&= (e,(e,e)p^{-1}a)(f,(f,f)p^{-1}) \quad \text{by Definition 1.1}\\
&= (f,(e,f)p(((e,e)p^{-1}a)\bullet f)(e\bullet(f,f)p^{-1}))\\
&= (f,(e,f)p((e,e)p^{-1}\bullet f)(a\bullet f)(f,f)p^{-1})\\
&\qquad \text{since } e\bullet(f,f)p = (f,f)p \text{ by Lemma 2.13(ii)}\\
&= (f,(a\bullet f)(f,f)p^{-1}) \quad \text{since } (e,e)p\bullet f = (e,f)p \text{ by Lemma 2.13(ii)}\\
&= (f,(f,f)p^{-1}(aA(e,f))) \quad \text{by Lemma 2.13(i)}\\
&= a\mathbf{A}(e,f)U_f.
\end{aligned} \tag{2.69}$$

Similarly we prove the diagram is commutative for other cases $e\mathbf{R}f$ and $e\mathbf{L}f$ also. Hence, by Definition 2.1, (T_p,π_p,U_p) is an extension of S by $\mathbf{A}$. The proof of Theorem 2.15 is complete. □

We denote the extension (T_p,π_p,U_p) by $(S,\sigma,p,\mathbf{A})$, and call *the crossed extension of S by* $\mathbf{A}$ determined by the crossed pair (σ,p).

Theorem 2.16. *Let $\Psi : S \to \operatorname{Reg}_E(\mathbf{A})/\operatorname{Inn}_E(\mathbf{A})$ be an abstract kernel and let (σ,p) be a crossed pair, with $(x)\sigma \in (x)\Psi$ for every $x \in S$. Then the abstract kernel of the crossed extension $(S,\sigma,p,\mathbf{A})$ coincides with Ψ.*

Proof. Define $j : S \to T_p$ by $(x)j = (x,1_x)$, where 1_x denotes the identity element of $\mathbf{A}_x$. For each $(x,a) \in T_p$, let

$$(x,a)^* = (x^*,x^*\bullet((a^{-1}\bullet x^*)(x,x^*)p^{-1}(xx^*,xx^*)p^{-1})). \tag{2.70}$$

Then the proof of Theorem 2.15 shows that $(x,a)^* \in V(x,a)$. Let $\overline{\mu} : T_p \to \operatorname{Reg}_E(\mathbf{A})$ be the idempotent-separating homomorphism defined by (2.8). Then

$$(x)j\overline{\mu} = \big[\beta(xj,(xj)^*),\Psi(xj,(xj)^*)\big], \quad x \in S. \tag{2.71}$$

The proof of the theorem follows once we show that the representative $(\beta(xj,(xj)^*),\Psi(xj,(xj)^*)) : xx^* \to x^*x$ of $(x)j\overline{\mu}$ and the representative $(\alpha(x),\phi(x)) : xx^* \to x^*x$ of $(x)\sigma$ in $\mathbf{G}(\mathbf{A})$ are equal. From Remark 2.8(ii) and (2.13) it is clear that

$$\beta(xj,(xj)^*) = \alpha(x) \tag{2.72}$$

we next show that $\Psi(xj,(xj)^*) = \phi(x)$. To prove this we must show that $\Psi_e(xj,(xj)^*) = \phi_e(x) : \mathbf{A_e} \to \mathbf{A}_{x^*ex}$ for each $e \in \omega(xx^*)$. For this purpose we first make some calculations.

Let $e \in \omega(xx^*)$. Put $d = (x,x^*)p^{-1}(xx^*,xx^*)p^{-1}$. Then, by Lemmas 2.13(ii) and 2.14(i),

$$(x^* \bullet d) \bullet ex = (x^*x \bullet (x^*,ex)p^{-1})(x^* \bullet (x,x^*ex)p^{-1}). \tag{2.73}$$

Therefore, since $x^*x \bullet (x^*,ex)p^{-1} = (x^*,ex)p^{-1}$ by Lemma 2.13(ii),

$$(x^* \bullet d) \bullet ex = (x^*,ex)p^{-1}(x^* \bullet (x,x^*ex)p^{-1}). \tag{2.74}$$

Putting $x = x^*e$, $y = x$, $z = x^*ex$ in (2.48),

$$(x^*ex,x^*ex)p(x^*e,x)p \bullet x^*ex = (x^*e,ex)p(x^*e \bullet (x,x^*ex)p). \tag{2.75}$$

Since $(x^*e,x)p \bullet x^*ex = (x^*ex,x^*ex)p^{-1}(x^*e,x)p(x^*ex,x^*ex)p$ by Lemma 2.13(i) and $x^*e \bullet (x,x^*ex)p = (x^* \bullet (e \bullet (x,x^*ex)p)) = x^* \bullet (x,x^*ex)p$ by Lemma 2.13(ii), the above equation becomes

$$(x^*e,x)p(x^*ex,x^*ex)p = (x^*e,ex)p(x^* \bullet (x,x^*ex)p) \tag{2.76}$$

or

$$(x^* \bullet (x,x^*ex)p^{-1}) = (x^*ex,x^*ex)p^{-1}(x^*e,x)p^{-1}(x^*e,ex)p. \tag{2.77}$$

Since

$$\begin{aligned}
(x,x^*e)p \bullet e &= (e,e)p^{-1}(x,x^*e)p(e,e)p \quad \text{by Lemma 2.13(i)},\\
(e,e)p(x,x^*e)p \bullet e &= (x,x^*e)p(x \bullet (x^*e,e)p) \quad \text{by (2.48)}\\
\Longrightarrow (x,x^*e)p(e,e)p &= (x,x^*e)p(x \bullet (x^*e,e)p) \Longrightarrow (e,e)p^{-1} = (x \bullet (x^*e,e)p^{-1})\\
\Longrightarrow (x^* \bullet ((e,e)p^{-1} \bullet x)) &= x^*x \bullet ((x^*e,e)p^{-1} \bullet x) = (x^*e,e)p^{-1} \bullet x.
\end{aligned} \tag{2.78}$$

Also since

$$\begin{aligned}
x^*e \bullet (e,x)p &= x^* \bullet (e \bullet (e,x)p) = x^* \bullet (e,x)p \quad \text{by Lemma 2.13(ii)},\\
(x^*e,x)p(x^*e,e)p \bullet x &= (x^*e,ex)p(x^*e \bullet (e,x)p) \quad \text{by (2.48)}\\
\Longrightarrow x^* \bullet (e,x)p &= (x^*e,ex)p^{-1}(x^*e,x)p(x^*e,e)p \bullet x\\
\Longrightarrow (x^*e,e)p \bullet x &= (x^*e,x)p^{-1}(x^*e,ex)p(x^* \bullet (e,x)p).
\end{aligned} \tag{2.79}$$

For any $a \in \mathbf{A}_e$, by (2.74), (2.77), (2.78), (2.79), and Lemma 2.13(i), it is easy to show

$$(xj)^*(e,(e,e)p^{-1}a)(xj) = (x^*ex,(x^*ex,x^*ex)p^{-1})(x^* \bullet (a \bullet x)). \tag{2.80}$$

But

$$\begin{aligned}
x^* \bullet (a \bullet x) &= x^* \bullet (a\mathbf{A}(D(e,e,xx^*))\phi_e(x)\mathbf{A}(L(x^*ex,(ex)^*ex)))\\
&= x^* \bullet (a\phi_e(x)\mathbf{A}(L(x^*ex,(ex)^*ex))) \quad \text{since } D(e,e,xx^*) = 1_e\\
&= a\phi_e(x)\mathbf{A}(L(x^*ex,(ex)^*ex))\mathbf{A}(L((ex)^*ex,x^*ex)) \quad \text{by (2.39)}\\
&= a\phi_e(x) \quad \text{by Lemma 2.13(i)}.
\end{aligned} \tag{2.81}$$

Hence, $(xj)^*(e,(e,e)p^{-1}a)(xj) = (x^*ex,(x^*ex,x^*ex)p^{-1})(a)\phi_e(x)$. This implies $\Psi_e(xj,(xj)^*) = \phi_e(x)$ for every $e \in \omega(xx^*)$, where we have identified $a \in \mathbf{A}_e$ with $(e,(e,e)p^{-1}a)$ under the isomorphism $(U_p)_e : \mathbf{A}_e \to \mathbf{A}_e{}^{\pi P}$. Hence,

$$\Psi(xj,(xj)^*) = \phi(x). \tag{2.82}$$

The result now follows from (2.72) and (2.82). □

Lemma 2.17 [9, Lemma 4.2]. *Let $(T,\pi,\mathbf{1})$ be an extension of S by $\mathbf{A}$. Let $j : S \to T$ be a map such that $j\pi = \mathbf{1}_S$ and let $\bullet$ denote the biaction of S on $\mathbf{A}$ induced by the composite*

$$\sigma : S \xrightarrow{j} T \xrightarrow{\overline{\mu}} \mathrm{Reg}_E(\mathbf{A}), \tag{2.83}$$

where $\overline{\mu}$ is as in (2.16). Then $(xj)a(yj)b = (xj)(yj)(a \bullet y)(x \bullet b)$ for $x,y \in S$, $a \in \mathbf{A}_{\mathbf{x}} = \mathbf{A}_{x^*x}$, $b \in \mathbf{A}_{\mathbf{y}} = \mathbf{A}_{y^*y}$.

Theorem 2.18. *Let $\varepsilon_T = (T,\pi,\mathbf{1})$ be an extension of S by $\mathbf{A}$ with abstract kernel $\Psi : S \to \mathrm{Reg}_E(\mathbf{A})/\mathrm{Inn}_E(\mathbf{A})$. Let $\sigma : S \to \mathrm{Reg}_E(\mathbf{A})$ be a map such that $(x)\sigma \in (x)\Psi$ for each $x \in S$. Then ε_T is equivalent to a crossed extension of the form $(S,\sigma,p,\mathbf{A})$ with abstract kernel Ψ.*

Proof. Let $\overline{\mu} : T \to \mathrm{Reg}_E(\mathbf{A})$ be the idempotent-separating homomorphism defined by (2.16). Using the commutativity of diagram (2.20), it is easy to see that every element in the class $(x)\Psi$ is of the form $[\beta(u,u'),\Psi(u,u')]$ for some $u \in T$, $u' \in V(u)$, with $u\pi = x$. So there is a map $j : S \to T$, with $j\pi = 1_S$, such that $\overline{j\mu} = \sigma$; in particular $(x)\sigma \in (x)\Psi$ for every $x \in S$. Since $((xj)(yj))\pi = (x)j\pi(y)j\pi = xy = (xy)j\pi$, Lemma 1.6 defines a function $p : S \times S \to \mathbf{A}$, $(x,y)p \in \mathbf{A}_{xy}$, such that $(xj)(yj) = (xy)j(x,y)p$. This implies, for $x,y \in S$, $(x)\sigma(y)\sigma = (x)j\overline{\mu}(y)j\overline{\mu} = ((xj)(yj))\overline{\mu} = (xy)j\overline{\mu}(x,y)p\overline{\mu} = (xy)\sigma(x,y)p\eta$, by (2.19). Again for $x,y,z \in S$, we have by Lemma 2.17, $(xj)((yj)(zj)) = (xj)(yz)j(y,z)p = (xj)(yz)j(x \bullet (y,z)p) = (xyz)j(x,yz)p(x \bullet (y,z)p)$, and $((xj)(yj))(zj) = (xy)j(x,y)p(zj) = (xy)j(zj)((x,y)p \bullet z) = (xyz)j(xy,z)p((x,y)p \bullet z)$, where $\bullet$ denotes the biaction of S on $\mathbf{A}$ induced by σ. Since the multiplication in T is associative, by Lemma 1.6,

$$(xy,z)p((x,y)p \bullet z) = (x,yz)p(x \bullet (y,z)p). \tag{2.84}$$

Thus (σ,p) is a crossed pair.

Next we show that the extension ε_T is equivalent to a crossed extension $(S,\sigma,p,\mathbf{A})$. Define $\theta : T_p \to T$ by $(x,a)\theta = (xj)a$. Then, by Lemma 2.17, θ is a homomorphism: $((x,a)(y,b))\theta = (xy,(x,y)p(a \bullet y)(x \bullet b))\theta = (xy)j(x,y)p(a \bullet y)(x \bullet b) = (x)j(y)j(a \bullet y)(x \bullet b) = (xj)a(yj)b = (x,a)\theta(y,b)\theta$. From Lemma 1.6, we see that θ is a bijection and therefore an isomorphism. $\theta_\pi = \pi_p$, since $(x,a)\theta_\pi = ((xj)a)\pi = x = (x,a)\pi_p$. Finally the diagram

$$\begin{tikzcd} & \mathbf{A}_e \arrow[dl, "U_p"'] \arrow[dr, equal] & \\ \mathbf{A}_e{}^{\pi p} \arrow[rr, "\theta"] & & \mathbf{A}_e{}^{\pi} \end{tikzcd} \tag{2.85}$$

is commutative, since $(ej)(ej) = (ej)(e,e)p$ implies $ej = (e,e)p$ and hence for $a \in \mathbf{A}_e$, $(a)U_p\theta = (e,(e,e)p^{-1}a)\theta = (ej)(e,e)p^{-1}a = (e,e)p(e,e)p^{-1}a = a$. Hence, ε_T is equivalent to a crossed extension $(S,\sigma,p,\mathbf{A})$.This completes the proof of Theorem 2.18. □

Combining Theorems 2.15, 2.16, and 2.18, we obtain a complete description of extensions of S by A which induce the given abstract kernel Ψ in terms of the crossed pairs (σ,p).

3. Obstructions to extensions

Let S^I be the regular semigroup obtained from S by adjoining an identity element I ($I \notin S$). Extend the map $* : S \to S$ (see (2.36)) to S^I by defining $I^* = I$. Now recall the category $D(S^I)$ [5] as follows. The objects are elements of S^I and morphisms are the triples $\langle u,x,v\rangle : x \to y$ such that $uxv = y$. The morphism composition is defined by $\langle u,x,v\rangle\langle u', uxv,v'\rangle = \langle u'u,x,vv'\rangle$. Let $\mathbf{F} : D(S^I) \to D(S^I)$ be the functor defined by $\mathbf{F}(x) = x^*x$ on objects of $D(S^I)$ and $\mathbf{F}\langle u,x,v\rangle = [x^*x,x^*xvy^*y,y^*ux]$ on morphisms $\langle u,x,v\rangle : x \to y$ of $D(S^I)$ [9]. A functor $\mathbf{G} : D(S^I) \to \mathbf{Ab}$ is called a $D(S^I)$-*module*. For $D(S^I)$-modules $\mathbf{G}$ and $\mathbf{H}$, $D(S^I)$-homomorphism $\phi : \mathbf{G} \to \mathbf{H}$ is a natural transformation of functors. We denote by $\hom_{D(S^I)}(\mathbf{G},\mathbf{H})$ the abelian group of all $D(S^I)$-homomorphisms from $\mathbf{G}$ to $\mathbf{H}$ and by $\mathrm{Mod}(D(S^I))$ the category of $D(S^I)$-modules and $D(S^I)$-homomorphisms. $\mathrm{Mod}(D(S^I))$ is an abelian category with enough injectives and projectives. Let $D(S^I)_0$ be the subcategory of $D(S^I)$ defined by the identity morphisms of $D(S^I)$. A $D(S^I)_0$-set is a functor $\Gamma : D(S^I)_0 \to \mathrm{Sets}$ from $D(S^I)_0$ to the category of sets, and $D(S^I)_0$-map is a natural transformation between two $D(S^I)_0$-sets. A $D(S^I)$-module (resp., $D(S^I)$-homomorphism) defines a $D(S^I)_0$-set (resp., $D(S^I)_0$-map) in an obvious manner. For more details, refer to [5].

If Γ is a $D(S^I)_0$-set, then the free $D(S^I)$-module on Γ is the $D(S^I)$-module $\mathbf{G}$ such that, for each object y of $D(S^I)$, $\mathbf{G}_y$ is the free abelian group generated by elements of the form $(a,\langle u,x,u'\rangle)$, $a \in \Gamma_x$, $x \in$ object $D(S^I)$, $uxu' = y$. If $\langle v,y,v'\rangle : y \to z$ is a morphism of $D(S^I)$, then $\mathbf{G}\langle v,y,v'\rangle : \mathbf{G}_y \to \mathbf{G}_z$ is defined by

$$(a,\langle u,x,u'\rangle)\mathbf{G}\langle v,y,v'\rangle = (a,\langle vu,x,u'v'\rangle). \tag{3.1}$$

We identify $a \in \Gamma_x$ with $(a,\langle 1,x,1\rangle)$ in G_x. For $n \geq 0$, let x_n be the free $D(S^I)$-module on the $D(S^I)_0$-set Γ_n, where, for $n \geq 1$,

$$\Gamma_n(x) = \{[u_1,\dots,u_n] \in (S^I)^n : u_1u_2\cdots u_n = x\} \tag{3.2}$$

and, for $n = 0$,

$$\Gamma_0(x) = \begin{cases} \{[1]\} & \text{if } x = 1, \\ 0 & \text{if } x \neq 1,\ x \in S^I. \end{cases} \tag{3.3}$$

Now we recall [5, Theorem 2.3]: the complex

$$X \cdots \longrightarrow X_n \xrightarrow{\partial n} X_{n-1} \xrightarrow{\partial_{n-1}} \cdots \xrightarrow{\partial_3} X_2 \xrightarrow{\partial_2} X_1 \xrightarrow{\partial_1} X_0 \xrightarrow{\varepsilon} \mathbf{Z}_{S^I} \longrightarrow 0 \tag{3.4}$$

is called the *standard resolution of* $\mathbf{Z}_{S^I}$.

Let $\overline{\Gamma}_n(x) = \{[u_1,\dots,u_n] \in \Gamma_n(x) : u_i \neq 1,\ i = 1,2,\dots,n\}$, $n \geq 1$, and $\overline{\Gamma}_n = \cup\overline{\Gamma}_n(x)$, $x \in S^I$. Then the $D(S^I)_0$-set $\overline{\Gamma}_n$ freely generates a $D(S^I)$-submodule $\overline{X}_n$ of X_n. Put $\overline{X}_0 = X_0$. Define ∂_n as before, putting $[u_1,\dots,u_n] = 0$ whenever one of the u_i is one. Then we obtain another projective resolution

$$\overline{X} \cdots \longrightarrow \overline{X}_n \xrightarrow{\partial_n} \overline{X}_{n-1} \xrightarrow{\partial_{n-1}} \cdots \xrightarrow{\partial_3} \overline{X}_2 \xrightarrow{\partial_2} \overline{X}_1 \xrightarrow{\partial_1} \overline{X}_0 \xrightarrow{\varepsilon} \mathbf{Z}_{S^I} \longrightarrow 0 \tag{3.5}$$

of $\mathbf{Z}_{S^I}$, called *the normalised standard resolution of* $\mathbf{Z}_{S^I}$.

Let $\mathbf{G} \in \mathrm{Mod}(\mathbf{D}(S^I))$ and let

$$\begin{aligned} &\hom_{D(S^I)}(\overline{X},\mathbf{G}) : 0 \longrightarrow \hom_{D(S^I)}(\overline{X}_0,\mathbf{G}) \xrightarrow{\partial_1{}^*} \hom_{D(S^I)}(\overline{X}_1,\mathbf{G}) \xrightarrow{\partial_2{}^*} \cdots \\ &\xrightarrow{\partial_{n-1}{}^*} \hom_{D(S^I)}(\overline{X}_{n-1},\mathbf{G}) \xrightarrow{\partial_n{}^*} \hom_{D(S^I)}(\overline{X}_n,\mathbf{G}) \xrightarrow{\partial_{n+1}{}^*} \cdots. \end{aligned} \tag{3.6}$$

Definition 3.1. The nth *cohomology group of S^I with coefficients in* $\mathbf{G}$, denoted by $\mathbf{H}^n(S^I, \mathbf{G})$, is defined by

$$\mathbf{H}^n(S^I,\mathbf{G}) = \mathbf{H}^n[\hom_{D(S^I)}(\overline{X},\mathbf{G})] = \mathrm{Ker}\,\partial_{n+1}{}^*/\mathrm{Im}\,\partial_n{}^*. \tag{3.7}$$

The elements of $\hom_{D(S^I)}(\overline{X},\mathbf{G})$ are called *(normalized) n-cochains*. The elements of $\mathrm{Ker}\,\partial_{n+1}{}^*$ are called *(normalized) n-cocycles* and the elements of $\mathrm{Im}\,\partial_n{}^*$ are called *(normalized) n-coboundaries*. Two n-cocycles $k_1,k_2 \in \mathrm{Ker}\,\partial_{n+1}{}^*$ are called *cohomologous* if they differ by a coboundary.

Let $\mathbf{A} : \underline{C}(E) \to \mathbf{GR}$ be a group E-diagram that factors through $\mathbf{D}(B(E))$ and let $\mathbf{Z}(\mathbf{A})$ be the centre of $\mathbf{A}$. For each $x \in S$, let $\overline{\mathbf{Z}(\mathbf{A})}_x = \mathbf{Z}(\mathbf{A})_{x^*x}$ and let

$$\overline{\mathbf{Z}(\mathbf{A})} = \bigcup_{x\in S} \overline{\mathbf{Z}(\mathbf{A})}_x \tag{3.8}$$

be the disjoint union of $\overline{\mathbf{Z}(\mathbf{A})}_x$'s. Remark that $\overline{\mathbf{Z}(\mathbf{A})}_x$ is contained in the centre of $\overline{\mathbf{A}}_x$, where as in the previous section $\overline{\mathbf{A}} = \bigcup_{x\in S}\overline{\mathbf{A}}_x$ with $\mathbf{A}_x = \overline{\mathbf{A}}_{x^*x}$.

Suppose $\Psi : S \to \mathrm{Reg}_E(\mathbf{A})/\mathrm{Inn}_E(\mathbf{A})$ is an abstract kernel. Then the composite $S \xrightarrow{\Psi} (\mathrm{Reg}_E(\mathbf{A})/\mathrm{Inn}_E(\mathbf{A})) \xrightarrow{V} \mathrm{Reg}_E(\mathbf{Z}(\mathbf{A}))$ is an idempotent-separating homomorphism. Since $\mathbf{A}$ and hence $\mathbf{Z}(\mathbf{A})(: \underline{C}(E) \to \mathbf{Ab})$ factors through $D(B(E))$, by Remark 2.9, $\Psi\nu$ induces a functor $\check{\mathbf{Z}}(\mathbf{A}) = \check{\mathbf{Z}}(\mathbf{A})_{\Psi\nu} : \mathbf{D}(S) \to \mathbf{Ab}$. Let $\check{\mathbf{Z}}(\mathbf{A})^0 : \mathbf{D}(S^I) \to \mathbf{Ab}$ be the extension of $\check{\mathbf{Z}}(\mathbf{A})$ such that $\check{\mathbf{Z}}(\mathbf{A})_I{}^0 = \{0\}$ and let $\check{\mathbf{Z}}(\mathbf{A})^0\mathbf{F}$ be the composite $D(S^I) \xrightarrow{\mathbf{F}} \mathbf{D}(S^I) \xrightarrow{\check{\mathbf{Z}}(\mathbf{A})^0} \mathbf{Ab}$. In this section, we associate with the abstract kernel Ψ a 3-dimensional cohomology class $[k] \in \mathbf{H}^3(S^I,\check{\mathbf{Z}}(\mathbf{A})^0\mathbf{F})$ and show that Ψ admits an extension if and only if $[k] = 0$. We also show that if Ψ has an extension, then the set of all equivalence classes of extensions of S by A is in bijective correspondence with the set $\mathbf{H}^2(S^I,\check{\mathbf{Z}}(\mathbf{A})^0\mathbf{F})$.

Let $\sigma : S \to \mathrm{Reg}_E(\mathbf{A})$ be any map such that $(x)\sigma \in (x)\Psi$. As before for each $x \in S$, let $(\alpha(x),\phi(x)) : xx^* \to x^*x$ denote the unique representative of $(x)\sigma$ in $\mathbf{G}(\mathbf{A})$ with domain xx^* and range x^*x and let $(\alpha(x),\overline{\phi(x)}) : xx^* \to x^*x$ denote the element of $\mathbf{G}(\mathbf{Z}(\mathbf{A}))$ determined by $(\alpha(x),\phi(x))$ (see (2.13)) so that $(x)\Psi\nu = (x)\sigma u = [\alpha(x),\overline{\phi(x)}]$. The biaction of S on $\mathbf{A}$ defined by σ induces by restriction a biaction of S on $\overline{\mathbf{Z}(\mathbf{A})}$ which coincides with the one induced by the composite $\Psi\nu = \sigma u : S \to \mathrm{Reg}_E(\mathbf{A}) \to \mathrm{Reg}_E(\mathbf{Z}(\mathbf{A}))$. In particular, the induced biaction of S on $\overline{\mathbf{Z}(\mathbf{A})}$ is independent of the chosen σ. We next see the relation between this biaction and the functor $\check{\mathbf{Z}}(\mathbf{A})^0\mathbf{F} : D(S^I) \to \mathbf{Ab}$. Let $x \in S$, $a \in \overline{\mathbf{Z}(\mathbf{A})}_y = \mathbf{Z}(\mathbf{A})_{y^*y} = (\check{\mathbf{Z}}(\mathbf{A})^0\mathbf{F})_y$. Then by (2.23) and (2.39) we have

$$\begin{aligned}
a(\check{\mathbf{Z}}(\mathbf{A})^0\mathbf{F}\langle x,y,I\rangle) &= a\check{\mathbf{Z}}(\mathbf{A})^0[y^*y, y^*y(xy)^*xy,(xy)^*xy] \\
&= a\check{\mathbf{Z}}(\mathbf{A})[y^*y, y^*y(xy)^*xy,(xy)^*xy] \\
&= a\mathbf{Z}(\mathbf{A})(L(y^*y,(xy)^*xy)) = x \bullet a, \\
a(\check{\mathbf{Z}}(\mathbf{A})^0F\langle I,y,x\rangle) &= a\check{\mathbf{Z}}(\mathbf{A})^0[y^*y, y^*yx,(yx)^*y] \\
&= a\check{\mathbf{Z}}(\mathbf{A})[y^*y, y^*yx,(yx)^*y] \\
&= a\check{\mathbf{Z}}(\mathbf{A})[y^*y, y^*yx,(yx)^*yh] \quad \text{by (1.2)} \\
&= a\check{\mathbf{Z}}(\mathbf{A})\{[y^*y, y^*yxx^*,h] \\
&\qquad \times [hxx^*,hx,x^*hxx^*][x^*hx,x^*hx,(yx)^*yx]\} \\
&= a\mathbf{Z}(\mathbf{A})(D(h,y^*y,xx^*))\check{\mathbf{Z}}(\mathbf{A})[hxx^*,hx,x^*hxx^*] \\
&\quad \times \mathbf{Z}(\mathbf{A})(L(x^*hx,(yx)^*yx)) \quad \text{by (2.23)} \\
&= a\mathbf{Z}(\mathbf{A})(D(h,y^*y,xx^*))\overline{\phi}_{hxx^*}(hx)\mathbf{Z}(\mathbf{A})(L(x^*hx,(yx)^*yx)) \\
&\qquad \text{by Remark 2.9} \\
&= a\mathbf{Z}(\mathbf{A})(D(h,y^*y,xx^*))\overline{\phi}_{hxx^*}(x)\mathbf{Z}(\mathbf{A})(L(x^*hx,(yx)^*yx)) \\
&= a \bullet x \quad \text{by (2.40)},
\end{aligned} \tag{3.9}$$

where $h \in S(y^*y,xx^*)$, and the components $\overline{\phi}_{hxx^*}(hx)$ of $\overline{\phi}(hx)$ and $\overline{\phi}_{hxx^*}(x)$ of $\overline{\phi}(x)$ are equal since $[\alpha(hx),\overline{\phi}(hx)] = (hx)\Psi\nu = (hxx^*)\Psi\nu(x)\Psi\nu = [1_{hxx^*},\mathbf{1}_{hxx^*}][\alpha(x),\overline{\phi}(x)] = [hxx^* * (\alpha(x),\phi(x))]$. Thus we have $x \bullet a = a(\check{\mathbf{Z}}(\mathbf{A})^0\mathbf{F}\langle x,y,I\rangle)$ and $a \bullet x = a(\check{\mathbf{Z}}(\mathbf{A})^0\mathbf{F}\langle I,y,x\rangle)$.

Next we describe the cohomology groups. Consider the normalized standard resolution (3.5). Since the $D(S^I)$-module $\overline{X}_n$'s are free on $\overline{\Gamma}_n$'s and since $(\check{\mathbf{Z}}(\mathbf{A})^0\mathbf{F})_I = \{0\}$, we have

$$\begin{aligned}
\hom_{D(S^I)}(\overline{X}_n,\check{\mathbf{Z}}(\mathbf{A})^0\mathbf{F}) &= \hom_{D(S^I)_0}(\overline{\Gamma}_n,\check{\mathbf{Z}}(\mathbf{A})^0\mathbf{F}) \\
&= \Big\{\alpha : \underset{(n\text{ times})}{S \times S} \times \cdots \times S \longrightarrow \overline{\mathbf{Z}(\mathbf{A})} : (x_1,x_2,\ldots,x_n) \in \overline{\mathbf{Z}(\mathbf{A})}_{x_1x_2\cdots x_n}\Big\}.
\end{aligned} \tag{3.10}$$

Hence, we may regard an n-cochain as a map $\alpha : \underbrace{S\times S}_{(n \text{ times})} \times \cdots \times S \to \overline{\mathbf{Z}(\mathbf{A})}$, with $(x_1,x_2,\ldots,x_n) \in \overline{\mathbf{Z}(\mathbf{A})}_{x_1x_2\cdots x_n}$. The coboundary $\partial_n{}^*\alpha$ of an $n-1$ cochain α is given by the formula

$$\begin{aligned}(x_1,x_2,\ldots,x_n)\partial_n{}^*\alpha &= (x_2,x_3,\ldots,x_n)\alpha\check{\mathbf{Z}}(\mathbf{A})^0\mathbf{F}\langle x_1,x_2,\ldots,x_n,I\rangle \\ &\quad+\sum_{i=1}^{n-1}(-1)^i(x_1,x_2,\ldots,x_ix_{i+1},\ldots,x_n)\alpha \\ &\quad+(-1)^n(x_1,x_2,\ldots x_{n-1})\alpha\check{\mathbf{Z}}(\mathbf{A})^0\mathbf{F}\langle I,x_1,x_2,\ldots,x_{n-1},x_n\rangle \\ &= x_1\bullet(x_2,x_3,\ldots,x_n)\alpha+\sum_{i=1}^{n-1}(-1)^i(x_1,x_2,\ldots,x_ix_{i+1},\ldots,x_n)\alpha \\ &\quad+(-1)^n(x_1,x_2,\ldots,x_{n-1})\alpha\bullet x_n.\end{aligned} \tag{3.11}$$

From now on we write the group operation as multiplication. Note that a 2-cochain $\alpha : S\times S \to \overline{\mathbf{Z}(\mathbf{A})}$, $(x,y)\in\overline{\mathbf{Z}(\mathbf{A})}_{xy}$, is a 2-cocycle if

$$(xy,z)\alpha\big((x,y)\alpha\bullet z\big) = (x,yz)\alpha\big(x\bullet(y,z)\alpha\big) \tag{3.12}$$

for all $x,y,z\in S$; α is a coboundary if and only if there exists a 1-cochain $\beta : S\to\overline{\mathbf{Z}(\mathbf{A})}$, $(x)\beta\in\overline{\mathbf{Z}(\mathbf{A})}_x$, such that

$$(x,y)\alpha = \big(x\bullet(y)\beta\big)(xy)\beta^{-1}\big((x)\beta\bullet y\big) \tag{3.13}$$

for all $x,y\in S$. Similarly a 3-cocycle k is a map $k : S\times S\times S\to\overline{\mathbf{Z}(\mathbf{A})}$, $(x,y,z)k\in\overline{\mathbf{Z}(\mathbf{A})}_{xyz}$, such that

$$(xy,z,t)k(x,y,zt)k = \big((x,y,z)k\bullet t\big)(x,yz,t)k\big(x\bullet(y,z,t)k\big) \tag{3.14}$$

for all $x,y,z,t\in S$; k is a coboundary if and only if there exists a 2-cochain $\alpha : S\times S\to\overline{\mathbf{Z}(\mathbf{A})}$, $(x,y)\in\overline{\mathbf{Z}(\mathbf{A})}_{xy}$, such that

$$(x,y,z)k = \big(x\bullet(y,z)\alpha\big)(xy,z)\alpha^{-1}(x,yz)\alpha\big((x,y)\alpha\bullet z\big)^{-1} \tag{3.15}$$

for all $x,y,z\in S$. For $n=2,3$, let $\mathbf{Z}^n(\mathbf{S}^I,\check{\mathbf{Z}}(\mathbf{A})^0\mathbf{F})$ denote the abelian group of all n-cocycles and let $\mathbf{B}^n(\mathbf{S}^I,\check{\mathbf{Z}}(\mathbf{A})^0\mathbf{F})\subseteq\mathbf{Z}^n(\mathbf{S}^I,\check{\mathbf{Z}}(\mathbf{A})^0\mathbf{F})$ be the subgroup of all coboundaries. Then

$$\mathbf{H}^n(S^I,\check{\mathbf{Z}}(\mathbf{A})^0\mathbf{F}) = \frac{\mathbf{Z}^n(\mathbf{S}^I,\check{\mathbf{Z}}(\mathbf{A})^0\mathbf{F})}{\mathbf{B}^n(\mathbf{S}^I,\check{\mathbf{Z}}(\mathbf{A})^0\mathbf{F})}. \tag{3.16}$$

Now we proceed to show that the abstract kernel $\Psi : S\to\mathrm{Reg}_E(\mathbf{A})/\mathrm{Inn}_E(\mathbf{A})$ defines an element in the cohomology group $\mathbf{H}^3(\mathbf{S}^I,\check{\mathbf{Z}}(\mathbf{A})^0\mathbf{F})$, the vanishing of which is necessary and sufficient for the existence of extensions of Ψ.

We fix a map $\sigma : S \to \mathrm{Reg}_E(\mathbf{A})$ such that $(x)\sigma \in (x)\Psi$ for all $x \in S$. Let $\bullet$ denote the biaction of S on $\mathbf{A}$ induced by σ. As before we denote by $(\alpha(x),\phi(x)) : xx^* \to x^*x$ the unique representative of $(x)\sigma$ in $\mathbf{G}(\mathbf{A})$ with domain xx^* and range x^*x. Since $(x\sigma)(y\sigma)$ and $(xy)\sigma$ both belong to the same class $(xy)\Psi$, we can choose a function $p : S\times S \to \mathbf{A}$, $(x,y)p \in \mathbf{A}_{xy}$, such that

$$(x)\sigma(y)\sigma = (xy)\sigma\big((x,y)p\big)\eta, \tag{3.17}$$

where $\eta : \mathbf{A}_{xy,} = \mathbf{A}_{(xy)^*xy} \to \mathrm{Reg}_E(\mathbf{A})$ is as before.

Before proceeding further, let us first prove the following.

Lemma 3.2. *For $a \in \mathbf{A}_x$, $b \in \mathbf{A}_y$,*

$$(x\sigma)(a)\eta(y\sigma)(b)\eta = (xy)\sigma\big((x,y)p(a\bullet y)(x\bullet b)\big)\eta. \tag{3.18}$$

Proof. Consider the diagram

$$\begin{tikzcd}
xhx^* \arrow[r, "(\alpha,\phi)"] \arrow[d, "(\alpha^{c_2},\phi^{c_2})"'] & x^*xh \arrow[r, "(\alpha^{c_1},\phi^{c_1})"] & hyy^* \arrow[r, "(\beta,\Psi)"] & y^*hy \arrow[r, "(1_{y^*hy},\eta^{(aG)(bH)})"] \arrow[d, "(\alpha^{c_3},\phi^{c_3})"'] & y^*hy \arrow[d, "(\alpha^{c_3},\phi^{c_3})"'] \\
xy(xy)^* \arrow[rrr, "(\alpha(xy),\phi(xy))(1_{(xy)^*xy},\eta^{(x,y)p})"] & & & (xy)^*xy \arrow[r, "(1_{(xy)^*xy},\eta^{(a\bullet y)(x\bullet b)})"] & (xy)^*xy
\end{tikzcd} \tag{3.19}$$

where

$$\begin{gathered}
h \in S(x^*x, yy^*), \qquad c_1 = c(x^*xh, h, hyy^*), \qquad c_2 = c(xhx^*, xy(xy)^*), \\
c_3 = c(y^*hy, (xy)^*xy), \\
\varepsilon(c_1) = (\alpha^{c_1},\phi^{c_1}), \qquad \varepsilon(c_2) = (\alpha^{c_2},\phi^{c_2}), \qquad \varepsilon(c_3) = (\alpha^{c_3},\phi^{c_3}), \\
(\alpha,\phi) = (\alpha(x),\phi(x)) * x^*xh : xhx^* \longrightarrow x^*xh, \\
(\beta,\Psi) = hyy^* * (\alpha(y),\phi(y) : hyy^*) \longrightarrow y^*hy,
\end{gathered} \tag{3.20}$$

$$\begin{aligned}
H &= \mathbf{A}(L(y^*y, y^*hy)), \\
G &= \mathbf{A}(L(x^*x, x^*xh))(\phi^{c_1})_{x^*xh}\Psi_{hyy^*} \\
&= \mathbf{A}(L(x^*x, x^*xh))\mathbf{A}(D(h, x^*xh, yy^*))\Psi_{hyy^*} \\
&= \mathbf{A}(D(h, x^*x, yy^*))\Psi_{hyy^*} \quad \text{by Lemma 2.10(ii)},
\end{aligned} \tag{3.21}$$

$$\begin{gathered}
a\bullet y = (a)G\mathbf{A}(L(y^*hy,(xy)^*xy)) = (a)G(\phi^{c_3})_{y^*hy} \quad \text{by [9, (1.8)]}, \\
x\bullet b = (b)\mathbf{A}(L(y^*y,(xy)^*xy)) = (b)\mathbf{A}(L(y^*hy,(xy)^*xy)) = (b)H(\phi^{c_3})_{y^*hy} \\
\text{by [10, (1.8)]}.
\end{gathered} \tag{3.22}$$

Since $(x\sigma)(y\sigma)=(xy)\sigma((x,y)p)\eta$, the first rectangle is commutative. The second rectangle is also commutative, since for $d\in\mathbf{A}_{y^*hy}$,

$$\begin{aligned}
&d(\eta^{(aG)(bH)})_{y^*hy}(\phi^{c_3})_{y^*hy}\\
&\quad=(((aG)(bH))^{-1}d(aG)(bH))(\phi^{c_3})_{y^*hy}\\
&\quad=((aG)(bH))^{-1}(\phi^{c_3})_{y^*hy}d(\phi^{c_3})_{y^*hy}((aG)(bH))(\phi^{c_3})_{y^*hy}\\
&\quad=((a\bullet y)(x\bullet b))^{-1}(d\phi^{c_3})_{y^*hy}((a\bullet y)(x\bullet b))\\
&\quad=(d)(\phi^{c_3})_{y^*hy}(\eta^{(a\bullet y)(x\bullet b)})(xy)^*xy\\
&\quad=(d)(\phi^{c_3}(\underline{C}(\alpha^{c_3})\eta^{(a\bullet y)(x\bullet b)}))_{y^*hy}.
\end{aligned}\tag{3.23}$$

Hence, the outer diagram is commutative. Now

$$\begin{aligned}
(x\sigma)(a)\eta(y)\sigma(b)\eta&=[\alpha(x),\phi(x)][1_{x^*x},\eta^a][\alpha(y),\phi(y)][1_{y^*y},\eta^b]\\
&=[(\alpha,\phi)((1_{x^*x},\eta^a)*x^*xh)\varepsilon(c_1)(\beta,\Psi)(y^*hy^*(1_{y^*y},\eta^b))]\quad\text{by (1.10)}\\
&=[(\alpha,\phi)\varepsilon(c_1)(\beta,\Psi)(1_{y^*hy},\eta^{aG})(1_{y^*hy},\eta^{bH})]\\
&=[(\alpha(xy),\phi(xy))(1_{(xy)^*xy},\eta^{(x,y)p(a\bullet y)(x\bullet b)})]\quad\text{by the diagram}\\
&=[\alpha(xy),\phi(xy)][1_{(xy)^*xy},\eta^{(x,y)p(a\bullet y)(x\bullet b)}]\\
&=(xy)\sigma((x,y)p(a\bullet y)(x\bullet b))\eta.
\end{aligned}\tag{3.24}$$

Hence, the proof of the lemma is complete. □

Let σ and p be as before. Using (3.17) and Lemma 3.2, we get

$$\begin{aligned}
((x\sigma)(y\sigma))(z\sigma)&=(xy)\sigma((x,y)p)\eta(z\sigma)=(xyz)\sigma((xy,z)p((x,y)p\bullet z))\eta,\\
(x\sigma)((y\sigma)(z\sigma))&=(x\sigma)(yz)\sigma((y,z)p)\eta=(xyz)\sigma((x,yz)p(x\bullet(y,z)p))\eta.
\end{aligned}\tag{3.25}$$

Since the multiplication in $\mathrm{Reg}_E(\mathbf{A})$ is associative, by Lemma 1.6,

$$((xy,z)p((x,y)p\bullet z))\eta=((x,yz)p(x\bullet(y,z)p))\eta.\tag{3.26}$$

The exactness of the sequence in Proposition 2.5 gives us a 3-cochain $k:S\times S\times S\to\overline{\mathbf{Z}(\mathbf{A})}$ such that

$$(xy,z)p((x,y)p\bullet z)=(x,yz)p(x\bullet(y,z)p)(x,y,z)k\tag{3.27}$$

for all $x,y,z\in S$.

Lemma 3.3. *The map* $k:S\times S\times S\to\overline{\mathbf{Z}(\mathbf{A})}$ *is a 3-cocycle.*

Proof. We must show that k satisfies (3.14). Let $x,y,z,t\in S$. Following [11], it is easy to calculate the expression

$$L=(xyz,t)p[(xy,z)p((x,y)p\bullet z)]\bullet t\tag{3.28}$$

in two ways. In the first way using (3.27) and Lemma 2.14, we easily get

$$L=(x,yzt)p\big(x\bullet(y,zt)p\big)\big(xy\bullet(z,t)p\big)\big(x\bullet(y,z,t)k\big)(x,yz,t)k\big((x,y,z)k\bullet t\big). \tag{3.29}$$

In the second way also using Lemma 2.14(iii) to the term $((x,y)p\bullet z)\bullet t$, we get

$$L=(xyz,t)p\big((xy,z)p\bullet t\big)\big(xy\bullet(z,t)p\big)^{-1}\big((x,y)p\bullet zt\big)\big(xy\bullet(z,t)p\big). \tag{3.30}$$

Using (3.27) to the first two terms and since $(xy,z,t)k\in\overline{\mathbf{Z}(\mathbf{A})}_{xyzt}$, $(x,y,zt)k\in\overline{\mathbf{Z}(\mathbf{A})}_{xyzt}$, we finally get

$$L=(x,yzt)p\big(x\bullet(y,zt)p\big)\big(xy\bullet(z,t)p\big)(x,y,zt)k(xy,z,t)k. \tag{3.31}$$

Comparison gives

$$(xy,z,t)k(x,y,zt)k=\big((x,y,z)k\bullet t\big)(x,yz,t)k\big(x\bullet(y,z,t)k\big). \tag{3.32}$$

Hence, by (3.14) k is a 3-cocycle. □

Definition 3.4. The cocycle k satisfying (3.27) is called an *obstruction of the abstract kernel* $\Psi:S\to\mathrm{Reg}_E(\mathbf{A})/\mathrm{Inn}_E(\mathbf{A})$. The following lemma shows that the cohomology class defined by k is independent of chosen σ and p.

LEMMA 3.5. (i) *For a given σ, a change in the choice of p in (3.17) replaces k by a cohomologous cocycle. By suitably changing the choice of p, k may be replaced by any cohomologous cocycle.*

(ii) *A change in the choice of σ may be followed by a suitable new selection of p so as to leave the obstruction cocycle k unchanged.*

Proof. (i) Suppose p' is another choice of p and let k' be the corresponding 3-cocycle so that

$$(xy,z)p'\big((x,y)p'\bullet z\big)=(x,yz)p'\big(x\bullet(y,z)p'\big)(x,y,z)k' \tag{3.33}$$

for all $x,y,z\in S$. We will show that k,k' are cohomologous. Since p and p' satisfy (3.17), by Lemma 1.6, $((x,y)p)\eta=((x,y)p')\eta$. So the exactness of the sequence in Proposition 2.5 gives rise to a 2-cochain $\tau:S\times S\to\overline{\mathbf{Z}(\mathbf{A})}$ such that

$$(x,y)p'=(x,y)p(x,y)\tau. \tag{3.34}$$

Substituting (3.34) in (3.33) and using (3.17), we get

$$(x,y,z)kk'^{-1}=(x,y,z)k(x,y,z)k'^{-1}=\big(x\bullet(y,z)\tau\big)(xy,z)\tau^{-1}(x,yz)\tau\big((x,y)\tau\bullet z\big)^{-1} \tag{3.35}$$

for all $x,y,z\in S$. Thus by (3.15) k and k' are cohomologous. To prove the second statement, take any 3-cocycle k' that is cohomologous to k. Then there is a 2-cochain $\tau:S\times S\to\overline{\mathbf{Z}(\mathbf{A})}$ such that (3.35) holds. If we put $(x,y)p'=(x,y)p(x,y)\tau$, $x,y\in S$, then p' satisfies (3.17) and (3.33), and so k' is the 3-cocycle defined by p'.

(ii) Let $\sigma' : S \to \mathrm{Reg}_E(\mathbf{A})$ be another map such that $(x)\sigma' \in (x)\Psi$ for all $x \in S$, and let $\circ$ denote the biaction of S on A induced by σ'. Then by Lemma 2.11(ii) there exists a map $\beta : S \to \mathbf{A}$, $(x)\beta \in A_x$, such that $(x)\sigma' = (x)\sigma((x)\beta)\eta$ for all $x \in S$. This implies by Lemma 3.2

$$(x\sigma')(y\sigma') = (xy)\sigma'\big((xy)\beta^{-1}(x,y)p\big((x)\beta \bullet y\big)\big(x \bullet (y)\beta\big)\big)\eta. \tag{3.36}$$

Put $(x,y)p' = (xy)\beta^{-1}(x,y)p((x)\beta \bullet y)(x \bullet (y)\beta)\eta$. Then, by Lemma 2.11(i) and (ii),

$$(x,y)p' = (xy)\beta^{-1}(x,y)p\big(x \circ (y)\beta\big)\big((x)\beta \circ y\big). \tag{3.37}$$

By Lemma 2.14, (3.37), and by the relation $x \circ (y \circ (z)\beta = xy \circ (z)\beta)$, we have

$$(xyz)\beta(xy,z)p'\big((x,y)p' \circ z\big) = (xyz)\beta(x,yz)p'\big(x \circ (y,z)p'\big)(x,y,z)k. \tag{3.38}$$

Hence,

$$(xy,z)p'\big((x,y)p' \circ z\big) = (x,yz)p'\big(x \circ (y,z)p'\big)(x,y,z)k. \tag{3.39}$$

Thus the obstruction cocycle determined by p' coincides with k. □

From Lemmas 3.3 and 3.5, we obtain the first part of the following.

Theorem 3.6. *Let $\Psi : S \to \mathrm{Reg}_E(\mathbf{A})/\mathrm{Inn}_E(\mathbf{A})$ be an abstract kernel. Then Ψ defines a well-defined element $[k]$ of $\mathbf{H}^3(S^I, \check{\mathbf{Z}}(\mathbf{A})^0\mathbf{F})$. Further, Ψ has an extension of S by $\mathbf{A}$ if and only if $[k] = 0$.*

Proof. If Ψ has an extension, then by Theorem 2.18 there is an extension of the form $(S,\sigma,p,\mathbf{A})$ with abstract kernel Ψ and crossed pair (σ,p). Then, since (σ,p) satisfies (2.48), it is clear from (3.27) that $[k] = 0$. Conversely, suppose $[k] = 0$. In view of Lemma 3.5(i), we can assume without loss of generality that $k = 0$, the zero 3-cocycle. Then (σ,p) is a crossed pair by (3.17) and (3.27), and the crossed extension $(S,\sigma,p,\mathbf{A})$ is an extension of S by $\mathbf{A}$ with abstract kernel Ψ by Theorems 2.15 and 2.16. □

Theorem 3.7. *Let $(S,\sigma,p,\mathbf{A})$ and $(S,\sigma,q,\mathbf{A})$ be two crossed extensions of S by $\mathbf{A}$ with abstract kernel Ψ. Then $(S,\sigma,p,\mathbf{A})$ is equivalent to $(S,\sigma,q,\mathbf{A})$ if and only if there exists an 1-cochain $\beta : S \to \overline{\mathbf{Z}(\mathbf{A})}$ such that*

$$(x,y)p(x,y)q^{-1} = \big((x)\beta \bullet y\big)\big(x \bullet (y)\beta\big)(xy)\beta^{-1} \tag{3.40}$$

for all $x,y \in S$.

Proof. Suppose $(S,\sigma,p,\mathbf{A})$ and $(S,\sigma,q,\mathbf{A})$ are equivalent extensions and let $\theta : T_p \to T_q$ be an isomorphism such that $\theta\pi_q = \pi_p$ and $(a)U_p\theta = (a)U_q$, $a \in \mathbf{A}_e$, $e \in E$. Define maps $j_1 : S \to T_p$ and $j_2 : S \to T_q$ by $(x)j_1 = (x,1_x)$; $(x)j_2 = (x,1_x)$, where 1_x denotes the identity element of $\mathbf{A}_x$ for all $x \in S$. Then $j_1\pi_p = 1_S = j_2\pi_q$. For $x,y \in S$ and by Lemma 2.13(i) and (ii), we can easily show that $(xy)j_1((x,y)p)U_p = (xj_1)(yj_1)$. That is, $(xj_1)(yj_1) = (xy)j_1((x,y)p)U_p$. Similarly $(xj_2)(yj_2) = (xy)j_2((x,y)q)U_q$. The proof of Theorem 2.16 gives $(x)j_1\overline{\mu}_1 = (x)\sigma(x)j_2\overline{\mu}_2$ for all $x \in S$, where $\overline{\mu}_1 : T_p \to \mathrm{Reg}_E(\mathbf{A})$ and $\overline{\mu}_2 : T_q \to \mathrm{Reg}_E(\mathbf{A})$

are defined by (2.16). If we denote the composite $j_1\theta : S \to T_q$ by j, then $j\pi_q = 1_S$, $(xj)(yj) = (xy)j((x,y)p)U_q$, $(x)\sigma = (x)j\overline{\mu}_2$ for all $x,y \in S$. Since $j\pi_q = 1_S = j_2\pi_q$, by Lemma 1.6 there exists a map $\beta : S \to \mathbf{A}$, $(x)\beta \in \mathbf{A}_x$, such that

$$(x)j = (x)j_2((x)\beta)U_q, \tag{3.41}$$

and so $(x)\sigma = (x)\overline{j\mu_2} = (x)j_2\overline{\mu}_2((x)\beta)U_q\overline{\mu}_2 = (x)\sigma((x)\beta)\eta$ for all $x \in S$. Then, by Lemma 1.6, $(x)\beta \in \operatorname{Ker}\eta$ or $((x)\beta)\eta = (1_x)\eta$ (where 1_x is the identity element of $\mathbf{A}_x$) and therefore by Proposition 2.5, $(x)\beta \in \overline{\mathbf{Z}(\mathbf{A})}_x$ for all $x \in S$. Thus β is a 1-cochain. By using Lemma 2.17 and $((x)\beta \bullet y)(x \bullet (y)\beta) \in \overline{\mathbf{Z}(\mathbf{A})}_{xy}$, we easily derive

$$(xy)j = (xy)j\big[(xy)\beta^{-1}(x,y)q(x,y)p^{-1}\big((x)\beta \bullet y\big)\big(x \bullet (y)\beta\big)\big]U_q. \tag{3.42}$$

Then, by Lemma 1.6, $(xy)\beta^{-1}(x,y)q(x,y)p^{-1}((x)\beta \bullet y)(x \bullet (y)\beta) = 1_{xy}$ or, since β takes values in $\overline{\mathbf{Z}(\mathbf{A})}$,

$$(x,y)p(x,y)q^{-1} = \big((x)\beta \bullet y\big)\big(x \bullet (y)\beta\big)(xy)\beta^{-1}. \tag{3.43}$$

Conversely, let $\beta : S \to \overline{\mathbf{Z}(\mathbf{A})}$ be a 1-cochain such that (3.40) holds. This implies, in particular, $(x)\beta$ commutes with every element of $\mathbf{A}_x$, $x \in S$. Define a map $\theta : T_p \to T_q$ by $(x,a)\theta = (x,a(x)\beta)$ for all $(x,a) \in T_p$. Then clearly $\theta\pi_q = \pi_p$. Moreover, for $e \in E(S)$ and $a \in \mathbf{A}_e$, by (2.65) we get $(a)U_p\theta = (e,(e,e)p^{-1}a)\theta = (e,(e,e)p^{-1}(e)\beta a) = (e,(e,e)q^{-1}a) = (a)U_q$ since (3.40) implies $(e,e)q^{-1} = (e,e)p^{-1}(e)\beta \bullet e$, and by Lemma 2.13(i), $(e)\beta \bullet e = (e)\beta$. Using (3.40) we can easily verify θ is an isomorphism. Hence, $(S,\sigma,p,\mathbf{A})$ and $(S,\sigma,q,\mathbf{A})$ are equivalent. □

Theorem 3.8. *If the abstract kernel $\Psi : S \to \operatorname{Reg}_E(\mathbf{A})/\operatorname{Inn}_E(\mathbf{A})$ has an extension, then the set $\varepsilon(S,\mathbf{A})$ of equivalence classes of extensions of S by $\mathbf{A}$ with abstract kernel Ψ is in one-to-one correspondence with the set $\mathbf{H}^2(S^I,\check{\mathbf{Z}}(\mathbf{A})^0\mathbf{F})$.*

Proof. Since Ψ admits an extension of S by $\mathbf{A}$, by Theorem 2.18, there is an extension of the form $(S,\sigma,p,\mathbf{A})$ with abstract kernel Ψ. Keep σ fixed. Let $\alpha : S \times S \to \overline{\mathbf{Z}(\mathbf{A})}$ be a 2-cocycle so that $(xy,z)\alpha((x,y)\alpha \bullet z) = (x,yz)\alpha(x \bullet (y,z)\alpha)$ for all $x,y,z \in S$. Define $p\alpha : S \times S \to A$ by $(x,y)p\alpha = (x,y)p(x,y)\alpha$. Then $(\sigma,p\alpha)$ is a crossed pair and hence defines a crossed extension $(S,\sigma,p\alpha,\mathbf{A})$ with abstract kernel Ψ. If α' is another 2-cocycle, then

$$\begin{aligned}(x,y)\alpha^{-1}(x,y)\alpha' &= (x,y)\alpha^{-1}(x,y)p^{-1}(x,y)p(x,y)\alpha' \\ &= \big((x,y)p(x,y)\alpha^{-1}\big)(x,y)p(x,y)\alpha' \\ &= \big((x,y)p\alpha\big)^{-1}(x,y)p\alpha'.\end{aligned} \tag{3.44}$$

Therefore, using Theorem 3.7, it is easy to see that α, α' are cohomologous if and only if $(S,\sigma,p\alpha,\mathbf{A})$ and $(S,\sigma,p\alpha',\mathbf{A})$ are equivalent. Hence, we have a well-defined injective map

$$\xi : [\alpha] \longrightarrow [S,\sigma,p\alpha,\mathbf{A}] : \mathbf{H}^2(S^I,\check{\mathbf{Z}}(\mathbf{A})^0\mathbf{F}) \longrightarrow \varepsilon(S,\mathbf{A}), \tag{3.45}$$

where $[S,\sigma,p\alpha,\mathbf{A}]$ denotes the equivalence class of $(S,\sigma,p\alpha,\mathbf{A})$. Let $(S,\sigma,q,\mathbf{A})$ be an extension of S by $\mathbf{A}$ with abstract kernel Ψ. Then by (2.23), Lemma 1.6, and Proposition 2.5, we prove $(x,y)q(x,y)p^{-1} \in \mathbf{Z}(\mathbf{A})_{(xy)^*xy} = \overline{\mathbf{Z}(\mathbf{A})}_{xy}$. Put $(x,y)\alpha = (x,y)q(x,y)p^{-1}$. Then $\alpha : S\times S \to \overline{\mathbf{Z}(\mathbf{A})}$, $(x,y)\alpha \in \overline{\mathbf{Z}(\mathbf{A})}_{xy}$ is a 2-cochain. α is indeed a 2-cocycle. So $[\alpha] \in \mathbf{H}^2(S^I, \check{\mathbf{Z}}(\mathbf{A})^0\mathbf{F})$ and $[\alpha]\xi = [S,\sigma,p\alpha,\mathbf{A}] = [S,\sigma,q,\mathbf{A}]$. Since every extension of S by $\mathbf{A}$ with abstract kernel Ψ is equivalent to an extension of the form $(S,\sigma,q,\mathbf{A})$ by Theorem 2.18, it follows that ξ is surjective. The proof of the theorem is complete. $\square$

Theorems 3.6 and 3.8 generalize the corresponding results for inverse semigroups due to Lausch [4].

References

[1] A. H. Clifford and G. B. Preston, *The Algebraic Theory of Semigroups. Vol. I*, Mathematical Surveys, no. 7, American Mathematical Society, Rhode Island, 1961.

[2] S. Eilenberg and S. Mac Lane, *Cohomology theory in abstract groups. II. Group extensions with a non-Abelian kernel*, Ann. of Math. (2) **48** (1947), 326–341.

[3] G. Lallement, *Demi-groupes réguliers*, Ann. Mat. Pura Appl. (4) **77** (1967), 47–129 (French).

[4] H. Lausch, *Cohomology of inverse semigroups*, J. Algebra **35** (1975), no. 1-3, 273–303.

[5] J. Leech, *The D-category of a semigroup*, Semigroup Forum **11** (1975/76), no. 4, 283–296.

[6] ______, *The cohomology of monoids*, preprint, 1987.

[7] M. Loganathan, *Complementation and inner automorphism for regular semigroups*, Semigroup Forum **21** (1980), no. 2-3, 195–204.

[8] ______, *Cohomology of inverse semigroups*, J. Algebra **70** (1981), no. 2, 375–393.

[9] ______, *Cohomology and extensions of regular semigroups*, J. Austral. Math. Soc. Ser. A **35** (1983), no. 2, 178–193.

[10] M. Loganathan and A. Tamilarasi, *Construction and representation of regular semigroups*, Semigroup Forum **51** (1995), no. 2, 191–216.

[11] S. Mac Lane, *Homology*, Die Grundlehren der mathematischen Wissenschaften, vol. 114, Academic Press, New York; Springer, Berlin, 1963.

[12] ______, *Categories for the Working Mathematician*, Graduate Texts in Mathematics, vol. 5, Springer, New York, 1971.

[13] K. S. S. Nambooripad, *Structure of regular semigroups. I*, Mem. Amer. Math. Soc. **22** (1979), no. 224, vii+119.

[14] F. J. Pastijn and M. Petrich, *Regular Semigroups as Extensions*, Research Notes in Mathematics, vol. 136, Pitman, Massachusetts, 1985.

[15] M. Petrich, *Extensions normales de demi-groupes inverses*, Fund. Math. **112** (1981), no. 3, 187–203 (French).

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