

# A SEQUENTIAL RIESZ-LIKE CRITERION FOR THE RIEMANN HYPOTHESIS

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Received 7 June 2005 and in revised form 18 August 2005

Let  $c_k := \sum_{j=0}^k (-1)^j \binom{k}{j} (1/\zeta(2j+2))$ . We prove that the Riemann hypothesis is equivalent to  $c_k \ll k^{-3/4+\epsilon}$  for all  $\epsilon > 0$ ; furthermore, we prove that  $c_k \ll k^{-3/4}$  implies that the zeros of  $\zeta(s)$  are simple. This is closely related to M. Riesz's criterion which states that the Riemann hypothesis is equivalent to  $\sum_{k=1}^{\infty} ((-1)^{k+1} x^k / (k-1)! \zeta(2k)) \ll x^{1/4+\epsilon}$  as  $x \rightarrow +\infty$ , for all  $\epsilon > 0$ .

## 1. Introduction and preliminaries

The main theorem of this note is the following theorem.

**THEOREM 1.1.** *Let*

$$c_k := \sum_{j=0}^k (-1)^j \binom{k}{j} \frac{1}{\zeta(2j+2)}; \quad (1.1)$$

*then the Riemann hypothesis is true if and only if*

$$c_k \ll k^{-3/4+\epsilon}, \quad \forall \epsilon > 0. \quad (1.2)$$

*Furthermore, if  $c_k \ll k^{-3/4}$ , then the zeros of  $\zeta(s)$  are simple.*

The proof of this theorem is given in Section 3.

*Remark 1.2.* It will be seen below that unconditionally

$$c_k \ll k^{-1/2}, \quad (1.3)$$

and, on the other hand, that

$$c_k \not\ll k^{-3/4-\delta} \quad (\forall \delta > 0). \quad (1.4)$$

*Remark 1.3.* It is quite obvious how one can trivially modify the proof of the theorem to obtain a more general result.

**THEOREM 1.4.** *A necessary and sufficient condition for  $\zeta(s) \neq 0$  in the half-plane  $\Re(s) > 2(1 - \alpha)$  is*

$$c_k \ll k^{-\alpha+\epsilon} \quad (\forall \epsilon > 0). \tag{1.5}$$

However, we will eschew such gratuitous generalizing at this stage.

Necessary and sufficient conditions for the Riemann hypothesis depending only on values of  $\zeta(s)$  at positive integers have been known for a long time, for example those of Riesz [7] and Hardy and Littlewood [3]. Riesz’s criterion, for example, states that the Riemann hypothesis is true if and only if

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1} x^k}{(k-1)! \zeta(2k)} = O(x^{1/4+\epsilon}) \quad (x \rightarrow +\infty). \tag{1.6}$$

We believe our condition is new and it is definitely simpler, as it only involves finite rational combinations of the values  $\zeta(2h)$ , and seems well posed for numerical calculations. This work however did not originate as an attempt to simplify Riesz’s criterion. It arose rather as a consequence of our note [1] on Maslanka’s expression of the Riemann zeta function [4, 5] in the form

$$(s-1)\zeta(s) = \sum_{k=0}^{\infty} A_k P_k\left(\frac{s}{2}\right), \tag{1.7}$$

where

$$A_k = \sum_{j=0}^k (-1)^j \binom{k}{j} (2j+1)\zeta(2j+2), \tag{1.8}$$

and the  $P_k(s)$  are the *Pochhammer polynomials*

$$P_k(s) := \prod_{r=1}^k \left(1 - \frac{s}{r}\right), \tag{1.9}$$

which will appear prominently in the proof of Theorem 1.1. The necessary elementary facts about these polynomials are proved in Section 2.

In Section 4, we prove an *unconditional exact formula* for the coefficients  $c_k$ , stated in the following theorem, where we denote

$$R_k(\omega) := \text{Res} \left( \frac{1}{\zeta(s) P_k(s/2)}; s = \omega \right). \tag{1.10}$$

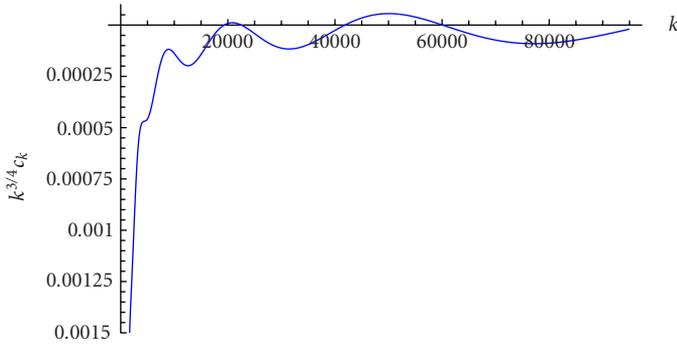


Figure 1.1

THEOREM 1.5 (explicit formula). For sufficiently large  $k$ ,

$$-2kc_{k-1} = \lim_{\nu \rightarrow \infty} \sum_{|\Im \rho| < T_\nu} R_k(\rho) + o(1), \tag{1.11}$$

where  $\{T_\nu\}_{\nu=0}^\infty$  is a certain sequence satisfying  $\nu < T_\nu < \nu + 1$ , and the  $\rho$  denote complex zeta zeros. If simple zeros are assumed, then the above series becomes

$$-2kc_{k-1} = \lim_{\nu \rightarrow \infty} \sum_{|\Im \rho| < T_\nu} \frac{1}{\zeta'(\rho) P_k(\rho/2)} + o(1). \tag{1.12}$$

Remark 1.6. The  $o(1)$  can be written explicitly in terms of the trivial zeros  $-2, -4, \dots$  of  $\zeta(s)$ . This representation is the initial step to prove this conjecture that the condition  $c_k \ll k^{-3/4}$  is both a necessary and sufficient condition for the Riemann hypothesis with simple zeros. The sufficiency is indeed true as will be seen below.

Some extended numerical computations of the  $c_k$  were kindly carried out for the author by K. Maslanka (personal communication). They seem to indicate good agreement with even  $c_k \ll k^{-3/4}$ . In fact, one sees  $c_k k^{3/4}$  increase monotonically at first and then begin to oscillate around zero, where the wavelength fits well with the imaginary part of the first critical zero of  $\zeta(s)$ , see Figure 1.1.

## 2. Elementary properties of the Pochhammer polynomials

We begin with

$$(-1)^k \binom{\frac{s}{2} - 1}{k} = P_k\left(\frac{s}{2}\right), \tag{2.1}$$

which is essentially a matter of notation.

The proofs of the following lemmas are rather standard, but we give them here for the sake of completeness.

The  $k$ th-degree polynomial  $P_k(s)$  grows like  $s^k$  for large  $|s|$ ; more precisely, we can state the following lemma.

LEMMA 2.1.

$$|P_k(s)| > \frac{|s|^k}{k!2^k} \quad (|s| > 2k). \quad (2.2)$$

*Proof.* The condition  $|s| > 2k$  implies that  $|1 - r/s| > 1/2$  for  $r = 1, 2, \dots, k$ , thus

$$|P_k(s)| = \prod_{r=1}^k \left| \frac{r-s}{r} \right| = \frac{|s|^k}{k!} \prod_{r=1}^k \left| 1 - \frac{r}{s} \right| > \frac{|s|^k}{k!2^k}. \quad (2.3)$$

□

The following is the fundamental limit relation connecting the Pochhammer polynomials to the gamma function.

LEMMA 2.2. *Uniformly on compact sets,*

$$\lim_{k \rightarrow \infty} P_k(s)k^s = \frac{1}{\Gamma(1-s)}. \quad (2.4)$$

*Proof.* From  $\sum_{r=1}^k (1/r) = \gamma + \log k + O(1/k)$  and the infinite product for the gamma function, we obtain

$$P_k(s)k^s = e^{sO(1/k)} e^{-\gamma s} \prod_{r=1}^k \left( 1 - \frac{s}{r} \right) e^{s/r} \rightarrow \frac{1}{-s\Gamma(-s)} = \frac{1}{\Gamma(1-s)}, \quad (2.5)$$

the convergence being *uniform on compact sets* of the plane. □

An immediate corollary of Lemma 2.2 is the following lemma.

LEMMA 2.3. *For every compact set  $H \subset \mathbb{C}$ , there is a positive constant  $C_H$ , not depending on  $k$ , such that*

$$|P_k(s)| \leq C_H k^{-\Re(s)} \quad (s \in H, k = 1, 2, \dots). \quad (2.6)$$

*Proof.* Write the uniform limit (2.4) as

$$\left| P_k(s)k^s - \frac{1}{\Gamma(1-s)} \right| \leq \epsilon_H(k) \rightarrow 0 \quad (s \in H, k = 1, 2, \dots); \quad (2.7)$$

therefore

$$|P_k(s)| \leq \left( \frac{1}{|\Gamma(1-s)|} + \epsilon_H(k) \right) k^{-\Re(s)}. \quad (2.8)$$

□

LEMMA 2.4. For  $\Re(s) < 0$ ,

$$\begin{aligned} \frac{1}{|P_k(s)|} &> \frac{1}{|P_{k+1}(s)|} \quad (k \geq 1), \\ \frac{1}{|P_k(s)|} &\rightarrow 0 \quad (k \rightarrow \infty). \end{aligned} \tag{2.9}$$

*Proof.* Lemma 2.2 clearly implies that

$$\frac{1}{|P_k(s)|} \rightarrow 0 \quad (\Re(s) < 0), \tag{2.10}$$

and for  $\Re s < 0$ , the trivial inequality  $|w| \geq |\Re w|$  yields

$$\left| \frac{P_{k+1}(s)}{P_k(s)} \right| = \left| 1 - \frac{s}{k+1} \right| \geq 1 - \frac{\Re(s)}{k+1} > 1, \tag{2.11}$$

so the sequence  $1/P_k(s)$  is strictly decreasing. □

The next lemma establishes an interesting connection between the partial fraction decomposition of  $1/P_k(s)$  and the iterated forward difference operator involved in definition (1.1).

LEMMA 2.5.

$$\frac{1}{P_k(s)} = \sum_{j=1}^k (-1)^j \binom{k}{j} \frac{j}{s-j}, \quad k \geq 1. \tag{2.12}$$

The proof of this lemma is an elementary exercise in computing

$$\operatorname{Res} \left( \frac{1}{P_k(s)}; s = j \right) = (-1)^j \binom{k}{j} j, \quad j = 1, 2, \dots, k. \tag{2.13}$$

**3. Proof of the main theorem (Theorem 1.1)**

**3.1. Proof of sufficiency.** The sufficiency of the condition (1.5) follows from writing  $1/\zeta(s)$  as a series of Pochhammer polynomials. We state it as a separate proposition as we believe it deserves special attention.

PROPOSITION 3.1. *If  $c_k \ll k^{-3/4+1/2\epsilon}$  for any  $\epsilon > 0$ , then*

$$\frac{1}{\zeta(s)} = \sum_{k=0}^{\infty} c_k P_k \left( \frac{s}{2} \right), \quad \Re(s) > \frac{1}{2}, \tag{3.1}$$

where the series converges uniformly on compact subsets of the half-plane. A fortiori,  $\zeta(s)$  does not vanish for  $\Re(s) > 1/2$ .

*Remark 3.2.* Since it will be shown that actually  $c_k \ll k^{-1/2}$ , it follows rather trivially that the representation (3.1) for  $1/\zeta(s)$  is *unconditionally* valid at least in the half-plane

$\Re(s) > 1$ . On the other hand, as announced in (1.4), Proposition 3.1 shows that

$$c_k \ll k^{-3/4-\delta} \quad (\forall \delta > 0), \tag{3.2}$$

since the contrary statement would imply by (3.1) that  $\zeta(s)$  has no zeros on the critical line.

We need a lemma before proving Proposition 3.1.

LEMMA 3.3. *Define*

$$q_k := \sum_{n=1}^{\infty} \frac{1}{n^2} \left(1 - \frac{1}{n^2}\right)^k, \tag{3.3}$$

then

$$q_k \ll k^{-1/2}. \tag{3.4}$$

*Proof.* The contribution of the terms with  $n > \sqrt{k}$  is trivially  $\ll k^{-1/2}$ , whereas the contribution of the remaining terms is

$$\sum_{n \leq \sqrt{k}} \frac{1}{n^2} \left(1 - \frac{1}{n^2}\right)^k \ll \int_1^{\infty} e^{-k/x^2} \frac{dx}{x^2} = \frac{k^{-1/2}}{2} \int_0^{\infty} e^{-u} u^{-1/2} du \ll k^{-1/2}. \tag{3.5}$$

□

*Proof of Proposition 3.1.* First, note that

$$\begin{aligned} c_k &= \sum_{j=0}^k (-1)^j \binom{k}{j} \frac{1}{\zeta(2j+2)} \\ &= \sum_{j=0}^k (-1)^j \binom{k}{j} \sum_{n=1}^{\infty} \frac{\mu(n)}{n^{2j+2}} \\ &= \sum_{n=1}^{\infty} \frac{\mu(n)}{n^2} \sum_{j=0}^k (-1)^j \binom{k}{j} \frac{1}{n^{2j}} \\ &= \sum_{n=1}^{\infty} \frac{\mu(n)}{n^2} \left(1 - \frac{1}{n^2}\right)^k. \end{aligned} \tag{3.6}$$

For  $\Re(s) > 1$ , we have

$$\begin{aligned} \frac{1}{\zeta(s)} &= \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^2} \left(\frac{1}{n^2}\right)^{s/2-1} \\ &= \sum_{n=1}^{\infty} \frac{\mu(n)}{n^2} \left(1 - \left(1 - \frac{1}{n^2}\right)\right)^{s/2-1} \\ &= \sum_{n=1}^{\infty} \frac{\mu(n)}{n^2} \sum_{k=0}^{\infty} (-1)^k \binom{\frac{s}{2}-1}{k} \left(1 - \frac{1}{n^2}\right)^k \\ &= \sum_{n=1}^{\infty} \frac{\mu(n)}{n^2} \sum_{k=0}^{\infty} P_k\left(\frac{s}{2}\right) \left(1 - \frac{1}{n^2}\right)^k. \end{aligned} \tag{3.7}$$

These summations can be interchanged because letting

$$S = \sum_{n=1}^{\infty} \sum_{k=0}^{\infty} \frac{1}{n^2} \left| P_k\left(\frac{s}{2}\right) \right| \left(1 - \frac{1}{n^2}\right)^k, \tag{3.8}$$

we see from Lemmas 2.3 and 3.3 that

$$\begin{aligned} S &= \sum_{k=0}^{\infty} \left| P_k\left(\frac{s}{2}\right) \right| \sum_{n=1}^{\infty} \frac{1}{n^2} \left(1 - \frac{1}{n^2}\right)^k \\ &= \sum_{k=0}^{\infty} \left| P_k\left(\frac{s}{2}\right) \right| q_k \ll \sum_{k=1}^{\infty} k^{-\Re(s)/2-1/2} < \infty. \end{aligned} \tag{3.9}$$

Thus, we proceed to interchange summations in (3.7), taking into account (3.6), to obtain, unconditionally for  $\Re(s) > 1$ ,

$$\frac{1}{\zeta(s)} = \sum_{k=0}^{\infty} P_k\left(\frac{s}{2}\right) \sum_{n=1}^{\infty} \frac{\mu(n)}{n^2} \left(1 - \frac{1}{n^2}\right)^k = \sum_{k=0}^{\infty} c_k P_k\left(\frac{s}{2}\right). \tag{3.10}$$

But Lemma 2.3, together with the hypothesis  $c_k \ll k^{-3/4+(1/2)\epsilon}$ , implies that the above series converges uniformly on compacts of the half-plane  $\Re(s) > 1/2 + \epsilon$ . Thus, the series extends  $1/\zeta(s)$  analytically to the half-plane  $\Re(s) > 1/2$ . We have thus proved the validity of (3.1).  $\square$

Finally, we prove the assertion on simple zeros in the main theorem (Theorem 1.1). Assume that  $c_k \ll k^{-3/4}$ . Take any *fixed*  $s = 1/2 + i\beta$  on the critical line and  $0 < h \leq \delta$  for a fixed, finite  $\delta > 0$ . By (3.1),

$$\left| \frac{1}{\zeta(s+h)} \right| \leq |c_0| + \sum_{k=1}^{\infty} O(k^{-3/4}) \left| P_k\left(\frac{s+h}{2}\right) \right|. \tag{3.11}$$

Now it is clear that

$$\alpha_1 = \sup_{0 \leq h \leq \delta} \frac{1}{|\Gamma(3/4 + h/2 - i(\beta/2))|} < \infty. \tag{3.12}$$

But by Lemma 2.2, there is a constant  $\alpha_2 > 0$  such that

$$\sum_{k=1}^{\infty} k^{-3/4} \left| P_k\left(\frac{s+h}{2}\right) \right| \leq \alpha_2 \sum_{k=1}^{\infty} \frac{k^{-1-h/2}}{|\Gamma(3/4 + h/2 - i(\beta/2))|} \leq \alpha_1 \alpha_2 \zeta\left(1 + \frac{h}{2}\right) \ll \frac{1}{h}. \tag{3.13}$$

Applying this in (3.11), we obtain

$$\left| \frac{1}{\zeta(s+h)} \right| \ll \frac{1}{h}. \tag{3.14}$$

This shows that if  $\zeta(s) = 0$ , then  $s$  can only be a simple zero.

### 3.2. Necessity of the condition

*Proof of the necessity of the condition.* Assume now that the Riemann hypothesis is true. If, as usual, we write

$$M(x) := \sum_{n \leq x} \mu(n), \quad (3.15)$$

we then have

$$M(x) \ll x^{1/2+2\epsilon} \quad (\forall \epsilon > 0), \quad (3.16)$$

which, actually, is well known to be equivalent to the Riemann hypothesis (see, e.g., [8, Theorem 14.25(C)]).

We can transform the second expression for  $c_k$  in (3.6) summing it by parts to obtain

$$\begin{aligned} c_k &= - \int_1^\infty M(x) \frac{d}{dx} \left( \frac{1}{x^2} \left( 1 - \frac{1}{x^2} \right)^k \right) dx \\ &= 2 \int_0^1 M\left(\frac{1}{x}\right) (1-x^2)^{k-1} (x - (k+1)x^3) dx. \end{aligned} \quad (3.17)$$

Therefore,

$$|c_k| \leq 2(k+1) \int_0^1 \left| M\left(\frac{1}{x}\right) \right| x^3 (1-x^2)^{k-1} dx + 2 \int_0^1 \left| M\left(\frac{1}{x}\right) \right| x (1-x^2)^{k-1} k dx, \quad (3.18)$$

but (on the Riemann hypothesis)

$$M\left(\frac{1}{x}\right) \ll x^{-1/2-2\epsilon} \quad (x \downarrow 0), \quad (3.19)$$

so that

$$c_k \ll k \int_0^1 x^{5/2-2\epsilon} (1-x^2)^{k-1} dx + \int_0^1 x^{1/2-2\epsilon} (1-x^2)^{k-1} dx. \quad (3.20)$$

On the other hand, for  $\Re(\lambda) > -1$ , a classical beta integral result (see, e.g., [2, Section 9.3]) gives

$$\int_0^1 x^\lambda (1-x^2)^{k-1} dx = \frac{1}{2} \Gamma\left(\frac{1}{2}(\lambda+1)\right) \frac{\Gamma(k)}{\Gamma(k + (1/2)(\lambda+1))} \ll k^{-1/2-\lambda/2}, \quad (3.21)$$

where the last estimate follows from Stirling's formula for the logarithm of the gamma function; hence (3.20) becomes

$$c_k \ll k^{-3/4+\epsilon}. \quad (3.22)$$

□

**4. An exact formula for  $c_k$**

The  $c_k$  have a nice exact expression as an integral in the complex plane, as shown by the following proposition.

PROPOSITION 4.1.

$$-2kc_{k-1} = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \frac{ds}{\zeta(s)P_k(s/2)} \quad (k \geq 2, 1 < a < 2), \tag{4.1}$$

where the integral is absolutely convergent. The path of integration is the line  $\Re(s) = a$  traversed in the upward direction.

*Proof.* Note first that for any  $\sigma > 1$ ,

$$\frac{1}{|\zeta(s)|} = \left| \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} \right| \leq \sum_{n=1}^{\infty} \frac{1}{n^\sigma} = \zeta(\sigma). \tag{4.2}$$

By Lemma 2.1, we may move the path of integration to a vertical line with any abscissa  $b > 2k$ . Calculating the residues with the help of Lemma 2.5, we get

$$\frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \frac{ds}{\zeta(s)P_k(s/2)} = \sum_{j=1}^k (-1)^j \binom{k}{j} \frac{2j}{\zeta(2j)} + \frac{1}{2\pi i} \int_{b-i\infty}^{b+i\infty} \frac{ds}{\zeta(s)P_k(s/2)}, \tag{4.3}$$

but

$$-2k \sum_{j=0}^{k-1} (-1)^j \binom{k-1}{j} \frac{1}{\zeta(2j+2)} = -2kc_{k-1}. \tag{4.4}$$

For fixed  $k$ , let  $b \rightarrow +\infty$ . By Lemma 2.1, this yields

$$\frac{1}{2\pi i} \int_{b-i\infty}^{b+i\infty} \frac{ds}{\zeta(s)P_k(s/2)} \rightarrow 0 \quad (b \rightarrow \infty), \tag{4.5}$$

so that (4.1) follows. □

*Proof of the explicit formula (Theorem 1.5).* We intend now to move the path of integration in (4.1) to the left of the critical strip. As this procedure is a little more delicate than that of the previous lemma, we will proceed in more detail. We begin with a fixed but arbitrary  $k \geq \max(4, A)$ , for some fixed  $A > 0$  to be determined later in the proof, so that, of course, to begin with Proposition 4.1 can be applied. For any  $T_\nu > 0$ , consider the integral

$$I(k, \nu) := \frac{1}{2\pi i} \int_{L_\nu} \frac{ds}{\zeta(s)P_k(s/2)}, \tag{4.6}$$

where  $L_\nu$  is the rectangle  $\{3/2 - iT_\nu, 3/2 + iT_\nu, -1 + iT_\nu, -1 - iT_\nu\}$  traversed in the positive direction. By the residue theorem, we have

$$I(k, \nu) = \sum_{|\Im \rho| < T_\nu} \operatorname{Res} \left( \frac{1}{\zeta(s)P_k(s/2)}; s = \rho \right), \tag{4.7}$$

where the finite sum runs over the zeros  $\rho$  of the zeta function in the interior of the rectangle  $L_\nu$ . The choice of the  $T_\nu$  is dictated by Theorem 9.7 in Titchmarsh’s monograph [8], where it is attributed to Valiron et al., independently. As a consequence of this *unconditional* theorem for some constant  $A > 0$ , there is a sequence  $T_\nu$  with  $\nu < T_\nu < \nu + 1$  such that

$$\frac{1}{|\zeta(\sigma + iT_\nu)|} < T_\nu^A \quad (-1 \leq \sigma \leq 2). \tag{4.8}$$

This estimate, together with Lemma 2.1, implies that the contribution of the horizontal rungs in  $I(k, \nu)$  tends to zero as  $\nu \rightarrow \infty$ .

On the other hand, it is clear that as  $\nu \rightarrow \infty$ , the integral on the right-hand vertical side of  $L_\nu$  tends to the absolutely convergent integral on the right-hand side of (4.1), thus to  $-2kc_{k-1}$ .

Likewise, to see that the contribution of the left-hand side of the rectangle  $L_\nu$  converges as  $\nu \rightarrow \infty$  to

$$J_k := -\frac{1}{2\pi i} \int_{-1-i\infty}^{-1+i\infty} \frac{ds}{\zeta(s)P_k(s/2)}, \tag{4.9}$$

it suffices to show that this integral is absolutely convergent. To prove this, note that the functional align implies that

$$\begin{aligned} \frac{1}{|\zeta(-1+it)|} &= \frac{1}{|2^{1-it}\pi^{-2-it} \cos(i\pi t/2)\Gamma(2-it)\zeta(2-it)|} \\ &\ll \frac{1}{e^{\pi t/2} + e^{-\pi t/2}} \frac{1}{|\Gamma(2-it)|} \ll |t|^{3/2}, \end{aligned} \tag{4.10}$$

where we used again (4.2) in writing  $|\zeta(2-it)^{-1}| \leq \zeta(2)$ , and the well-known estimates for the gamma function on vertical strips (see, e.g., formula (21.52) in Rademacher’s treatise [6]). Now (4.10) and the trivial bound (2.2) yield the absolute integrability of (4.9).

We have thus proved that the limit as  $\nu \rightarrow \infty$  of  $I(k, \nu)$  exists, arriving at

$$-2kc_{k-1} = \lim_{\nu \rightarrow \infty} \sum_{|\Re \rho| < T_\nu} \operatorname{Res} \left( \frac{1}{\zeta(s)P_k(s/2)}; s = \rho \right) + J_k, \tag{4.11}$$

where the limit of the summation has also been shown to exist. But  $J_k \rightarrow 0$  as  $k \rightarrow \infty$  by the monotone convergence theorem on account of Lemma 2.4. This completes the proof of (1.11) of Theorem 1.5, which immediately implies (1.12) under the assumption of simple zeros. □

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