ON THE LOCAL WELL-POSEDNESS OF A BENJAMIN-ONO-BOUSSINESQ SYSTEM

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Received 30 March 2005 and in revised form 25 September 2005

Consider a Benjamin-Ono-Boussinesq system $\eta_t + u_x + au_{xxx} + (u\eta)_x = 0$, $u_t + \eta_x + uu_x + c\eta_{xxx} - du_{xxt} = 0$, where a, c, and d are constants satisfying $a = c > 0$, $d > 0$ or $a < 0$, $c < 0$, $d > 0$. We prove that this system is locally well posed in Sobolev space $H^s(\mathbb{R}) \times H^{s+1}(\mathbb{R})$, with $s > 1/4$.

1. Introduction and main results

We consider the Cauchy problem for a Benjamin-Ono-Boussinesq system:

$$\begin{gathered}
\eta_t + u_x + au_{xxx} + (u\eta)_x = 0, \quad t > 0,\ x \in \mathbb{R}, \\
u_t + \eta_x + uu_x + c\eta_{xxx} - du_{xxt} = 0, \quad t > 0,\ x \in \mathbb{R}, \\
\eta|_{t=0} = f(x), \qquad u|_{t=0} = g(x),
\end{gathered} \tag{1.1}$$

where a, c, and d are constants satisfying

$$a = c > 0, \qquad d > 0 \quad \text{or} \quad a < 0, \qquad c < 0, \qquad d > 0. \tag{1.2}$$

The system is called a Benjamin-Ono-Boussinesq system because it can be reduced to a pair of equations whose linearization uncouples to a pair of linear Benjamin-Ono equations.

Equations of type (1.1) are a class of essential model equations appearing in physics and fluid mechanics. To describe two-dimensional irrotational flows of an inviscid liquid in a uniform rectangular channel, Boussinesq in 1871 derived from the Euler equation the classical Boussinesq system:

$$\begin{gathered}
\eta_t + u_x + (u\eta)_x = 0, \quad t > 0,\ x \in \mathbb{R}, \\
u_t + \eta_x + uu_x + \frac{1}{3}\eta_{xxt} = 0, \quad t > 0,\ x \in \mathbb{R}.
\end{gathered} \tag{1.3}$$

In [1], Bona et al. derived by considering first-order approximations to the Euler equation the following alternative (a four-parameter Boussinesq system) to the classical Boussinesq

International Journal of Mathematics and Mathematical Sciences 2005:22 (2005) 3609–3630
DOI: 10.1155/IJMMS.2005.3609

system:

$$\begin{aligned} \eta_t + u_x + a u_{xxx} + (u\eta)_x - b\eta_{xxt} &= 0, \\ u_t + \eta_x + u u_x + c\eta_{xxx} - d u_{xxt} &= 0, \end{aligned} \tag{1.4}$$

where the constants obey the relations

$$a+b = \frac{1}{2}\left(\theta^2 - \frac{1}{3}\right), \qquad c+d = \frac{1}{2}(1-\theta^2) \geq 0, \qquad a+b+c+d = \frac{1}{3}, \tag{1.5}$$

with $\theta \in [0,1]$. The system (1.1) is one of the four-parameter systems associated with $b = 0$. When $b = 0$, Bona et al. in [1] determined exactly that the four-parameter systems are linearly well posed if and only if a, c, and d satisfy the relation (1.2). The local well-posedness of the nonlinear system (1.1) is considered in [2]. They prove that the system (1.1) associated with (1.2) is locally well posed in the Sobolev space $H^s(\mathbb{R}) \times H^{s+1}(\mathbb{R})$, with $s \geq 1$. In this work, we will give some local well-posedness for the Cauchy problem (1.1) in the Sobolev space $H^s(\mathbb{R}) \times H^{s+1}(\mathbb{R})$, with $s > 1/4$, by using the so-called $L^p - L^q$ smoothing effect of the Strichartz type.

Denote by J the Fourier multiplier with symbol $(1+\xi^2)^{1/2}$, and denote by $\mathcal{H}$ the usual Hilbert transform. Our result is the following.

Theorem 1.1. *Fix $s > 1/4$. Then, for every $(f(x), g(x)) \in H^s(\mathbb{R}) \times H^{s+1}(\mathbb{R})$, there exist $T > 0$ depending only on $\|f(x)\|_{H^s} + \|g(x)\|_{H^{s+1}}$ and a unique solution of (1.1) on the time interval $[0,T]$ satisfying*

$$\begin{aligned} (J^{-1}\mathcal{H}\eta, u) &\in C([0,T],\ L^2(\mathbb{R}) \times L^2(\mathbb{R})), \\ (J^{-1}\mathcal{H}\eta_x, u_x) &\in C([0,T],\ L^\infty(\mathbb{R}) \times L^\infty(\mathbb{R})). \end{aligned} \tag{1.6}$$

Moreover, for any $R > 0$, there exists T depending on R such that the nonlinear map $(f(x), g(x)) \to (\eta, u)$ is continuous from the ball of radius R of $H^s(\mathbb{R}) \times H^{s+1}(\mathbb{R})$ to $C([0,T];$ $H^s(\mathbb{R}) \times H^{s+1}(\mathbb{R}))$.

In the sequel, we say that the pair $(p,q) \in \mathbb{R}^2$ is admissible if it satisfies

$$\frac{2}{p} + \frac{1}{q} = \frac{1}{2}, \quad p > 4. \tag{1.7}$$

We denote by J the Fourier multiplier with symbol $(1+\xi^2)^{1/2}$, denote by $\mathcal{H}$ the usual Hilbert transform, and denote by $m(D)$ the Fourier multiplier associated with symbol $m(\xi)$. We also denote the dyadic integers 2^k, $k \geq 0$, by λ or μ. Whenever a summation over λ or μ appears, it means that we sum over the dyadic integers. The notation $A \lesssim B$ (resp., $A \gtrsim B$) means that there exists a harmless positive constant C such that $A \leq CB$ (resp., $A \geq CB$). We denote by $L_T^p X$ (resp., $L_I^p X$) the space of X-valued measurable and p-integrable functions defined on $[0,T]$ (resp., I), equipped with the natural norm. We also use the notation $\|(u_1, u_2, \ldots, u_k)\|_X = \|u_1\|_X + \cdots + \|u_k\|_X$.

The rest of this paper is organized as follows. In Section 2, we prove some Strichartz-type estimates for smooth solutions of (1.1). In Section 3, we give the proof of the local well-posedness of the Cauchy problem (1.1).

2. Some estimates

In this section, we give some smoothing effects for (1.1). These estimate will be the main ingredient in the proof of local well-posedness of the Cauchy problem (1.1). Consider the following linear system:

$$\begin{aligned} \eta_t + u_x + a u_{xxx} &= F(x,t), \quad t > 0,\ x \in \mathbb{R}, \\ u_t + \eta_x + c\eta_{xxx} - d u_{xxt} &= G(x,t), \quad t > 0,\ x \in \mathbb{R}. \end{aligned} \tag{2.1}$$

Let

$$\sigma(\xi) = \left[\frac{(a\xi^2 - 1)(c\xi^2 - 1)}{d\xi^2 + 1}\right]^{1/2}, \qquad h(\xi) = \left[\frac{(a\xi^2 - 1)(d\xi^2 + 1)}{c\xi^2 - 1}\right]^{1/2}. \tag{2.2}$$

Consider the change of variables

$$\eta = h(D)(v + w), \qquad u = v - w, \qquad \tilde{\eta} = h^{-1}(D)\eta = v + w, \tag{2.3}$$

where $h(D)$ (resp., $h^{-1}(D)$) is the Fourier multiplier with the symbol $h(\xi)$ (resp., $h^{-1}(\xi)$). Then we have

$$\begin{aligned} v_t + \sigma(D)\partial_x v &= \frac{1}{2}h^{-1}(D)F + \frac{1}{2}(1 + dD^2)^{-1}G, \\ w_t - \sigma(D)\partial_x w &= \frac{1}{2}h^{-1}(D)F - \frac{1}{2}(1 + dD^2)^{-1}G. \end{aligned} \tag{2.4}$$

Rewrite $\sigma(\xi) = (ac/d)^{1/2}|\xi| + \gamma(\xi)$ with

$$\gamma(\xi) = \frac{d - (ac + ad + dc)\xi^2}{d\left[(a\xi^2 - 1)(d\xi^2 + 1)(c\xi^2 - 1)\right]^{1/2} + (acd)^{1/2}|\xi|(d\xi^2 + 1)}. \tag{2.5}$$

Let $\Gamma(D)$ be the Fourier multiplier with the symbol $i\xi\gamma(\xi)$, which is a skew-adjoint operator in $L^2(\mathbb{R})$ and is bounded from $L^p(\mathbb{R})$ to $L^p(\mathbb{R})$ for $1 < p < +\infty$. Then

$$\begin{aligned} v_t + \left(\frac{ac}{d}\right)^{1/2}\mathcal{H}\partial_{xx}v &= \Gamma(D)v + \frac{1}{2}h^{-1}(D)F + \frac{1}{2}(1 + dD^2)^{-1}G, \\ w_t - \left(\frac{ac}{d}\right)^{1/2}\mathcal{H}\partial_{xx}w &= -\Gamma(D)w + \frac{1}{2}h^{-1}(D)F - \frac{1}{2}(1 + dD^2)^{-1}G. \end{aligned} \tag{2.6}$$

Using the Strichartz inequality for the Benjamin-Ono equation, we deduce from (2.6), for admissible pair (p,q),

$$\begin{aligned}\|v\|_{L_I^pL^q} &\lesssim \|v\|_{L_I^\infty L^2} + \left\| -\Gamma(D)v + \frac{1}{2}h^{-1}(D)F + \frac{1}{2}\left(1+dD^2\right)^{-1}G\right\|_{L_I^1L^2} \\ &\lesssim \|v\|_{L_I^\infty L^2} + \|v\|_{L_I^1L^2} + \left\|h^{-1}(D)F + \left(1+dD^2\right)^{-1}G\right\|_{L_I^1L^2} \\ &\lesssim (1+|I|)\|v\|_{L_I^\infty L^2} + \left\|h^{-1}(D)F\right\|_{L_I^1L^2} + \left\|\left(1+dD^2\right)^{-1}G\right\|_{L_I^1L^2},\end{aligned} \tag{2.7}$$

and similarly

$$\|w\|_{L_I^pL^q} \lesssim (1+|I|)\|w\|_{L_I^\infty L^2} + \left\|h^{-1}(D)F\right\|_{L_I^1L^2} + \left\|\left(1+dD^2\right)^{-1}G\right\|_{L_I^1L^2}. \tag{2.8}$$

Thus by (2.3),

$$\begin{aligned}\left\|(\widetilde{\eta},u)\right\|_{L_I^pL^q} \lesssim{} & (1+|I|)\left\|(\widetilde{\eta},u)\right\|_{L_I^\infty L^2} + \left\|h^{-1}(D)F(x,t)\right\|_{L_I^1L^2} \\ & + \left\|\left(1+dD^2\right)^{-1}G(x,t)\right\|_{L_I^1L^2}.\end{aligned} \tag{2.9}$$

Consider a standard Littlewood-Paley decomposition:

$$u = \sum_\lambda u_\lambda, \qquad \eta = \sum_\lambda \eta_\lambda, \qquad \widetilde{\eta} = \sum_\lambda \widetilde{\eta_\lambda}, \tag{2.10}$$

where

$$u_\lambda = \Delta_\lambda u, \qquad \eta_\lambda = \Delta_\lambda \eta, \qquad \widetilde{\eta_\lambda} = \Delta_\lambda \widetilde{\eta}, \tag{2.11}$$

Δ_λ are the Fourier multipliers with symbols $\phi(\xi/\lambda)$ when $\lambda = 2^k$ with $k \ge 1$, and $\chi(\xi)$ when $\lambda = 1$, and where the nonnegative functions $\chi \in C_0^\infty(\mathbb{R})$ and $\phi \in C_0^\infty(\mathbb{R})$ satisfy

$$\begin{gathered}\chi(\xi) + \sum_\lambda \phi(\xi/\lambda) = 1, \\ \phi(\xi) = \begin{cases} 0, & \text{if } |\xi| \le \dfrac{5}{8} \text{ or } |\xi| \ge 2, \\ 1, & \text{if } 1 \le |\xi| \le \dfrac{5}{4}. \end{cases}\end{gathered} \tag{2.12}$$

For a dyadic integer λ, we set

$$\widetilde{\Delta}_\lambda = \begin{cases} \Delta_{\lambda/2} + \Delta_\lambda + \Delta_{2\lambda}, & \text{if } \lambda > 1, \\ \Delta_1 + \Delta_2, & \text{if } \lambda = 1. \end{cases} \tag{2.13}$$

Lemma 2.1. *Let $\{a_\lambda\}$, $\{d_\lambda\}$, and $\{\delta_\lambda\}$ be three sequences indexed on positive dyadic integers λ. Assume that there exist two positive constants $1 < \kappa_1 < \kappa_2$ such that $\kappa_1\delta_\lambda \le \delta_{2\lambda} \le \kappa_2\delta_\lambda$.*

Then

$$\sum_{\lambda}\delta_\lambda\sum_{\mu\geq\lambda/8}a_\mu d_\lambda \lesssim \Big(\sum_{\lambda}\delta_\lambda^2 a_\lambda^2\Big)^{1/2}\Big(\sum_{\lambda}d_\lambda^2\Big)^{1/2}, \tag{2.14}$$

and hence by duality,

$$\sum_{\lambda}\delta_\lambda^2\Big(\sum_{\mu\geq\lambda/8}a_\mu\Big)^2 \lesssim \sum_{\lambda}\delta_\lambda^2 a_\lambda^2. \tag{2.15}$$

Proof. Note that $\kappa_1\delta_\lambda \leq \delta_{2\lambda} \leq \kappa_2\delta_\lambda$ implies $\delta_\lambda/\delta_{2^k\lambda} \leq \kappa_1^{-k}$. Using the Cauchy-Schwarz inequality, we get

$$\begin{aligned}\sum_{\lambda}\delta_\lambda\sum_{\mu\geq\lambda/8}a_\mu d_\lambda &= \sum_{\lambda}\delta_\lambda\sum_{k=-3,2^k\lambda\geq 1}^{\infty}a_{2^k\lambda}d_\lambda = \sum_{k=-3}^{\infty}\kappa_1^{-k}\sum_{\lambda\geq 2^{-k}}\delta_{2^k\lambda}a_{2^k\lambda}d_\lambda \\ &\leq \sum_{k=-3}^{\infty}\kappa_1^{-k}\Big(\sum_{\lambda\geq 2^{-k}}\delta_{2^k\lambda}^2 a_{2^k\lambda}^2\Big)^{1/2}\Big(\sum_{\lambda\geq 2^{-k}}d_\lambda^2\Big)^{1/2} \lesssim \Big(\sum_{\lambda}\delta_\lambda^2 a_\lambda^2\Big)^{1/2}\Big(\sum_{\lambda}d_\lambda^2\Big)^{1/2}.\end{aligned} \tag{2.16}$$

□

LEMMA 2.2. *Fix $T>0$ and $\sigma>1/2$. Let (η,u) be a smooth solution of the system (1.1). Then for every admissible pair (p,q),*

$$\begin{aligned}&\Big\{\sum_{\lambda}\lambda^{2\sigma}\|(\mathcal{H}\widetilde{\eta}_\lambda,u_\lambda)\|_{L_T^pL^q}^2\Big\}^{1/2} \\ &\quad\lesssim (1+T)^{1/p}\Big(1+\|J^\sigma u\|_{L_T^\infty L^2}\Big) \\ &\qquad\times\Big(1+\|(u,u_x,\mathcal{H}\widetilde{\eta},\partial_x\mathcal{H}\widetilde{\eta},\widetilde{\eta})\|_{L_T^1L^\infty}\Big)\Big(\sum_{\lambda}\lambda^{2/p+2\sigma}\|(\mathcal{H}\widetilde{\eta}_\lambda,u_\lambda)\|_{L_T^\infty L^2}^2\Big)^{1/2},\end{aligned} \tag{2.17}$$

where $\widetilde{\eta}=h^{-1}(D)\eta$, $\widetilde{\eta_\lambda}=h^{-1}(D)\Delta_\lambda\eta$, $h^{-1}(D)$ is the Fourier multiplier with the symbol $h^{-1}(\xi)$ defined in (2.2).

Proof. Let (η,u) be a smooth solution of the system (1.1). Then (η_λ,u_λ) satisfies the following system:

$$\begin{aligned}(\eta_\lambda)_t+(u_\lambda)_x+a(u_\lambda)_{xxx} &= -\big(uh(D)\widetilde{\eta}_\lambda+[\Delta_\lambda,uh(D)]\widetilde{\eta}_\lambda\big)_x, \quad t>0,\ x\in\mathbb{R}, \\ (u_\lambda)_t+(\eta_\lambda)_x+c(\eta_\lambda)_{xxx}-d(u_\lambda)_{xxt} &= -\Big(u(u_\lambda)_x+[\Delta_\lambda,u\partial_x]u\Big), \quad t>0,\ x\in\mathbb{R}.\end{aligned} \tag{2.18}$$

Using (2.9) to the system (2.18) and choosing the interval I satisfying $|I| \leq 1$ and $|I| \leq 1/\lambda$, we get

$$\begin{aligned}
&\|(\mathcal{H}\tilde{\eta}_\lambda, u_\lambda)\|_{L_I^p L^q} \\
&\quad\lesssim \|(\tilde{\eta}_\lambda, u_\lambda)\|_{L_I^p L^q} \\
&\quad\lesssim (1+|I|)\|(\tilde{\eta}_\lambda, u_\lambda)\|_{L_I^\infty L^2} + \|\partial_x h^{-1}(D)(uh(D)\tilde{\eta}_\lambda + [\Delta_\lambda, uh(D)]\tilde{\eta}_\lambda)\|_{L_I^1 L^2} \\
&\qquad + \left\|(1+dD^2)^{-1}(u(u_\lambda)_x + [\Delta_\lambda, u\partial_x]u)\right\|_{L_I^1 L^2} \\
&\quad\lesssim (1+|I|)\|(\tilde{\eta}_\lambda, u_\lambda)\|_{L_I^\infty L^2} + \|uh(D)\tilde{\eta}_\lambda\|_{L_I^1 L^2} \\
&\qquad + \|[\Delta_\lambda, uh(D)]\tilde{\eta}_\lambda\|_{L_I^1 L^2} + \|u(u_\lambda)_x\|_{L_I^1 L^2} + \|[\Delta_\lambda, u\partial_x]u\|_{L_I^1 L^2} \\
&\quad\lesssim (1+|I|)\|(\tilde{\eta}_\lambda, u_\lambda)\|_{L_I^\infty L^2} + \|u\|_{L_I^\infty L^\infty}\|h(D)\tilde{\eta}_\lambda\|_{L_I^1 L^2} \\
&\qquad + \|[\Delta_\lambda, uh(D)]\tilde{\eta}_\lambda\|_{L_I^1 L^2} + \|u\|_{L_I^\infty L^\infty}\|(u_\lambda)_x\|_{L_I^1 L^2} + \|[\Delta_\lambda, u\partial_x]u\|_{L_I^1 L^2}.
\end{aligned} \tag{2.19}$$

By the Sobolev embedding, $|I| \leq 1$ and $|I| \leq 1/\lambda$,

$$\begin{gathered}
\|u\|_{L_I^\infty L^\infty} \lesssim \|J^\sigma u\|_{L_I^\infty L^2}, \\
\|h(D)\tilde{\eta}_\lambda\|_{L_I^1 L^2} \lesssim \|\tilde{\eta}_\lambda\|_{L_I^1 L^2} + \|\partial_x \tilde{\eta}_\lambda\|_{L_I^1 L^2} \lesssim |I|\|\tilde{\eta}_\lambda\|_{L_I^\infty L^2} + |I|\lambda\|\tilde{\eta}_\lambda\|_{L_I^\infty L^2} \lesssim \|\tilde{\eta}_\lambda\|_{L_I^\infty L^2}, \\
\|\partial_x u_\lambda\|_{L_I^1 L^2} \lesssim |I||\lambda|\|u_\lambda\|_{L_I^\infty L^2} \lesssim \|u_\lambda\|_{L_I^\infty L^2}.
\end{gathered} \tag{2.20}$$

Then, we deduce from (2.19)

$$\begin{aligned}
\|(\mathcal{H}\tilde{\eta}_\lambda, u_\lambda)\|_{L_I^p L^q} \lesssim{}& \left(1 + \|J^\sigma u\|_{L_I^\infty L^2}\right)\|(\mathcal{H}\tilde{\eta}_\lambda, u_\lambda)\|_{L_I^\infty L^2} \\
&+ \|[\Delta_\lambda, uh(D)]\tilde{\eta}_\lambda\|_{L_I^1 L^2} + \|[\Delta_\lambda, u\partial_x]u\|_{L_I^1 L^2}.
\end{aligned} \tag{2.21}$$

Partition $[0,T] = \bigcup_k I_k$, where each interval I_k is of size $\leq \min\{1/\lambda, 1\}$. We can choose I_k such that their number is bounded by $(1+T)\lambda$. Therefore by (2.21), we obtain

$$\begin{aligned}
\|(\mathcal{H}\tilde{\eta}_\lambda, u_\lambda)\|_{L_T^p L^q} \lesssim{}& (1+T)^{1/p}\lambda^{1/p}\left(1 + \|J^\sigma u\|_{L_T^\infty L^2}\right) \\
&\times \left(\|(\mathcal{H}\tilde{\eta}_\lambda, u_\lambda)\|_{L_T^\infty L^2} + \|[\Delta_\lambda, uh(D)]\tilde{\eta}_\lambda\|_{L_T^1 L^2} + \|[\Delta_\lambda, u\partial_x]u\|_{L_T^1 L^2}\right).
\end{aligned} \tag{2.22}$$

To estimate the terms $\|[\Delta_\lambda, uh(D)]\tilde{\eta}_\lambda\|_{L_T^1 L^2}$ and $\|[\Delta_\lambda, u\partial_x]u\|_{L_T^1 L^2}$, we introduce the following estimates which come from [3, Lemmas 4.2 and 4.3]:

$$\|[\Delta_\lambda, v\partial_x]w\|_{L^2} \lesssim \|v_x\|_{L^\infty}\|w\|_{L^2}, \qquad \|[\Delta_\lambda, v]w\|_{L^2} \lesssim \|v\|_{L^\infty}\|w\|_{L^2}. \tag{2.23}$$

Then, for $\lambda \geq 4$,

$$\begin{aligned} \|[\Delta_\lambda, uh(D)]\widetilde{\Delta}_\lambda\widetilde{\eta}\|_{L^2} &= \left\|[\Delta_\lambda, u\partial_x]\frac{h(D)}{iD}\widetilde{\Delta}_\lambda\widetilde{\eta}\right\|_{L^2} \\ &\lesssim \|u_x\|_{L^\infty}\left\|\frac{h(D)}{iD}\widetilde{\Delta}_\lambda\widetilde{\eta}\right\|_{L^2} \lesssim \|u_x\|_{L^\infty}\|\widetilde{\Delta}_\lambda\widetilde{\eta}\|_{L^2}, \end{aligned} \tag{2.24}$$

and for $\lambda = 1, 2$,

$$\begin{aligned} \|[\Delta_\lambda, uh(D)]\widetilde{\Delta}_\lambda\widetilde{\eta}\|_{L^2} &= \|[\Delta_\lambda, u]h(D)\widetilde{\Delta}_\lambda\widetilde{\eta}\|_{L^2} \\ &\lesssim \|u\|_{L^\infty}\|h(D)\widetilde{\Delta}_\lambda\widetilde{\eta}\|_{L^2} \lesssim \|u\|_{L^\infty}\|\widetilde{\Delta}_\lambda\widetilde{\eta}\|_{L^2}, \end{aligned} \tag{2.25}$$

so we get

$$\|[\Delta_\lambda, uh(D)]\widetilde{\Delta}_\lambda\widetilde{\eta}\|_{L^1_TL^2} \lesssim \left(\|u\|_{L^1_TL^\infty} + \|u_x\|_{L^1_TL^\infty}\right)\|\widetilde{\Delta}_\lambda\widetilde{\eta}\|_{L^\infty_TL^2}. \tag{2.26}$$

Now we consider the term $\|[\Delta_\lambda, uh(D)](\mathrm{Id}-\widetilde{\Delta}_\lambda)\widetilde{\eta}\|_{L^2}$, that is, $\|\Delta_\lambda(uh(D)(\mathrm{Id}-\widetilde{\Delta}_\lambda)\widetilde{\eta})\|_{L^2}$. Note that the frequencies of order $\leq \lambda/8$ in the Littlewood-Paley decomposition of u do not contribute, therefore

$$\begin{aligned} \|\Delta_\lambda(uh(D)(\mathrm{Id}-\widetilde{\Delta}_\lambda)\widetilde{\eta})\|_{L^1_TL^2} &= \left\|\Delta_\lambda\sum_{\mu\geq\lambda/8}u_\mu h(D)(\mathrm{Id}-\widetilde{\Delta}_\lambda)\widetilde{\eta}\right\|_{L^1_TL^2} \\ &\lesssim \sum_{\mu\geq\lambda/8}\|\Delta_\lambda h(D)(\mathrm{Id}-\widetilde{\Delta}_\lambda)\widetilde{\eta}\|_{L^1_TL^\infty}\|u_\mu\|_{L^\infty_TL^2}. \end{aligned} \tag{2.27}$$

Denote

$$\begin{aligned} A(\xi) &= h(\xi) - \left(\frac{ad}{c}\right)^{1/2}|\xi| \\ &= \frac{(ad+ac-cd)\xi^2 - c}{c\left[(a\xi^2-1)(c\xi^2-1)(d\xi^2+1)\right]^{1/2} + (acd)^{1/2}|\xi|(c\xi^2-1)}. \end{aligned} \tag{2.28}$$

Let $\psi(\xi) \in C^\infty(\mathbb{R})$ be a nonnegative function satisfying $\psi(\xi) = 1$ for $|\xi| > 2$ and $\psi(\xi) = 0$ for $|\xi| \leq 1$. It is obvious that $\psi(\xi)A(\xi)\xi/|\xi| \in S^{-1}$; that is, $\psi(\xi)A(\xi)\xi/|\xi| \in C^\infty(\mathbb{R})$ satisfies

$$\left|\partial_\xi^\alpha\frac{\psi(\xi)A(\xi)\xi}{|\xi|}\right| \lesssim (1+\xi^2)^{(-1-\alpha)/2} \tag{2.29}$$

for any nonnegative integers α. Let Λ_γ be the Lipschitz space defined by

$$\Lambda_\gamma = \left\{f : \text{there exists a positive constant } A \text{ such that } \|f\|_{L^\infty} \leq A,\ \|\Delta_\lambda f\|_{L^\infty} \leq A\lambda^{-\gamma}\right\}. \tag{2.30}$$

Stein [4, Chapter 6, Section 5.3, Proposition 6] shows that $\psi(D)A(D)\mathcal{H}$ is a bounded mapping from $\Lambda_{1/2}$ to $\Lambda_{3/2}$. We have

$$\begin{aligned}
&\|\Delta_\lambda h(D)\psi(D)(\mathrm{Id}-\widetilde{\Delta}_\lambda)\widetilde{\eta}\|_{L^1_TL^\infty}\\
&\quad\lesssim \Big\|\Big(\Big(\frac{ad}{c}\Big)^{1/2}\partial_x\mathcal{H}\psi(D)+A(D)\psi(D)\Big)(\mathrm{Id}-\widetilde{\Delta}_\lambda)\widetilde{\eta}\Big\|_{L^1_TL^\infty}\\
&\quad\lesssim \|\psi(D)\partial_x(\mathrm{Id}-\widetilde{\Delta}_\lambda)\mathcal{H}\widetilde{\eta}\|_{L^1_TL^\infty}+\|A(D)\psi(D)\mathcal{H}(\mathrm{Id}-\widetilde{\Delta}_\lambda)\mathcal{H}\widetilde{\eta}\|_{L^1_TL^\infty}\\
&\quad\lesssim \|\partial_x(\mathcal{H}\widetilde{\eta})\|_{L^1_TL^\infty}+\|A(D)\psi(D)\mathcal{H}\mathcal{H}\widetilde{\eta}\|_{L^1_T\Lambda_{3/2}}\\
&\quad\lesssim \|\partial_x(\mathcal{H}\widetilde{\eta})\|_{L^1_TL^\infty}+\|\mathcal{H}\widetilde{\eta}\|_{L^1_T\Lambda_{1/2}}\lesssim\|\partial_x(\mathcal{H}\widetilde{\eta})\|_{L^1_TL^\infty}+\|\mathcal{H}\widetilde{\eta}\|_{L^1_TL^\infty},\\
&\|\Delta_\lambda h(D)(\mathrm{Id}-\psi(D))(\mathrm{Id}-\widetilde{\Delta}_\lambda)\widetilde{\eta}\|_{L^1_TL^\infty}\lesssim\|h(D)(\mathrm{Id}-\psi(D))\widetilde{\eta}\|_{L^1_TL^\infty}\lesssim\|\widetilde{\eta}\|_{L^1_TL^\infty},
\end{aligned}\tag{2.31}$$

because of $h(\xi)(1-\psi(\xi))\in C_0^\infty(\mathbb{R})$. Hence, we get

$$\begin{aligned}
&\|\Delta_\lambda h(D)(\mathrm{Id}-\widetilde{\Delta}_\lambda)\widetilde{\eta}\|_{L^1_TL^\infty}\\
&\quad\lesssim\|\Delta_\lambda h(D)\psi(D)(\mathrm{Id}-\widetilde{\Delta}_\lambda)\widetilde{\eta}\|_{L^1_TL^\infty}+\|\Delta_\lambda h(D)(\mathrm{Id}-\psi(D))(\mathrm{Id}-\widetilde{\Delta}_\lambda)\widetilde{\eta}\|_{L^1_TL^\infty}\\
&\quad\lesssim\|\partial_x(\mathcal{H}\widetilde{\eta})\|_{L^1_TL^\infty}+\|\mathcal{H}\widetilde{\eta}\|_{L^1_TL^\infty}+\|\widetilde{\eta}\|_{L^1_TL^\infty}.
\end{aligned}\tag{2.32}$$

A combination of (2.26) with (2.27) and (2.32) yields

$$\begin{aligned}
&\|\Delta_\lambda(uh(D)(\mathrm{Id}-\widetilde{\Delta}_\lambda)\widetilde{\eta})\|_{L^1_TL^2}\\
&\quad\lesssim\Big(\|\partial_x\mathcal{H}\widetilde{\eta}\|_{L^1_TL^\infty}+\|\mathcal{H}\widetilde{\eta}\|_{L^1_TL^\infty}+\|\widetilde{\eta}\|_{L^1_TL^\infty}\Big)\sum_{\mu\ge\lambda/8}\|u_\mu\|_{L^\infty_TL^2},
\end{aligned}\tag{2.33}$$

and so

$$\begin{aligned}
\|[\Delta_\lambda,uh(D)]\widetilde{\eta}_\lambda\|_{L^1_TL^2}&\lesssim\Big(\|u\|_{L^1_TL^\infty}+\|u_x\|_{L^1_TL^\infty}\Big)\|\widetilde{\Delta}_\lambda\widetilde{\eta}\|_{L^\infty_TL^2}\\
&\quad+\Big(\|\partial_x(\mathcal{H}\widetilde{\eta})\|_{L^1_TL^\infty}+\|\mathcal{H}\widetilde{\eta}\|_{L^1_TL^\infty}+\|\widetilde{\eta}\|_{L^1_TL^\infty}\Big)\sum_{\mu\ge\lambda/8}\|u_\mu\|_{L^\infty_TL^2}.
\end{aligned}\tag{2.34}$$

Similarly,

$$\|[\Delta_\lambda,u\partial_x]u\|_{L^1_TL^2}\lesssim\|u_x\|_{L^1_TL^\infty}\|\widetilde{\Delta}_\lambda u\|_{L^\infty_TL^2}+\|u_x\|_{L^1_TL^\infty}\sum_{\mu\ge\lambda/8}\|u_\mu\|_{L^\infty_TL^2}.\tag{2.35}$$

It follows from (2.22), (2.34), and (2.35) that

$$\begin{aligned}
&\|(\mathcal{H}\widetilde{\eta}_\lambda, u_\lambda)\|_{L^p_T L^q} \\
&\quad\lesssim (1+T)^{1/p}\lambda^{1/p}\left(1+\|J^\sigma u\|_{L^\infty_T L^2}\right) \\
&\qquad\times\left(\|(\mathcal{H}\widetilde{\eta}_\lambda, u_\lambda)\|_{L^\infty_T L^2}+\|[\Delta_\lambda, uh(D)]\widetilde{\eta}_\lambda\|_{L^1_T L^2}+\|[\Delta_\lambda, u\partial_x]u\|_{L^1_T L^2}\right) \\
&\quad\lesssim (1+T)^{1/p}\lambda^{1/p}\left(1+\|J^\sigma u\|_{L^\infty_T L^2}\right) \\
&\qquad\times\Bigg(\|(u,u_x)\|_{L^1_T L^\infty}\|(\widetilde{\Delta}_\lambda\mathcal{H}\widetilde{\eta}, \widetilde{\Delta}_\lambda u)\|_{L^\infty_T L^2}+\|(\mathcal{H}\widetilde{\eta}_\lambda, u_\lambda)\|_{L^\infty_T L^2} \\
&\qquad\qquad+\|(\mathcal{H}\widetilde{\eta}, \partial_x\mathcal{H}\widetilde{\eta}, \widetilde{\eta}, u_x)\|_{L^1_T L^\infty}\sum_{\mu\geq\lambda/8}\|u_\mu\|_{L^\infty_T L^2}\Bigg).
\end{aligned}\tag{2.36}$$

Hence,

$$\begin{aligned}
&\sum_\lambda \lambda^{2\sigma}\|(\mathcal{H}\widetilde{\eta}_\lambda, u_\lambda)\|^2_{L^p_T L^q} \\
&\quad\lesssim \sum_\lambda (1+T)^{2/p}\lambda^{2/p+2\sigma}\left(1+\|J^\sigma u\|_{L^\infty_T L^2}\right)^2 \\
&\qquad\times\Bigg(\|(u,u_x)\|^2_{L^1_T L^\infty}\|(\widetilde{\Delta}_\lambda\mathcal{H}\widetilde{\eta}, \widetilde{\Delta}_\lambda u)\|^2_{L^\infty_T L^2}+\|(\mathcal{H}\widetilde{\eta}_\lambda, u_\lambda)\|^2_{L^\infty_T L^2} \\
&\qquad\qquad+\|(\mathcal{H}\widetilde{\eta}, \partial_x\mathcal{H}\widetilde{\eta}, \widetilde{\eta}, u_x)\|_{L^1_T L^\infty}\left(\sum_{\mu\geq\lambda/8}\|u_\mu\|_{L^\infty_T L^2}\right)^2\Bigg) \\
&\quad\lesssim (1+T)^{2/p}\left(1+\|J^\sigma u\|_{L^\infty_T L^2}\right)^2\left(1+\|(u,u_x,\mathcal{H}\eta,\partial_x\mathcal{H}\eta,\widetilde{\eta})\|^2_{L^1_T L^\infty}\right) \\
&\qquad\times\Bigg(\sum_\lambda \lambda^{2/p+2\sigma}(\|(\mathcal{H}\widetilde{\eta}_\lambda, u_\lambda)\|^2_{L^\infty_T L^2}+\|(\widetilde{\Delta}_\lambda\mathcal{H}\widetilde{\eta}, \widetilde{\Delta}_\lambda u)\|_{L^\infty_T L^2}\Bigg) \\
&\qquad\qquad+\sum_\lambda \lambda^{2/p+2\sigma}\left(\sum_{\mu\geq\lambda/8}\|u_\mu\|_{L^\infty_T L^2}\right)^2\Bigg) \\
&\quad\lesssim (1+T)^{2/p}\left(1+\|J^\sigma u\|_{L^\infty_T L^2}\right)^2\left(1+\|(u,u_x,\mathcal{H}\widetilde{\eta},\partial_x\mathcal{H}\widetilde{\eta},\widetilde{\eta})\|^2_{L^1_T L^\infty}\right) \\
&\qquad\times\sum_\lambda \lambda^{2/p+2\sigma}\|(\mathcal{H}\widetilde{\eta}_\lambda, u_\lambda)\|^2_{L^\infty_T L^2},
\end{aligned}\tag{2.37}$$

where we have used the inequality due to Lemma 2.1:

$$\sum_\lambda \lambda^{2/p+2\sigma}\left(\sum_{\mu\geq\lambda/8}\|u_\mu\|_{L^\infty_T L^2}\right)^2 \lesssim \sum_\lambda \lambda^{2/p+2\sigma}\|u_\lambda\|^2_{L^\infty_T L^2},\tag{2.38}$$

and the inequality

$$\sum_{\lambda}\lambda^{2/p+2\sigma}\|(\widetilde{\Delta}_{\lambda}\mathcal{H}\widetilde{\eta},\widetilde{\Delta}_{\lambda}u)\|_{L_T^{\infty}L^2}\lesssim\sum_{\lambda}\lambda^{2/p+2\sigma}\|(\mathcal{H}\widetilde{\eta}_{\lambda},u_{\lambda})\|_{L_T^{\infty}L^2}^2. \tag{2.39}$$

Thus we get

$$\begin{aligned}\sum_{\lambda}\lambda^{2\sigma}\|(\mathcal{H}\widetilde{\eta}_{\lambda},u_{\lambda})\|_{L_T^pL^q}^2 &\lesssim (1+T)^{2/p}\left(1+\|J^{\sigma}u\|_{L_T^{\infty}L^2}\right)^2\\ &\quad\times\left(1+\|(u,u_x,\mathcal{H}\widetilde{\eta},\partial_x\mathcal{H}\widetilde{\eta},\widetilde{\eta})\|_{L_T^1L^{\infty}}^2\right)\sum_{\lambda}\lambda^{2/p+2\sigma}\|(\mathcal{H}\widetilde{\eta}_{\lambda},u_{\lambda})\|_{L_T^{\infty}L^2}^2.\end{aligned} \tag{2.40}$$

We complete the proof. □

Lemma 2.3. *Let $1\le\kappa_1\le\kappa_2$, and let $\{\delta_\lambda\}$ be the dyadic sequence of positive numbers satisfying $\kappa_1\delta_\lambda\le\delta_{2\lambda}\le\kappa_2\delta_\lambda$ and $\lambda\le\delta_\lambda\le\lambda^2$ for all dyadic integers λ. Then for all $\tau,t\in I$, the smooth solution (η,u) of (1.1) satisfies*

$$\sum_{\lambda}\delta_{\lambda}^2\|(\widetilde{\eta}_{\lambda}(t),u_{\lambda}(t))\|_{L^2}^2\lesssim\sum_{\lambda}\delta_{\lambda}^2\|(\widetilde{\eta}_{\lambda}(\tau),u_{\lambda}(\tau))\|_{L^2}^2\exp\left(2\|(u,u_x,\mathcal{H}\widetilde{\eta},\partial_x\mathcal{H}\widetilde{\eta})\|_{L_I^1L^{\infty}}\right). \tag{2.41}$$

Proof. Let $F=-(u\eta)_x$ and $G=-uu_x$ in (2.6), and without loss of generality assume $\tau<t$. Multiplying (2.6) by $(\Delta_\lambda v_\lambda,\Delta_\lambda w_\lambda)$ and integrating by parts, we get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\begin{pmatrix}\|v_\lambda(t)\|_{L^2}^2\\ \|w_\lambda(t)\|_{L^2}^2\end{pmatrix} &= \operatorname{Re}\frac{d}{dt}\begin{pmatrix}\int\widehat{v}_\lambda\overline{\widehat{v}_\lambda}\,d\xi\\ \int\widehat{w}_\lambda\overline{\widehat{w}_\lambda}\,d\xi\end{pmatrix}\\ &= \operatorname{Re}\begin{pmatrix}-\int\left[h^{-1}(D)((u\eta)_x)_\lambda+(1+dD^2)^{-1}(uu_x)_\lambda\right]v_\lambda\,dx\\ -\int\left[h^{-1}(D)((u\eta)_x)_\lambda-(1+dD^2)^{-1}(uu_x)_\lambda\right]w_\lambda\,dx\end{pmatrix},\end{aligned} \tag{2.42}$$

where we denote $(\widehat{v}_\lambda,\widehat{w}_\lambda)=\mathcal{F}_x(v_\lambda,v_\lambda)$. Using $(h^{-1}(D)\eta,u)=(v+w,v-w)=(\widetilde{\eta},u)$, we get

$$\begin{aligned}&\frac{d}{dt}\left(\|v_\lambda(t)\|_{L^2}^2+\|w_\lambda(t)\|_{L^2}^2\right)\\ &\quad=2\operatorname{Re}\left(-\int h^{-1}(D)((u\eta)_x)_\lambda\widetilde{\eta}_\lambda\,dx-\int(1+dD^2)^{-1}(uu_x)_\lambda u_\lambda\,dx\right),\end{aligned} \tag{2.43}$$

and so for the dyadic sequence $\{\delta_\lambda\}$, we get

$$
\begin{aligned}
\sum_\lambda \delta_\lambda^2 &\|(\tilde{\eta}_\lambda(t), u_\lambda(t))\|_{L^2}^2 \\
&\lesssim \sum_\lambda \delta_\lambda^2 \left(\|v_\lambda(t)\|_{L^2}^2 + \|w_\lambda(t)\|_{L^2}^2\right) \\
&\lesssim \sum_\lambda \delta_\lambda^2 \left(\|v_\lambda(\tau)\|_{L^2}^2 + \|w_\lambda(\tau)\|_{L^2}^2\right) + \int_\tau^t \left| \sum_\lambda \int \delta_\lambda^2 h^{-1}(D)\left((u(\sigma)\eta(\sigma))_x\right)_\lambda \tilde{\eta}(\sigma)_\lambda \, dx \right| d\sigma \\
&\quad + \int_\tau^t \left| \sum_\lambda \int \delta_\lambda^2 (1 + dD^2)^{-1} (u(\sigma)u_x(\sigma))_\lambda u(\sigma)_\lambda \, dx \right| d\sigma \\
&\lesssim \sum_\lambda \delta_\lambda^2 \|(\tilde{\eta}_\lambda(\tau), u_\lambda(\tau))\|_{L^2}^2 + I + II,
\end{aligned}
\tag{2.44}
$$

with

$$
\begin{aligned}
I &= \int_\tau^t \left| \sum_\lambda \int \delta_\lambda^2 h^{-1}(D)\left((u\eta)_x\right)_\lambda \tilde{\eta}_\lambda \, dx \right| d\sigma, \\
II &= \int_\tau^t \left| \sum_\lambda \int \delta_\lambda^2 (1 + dD^2)^{-1} (uu_x)_\lambda u_\lambda \, dx \right| d\sigma.
\end{aligned}
\tag{2.45}
$$

The estimate of II. Using $\lambda \le \delta_\lambda \le \lambda^2$, we have

$$
\begin{aligned}
II &\lesssim \sum_\lambda \int_\tau^t \|\delta_\lambda^2 u_\lambda\|_{L^2} \|\delta_\lambda^2 J^{-2} (uu_x)_\lambda\|_{L^2} \, d\sigma \\
&\lesssim \int_\tau^t \left(\sum_\lambda \delta_\lambda^2 \|u_\lambda\|_{L^2}^2 \right)^{1/2} \left(\sum_\lambda \delta_\lambda^2 \left\| J^{-2} (uu_x)_\lambda \right\|_{L^2}^2 \right)^{1/2} d\sigma \\
&\lesssim \int_\tau^t \left(\sum_\lambda \delta_\lambda^2 \|u_\lambda\|_{L^2}^2 \right)^{1/2} \left(\sum_\lambda \lambda^4 \left\| J^{-2} (uu_x)_\lambda \right\|_{L^2}^2 \right)^{1/2} d\sigma \\
&\lesssim \int_\tau^t \left(\sum_\lambda \delta_\lambda^2 \|u_\lambda\|_{L^2}^2 \right)^{1/2} \|uu_x\|_{L^2} \, d\sigma \lesssim \int_\tau^t \left(\sum_\lambda \delta_\lambda^2 \|u_\lambda\|_{L^2}^2 \right)^{1/2} \|u\|_{L^2} \|u_x\|_{L^\infty} \, d\sigma \\
&\lesssim \int_\tau^t \|u_x\|_{L^\infty} \sum_\lambda \delta_\lambda^2 \|u_\lambda\|_{L^2}^2 \, d\sigma.
\end{aligned}
\tag{2.46}
$$

The estimate of I. To estimate the term I, we introduce

$$
B(\xi) = h^{-1}(\xi) - \left(\frac{c}{ad}\right)^{1/2} |\xi|^{-1},
\tag{2.47}
$$

with

$$B(\xi) = \frac{(cd-ac-ad)\xi^2+c}{\{ad|\xi|[(a\xi^2-1)(d\xi^2+1)(c\xi^2-1)]^{1/2}+(acd)^{1/2}(a\xi^2-1)(d\xi^2+1)\}|\xi|}. \tag{2.48}$$

For $\lambda > 2$,

$$\begin{aligned} &\left|\sum_{\lambda>2}\int \delta_\lambda^2 h^{-1}(D)((u\eta)_x)_\lambda \tilde{\eta}_\lambda\,dx\right| \\ &\quad\lesssim \left|\sum_{\lambda>2}\int \delta_\lambda^2\left(\frac{c}{ad}\right)^{1/2}\frac{1}{|D|}((u\eta)_x)_\lambda\tilde{\eta}_\lambda\,dx\right| + \left|\sum_{\lambda>2}\int \delta_\lambda^2 B(D)((u\eta)_x)_\lambda\tilde{\eta}_\lambda\,dx\right|. \end{aligned} \tag{2.49}$$

Using $|\phi(\xi/\lambda)B(\xi)| \lesssim \lambda^{-3}$ for $\lambda > 2$, we have

$$\begin{aligned} &\left|\sum_{\lambda>2}\int \delta_\lambda^2 B(D)((u\eta)_x)_\lambda\tilde{\eta}_\lambda\,dx\right| \\ &\quad\lesssim \sum_{\lambda>2}\delta_\lambda^2\|B(D)((u\eta)_x)_\lambda\|_{L^2}\|\tilde{\eta}_\lambda\|_{L^2} \\ &\quad\lesssim \left\{\sum_{\lambda>2}\delta_\lambda^2\|B(D)((u\eta)_x)_\lambda\|_{L^2}^2\right\}^{1/2}\left\{\sum_{\lambda>2}\delta_\lambda^2\|\tilde{\eta}_\lambda\|_{L^2}^2\right\}^{1/2} \\ &\quad\lesssim \left\{\sum_{\lambda>2}\delta_\lambda^2\|\tilde{\eta}_\lambda\|_{L^2}^2\right\}^{1/2}\left\{\sum_{\lambda>2}\lambda^{-2}\|((u\eta)_x)_\lambda\|_{L^2}^2\right\}^{1/2} \\ &\quad\lesssim \left\{\sum_{\lambda>2}\delta_\lambda^2\|\tilde{\eta}_\lambda\|_{L^2}^2\right\}^{1/2}\|u\eta\|_{L^2}. \end{aligned} \tag{2.50}$$

By (2.28),

$$\begin{aligned} \|u\eta\|_{L^2} &= \|u\cdot h(D)\tilde{\eta}\|_{L^2} = \left\|u\cdot\left(A(D)+\left(\frac{ad}{c}\right)^{1/2}|D|\right)\tilde{\eta}\right\|_{L^2} \\ &\lesssim \|uA(D)\tilde{\eta}\|_{L^2} + \|u(|D|\tilde{\eta})\|_{L^2} \lesssim \|u\|_{L^\infty}\|A(D)\tilde{\eta}\|_{L^2} + \|u(\mathcal{H}\tilde{\eta})_x\|_{L^2} \\ &\lesssim \|u\|_{L^\infty}\|\tilde{\eta}\|_{L^2} + \|u(\mathcal{H}\tilde{\eta})\|_{H^1} + \|u_x(\mathcal{H}\tilde{\eta})\|_{L^2} \\ &\lesssim \|u\|_{L^\infty}\|\tilde{\eta}\|_{L^2} + \|u\|_{L^\infty}\|\mathcal{H}\tilde{\eta}\|_{H^1} + \|u_x\|_{L^\infty}\|\mathcal{H}\tilde{\eta}\|_{L^2} \\ &\lesssim \left(\|u\|_{L^\infty}+\|u_x\|_{L^\infty}\right)\left\{\sum_{\lambda}\delta_\lambda^2\|\tilde{\eta}_\lambda\|_{L^2}^2\right\}^{1/2}. \end{aligned} \tag{2.51}$$

A combination of (2.50) with (2.51) yields

$$\left|\sum_{\lambda>2}\int \delta_\lambda^2 B(D)\left((u\eta)_x\right)_\lambda \tilde{\eta}_\lambda\,dx\right| \lesssim \left(\|u\|_{L^\infty}+\|u_x\|_{L^\infty}\right)\sum_\lambda \delta_\lambda^2\|\tilde{\eta}_\lambda\|_{L^2}^2. \tag{2.52}$$

Using (2.28) again, we get

$$\begin{aligned}\left|\sum_{\lambda>2}\int \delta_\lambda^2\frac{1}{|D|}\left((u\eta)_x\right)_\lambda\cdot\tilde{\eta}_\lambda\,dx\right| &= \left|\sum_{\lambda>2}\delta_\lambda^2\int \mathcal{H}\left(u\left(A(D)+\left(\frac{ad}{c}\right)^{1/2}|D|\right)\tilde{\eta}\right)_\lambda\cdot\tilde{\eta}_\lambda\,dx\right|\\ &\lesssim \left|\sum_{\lambda>2}\delta_\lambda^2\int \mathcal{H}\left(u|D|\tilde{\eta}\right)_\lambda\cdot\tilde{\eta}_\lambda\,dx\right|\\ &\quad+\left|\sum_{\lambda>2}\delta_\lambda^2\int \mathcal{H}\left(uA(D)\tilde{\eta}\right)_\lambda\cdot\tilde{\eta}_\lambda\,dx\right|.\end{aligned} \tag{2.53}$$

$A(\xi)\in\mathcal{S}^{-1}$ and the inequality $\lambda\le\delta_\lambda\le\lambda^2$ imply

$$\begin{aligned}&\left|\sum_{\lambda>2}\delta_\lambda^2\int \mathcal{H}\left(uA(D)\tilde{\eta}\right)_\lambda\cdot\tilde{\eta}_\lambda\,dx\right|\\ &\quad\lesssim\left\{\sum_{\lambda>2}\delta_\lambda^2\|\mathcal{H}(uA(D)\tilde{\eta})_\lambda\|_{L^2}^2\right\}^{1/2}\left\{\sum_{\lambda>2}\delta_\lambda^2\|\tilde{\eta}_\lambda\|_{L^2}^2\right\}^{1/2}\lesssim\left\{\sum_{\lambda>2}\delta_\lambda^2\|\tilde{\eta}_\lambda\|_{L^2}^2\right\}^{1/2}\|uA(D)\tilde{\eta}\|_{H^2}\\ &\quad\lesssim\left\{\sum_{\lambda>2}\delta_\lambda^2\|\tilde{\eta}_\lambda\|_{L^2}^2\right\}^{1/2}\left(\|uA(D)\tilde{\eta}\|_{L^2}+\|\partial_x(u\cdot A(D)\tilde{\eta})\|_{H^1}\right)\\ &\quad\lesssim\left\{\sum_{\lambda>2}\delta_\lambda^2\|\tilde{\eta}_\lambda\|_{L^2}^2\right\}^{1/2}\left(\|u\|_{L^\infty}\|A(D)\tilde{\eta}\|_{L^2}+\|u_x\|_{L^\infty}\|A(D)\tilde{\eta}\|_{H^1}+\|u\|_{L^\infty}\|\partial_x(A(D)\tilde{\eta})\|_{H^1}\right)\\ &\quad\lesssim\left(\|u\|_{L^\infty}+\|u_x\|_{L^\infty}\right)\sum_\lambda\delta_\lambda^2\|\tilde{\eta}_\lambda\|_{L^2}^2,\end{aligned} \tag{2.54}$$

$$\begin{aligned}&\left|\sum_{\lambda>2}\delta_\lambda^2\int \mathcal{H}\left(u|D|\tilde{\eta}\right)_\lambda\cdot\tilde{\eta}_\lambda\,dx\right|\\ &\quad=\left|\sum_{\lambda>2}\delta_\lambda^2\int\left(u\partial_x\mathcal{H}\tilde{\eta}\right)_\lambda\cdot\mathcal{H}\tilde{\eta}_\lambda\,dx\right|\\ &\quad\le\left|\sum_{\lambda>2}\delta_\lambda^2\int u\partial_x\mathcal{H}\tilde{\eta}_\lambda\cdot\mathcal{H}\tilde{\eta}_\lambda\,dx\right|+\left|\sum_{\lambda>2}\delta_\lambda^2\int\left[\Delta_\lambda,u\partial_x\right]\mathcal{H}\tilde{\eta}\cdot\mathcal{H}\tilde{\eta}_\lambda\,dx\right|\\ &\quad\lesssim\left|\sum_{\lambda>2}\delta_\lambda^2\int\left(\left(\mathcal{H}(\tilde{\eta})\right)_\lambda\right)^2\cdot u_x\,dx\right|+\sum_{\lambda>2}\delta_\lambda^2\|[\Delta_\lambda,u\partial_x]\mathcal{H}\tilde{\eta}\|_{L^2}\cdot\|\mathcal{H}\tilde{\eta}_\lambda\|_{L^2}\\ &\quad\lesssim\|u_x\|_{L^\infty}\sum_{\lambda>2}\delta_\lambda^2\|\mathcal{H}\tilde{\eta}_\lambda\|_{L^2}^2+\sum_{\lambda>2}\delta_\lambda^2\|[\Delta_\lambda,u\partial_x]\mathcal{H}\tilde{\eta}\|_{L^2}\cdot\|\mathcal{H}\tilde{\eta}_\lambda\|_{L^2}.\end{aligned} \tag{2.55}$$

Using (2.23), we obtain

$$\begin{aligned} \|[\Delta_\lambda, u\partial_x]\mathcal{H}\tilde{\eta}\|_{L^2} &\lesssim \|[\Delta_\lambda, u\partial_x]\tilde{\Delta}_\lambda\mathcal{H}\tilde{\eta}\|_{L^2} + \|[\Delta_\lambda, u\partial_x](\mathrm{Id}-\tilde{\Delta}_\lambda)\mathcal{H}\tilde{\eta}\|_{L^2}, \\ \|[\Delta_\lambda, u\partial_x]\tilde{\Delta}_\lambda\mathcal{H}\tilde{\eta}\|_{L^2} &\lesssim \|u_x\|_{L^\infty}\|\tilde{\Delta}_\lambda\mathcal{H}\tilde{\eta}\|_{L^2}. \end{aligned} \tag{2.56}$$

Then we get

$$\sum_{\lambda>2}\delta_\lambda^2\|[\Delta_\lambda, u\partial_x]\tilde{\Delta}_\lambda\mathcal{H}\tilde{\eta}\|_{L^2}\|\mathcal{H}\tilde{\eta}_\lambda\|_{L^2} \lesssim \|u_x\|_{L^\infty}\sum_{\lambda>2}\delta_\lambda^2\|\mathcal{H}\tilde{\eta}_\lambda\|_{L^2}^2. \tag{2.57}$$

Now we estimate the term $\|[\Delta_\lambda, u\partial_x](\mathrm{Id}-\tilde{\Delta}_\lambda)\mathcal{H}\tilde{\eta}\|_{L^2}$. Let $u = \sum_\mu u_\mu$. We have

$$\begin{aligned} \|[\Delta_\lambda, u\partial_x](\mathrm{Id}-\tilde{\Delta}_\lambda)\mathcal{H}\tilde{\eta}\|_{L^2} &= \|\Delta_\lambda(u\partial_x(\mathrm{Id}-\tilde{\Delta}_\lambda)\mathcal{H}\tilde{\eta})\|_{L^2} \\ &\lesssim \sum_{\mu\geq\lambda/8}\|u_\mu\partial_x(\mathrm{Id}-\tilde{\Delta})\mathcal{H}\tilde{\eta}\|_{L^2} \lesssim \sum_{\mu\geq\lambda/8}\|\partial_x\mathcal{H}\tilde{\eta}\|_{L^\infty}\|u_\mu\|_{L^2}, \end{aligned} \tag{2.58}$$

and then by using Lemma 2.1,

$$\begin{aligned} &\sum_{\lambda>2}\delta_\lambda^2\|[\Delta_\lambda, u\partial_x](\mathrm{Id}-\tilde{\Delta}_\lambda)\mathcal{H}\tilde{\eta}\|_{L^2}\|\mathcal{H}\tilde{\eta}_\lambda\|_{L^2} \\ &\quad\lesssim \|\partial_x\mathcal{H}\tilde{\eta}\|_{L^\infty}\sum_{\lambda>2}\delta_\lambda^2\|\mathcal{H}\tilde{\eta}_\lambda\|_{L^2}\sum_{\mu\geq\lambda/8}\|u_\mu\|_{L^2} \\ &\quad\lesssim \|\partial_x\mathcal{H}\tilde{\eta}\|_{L^\infty}\left\{\sum_\lambda\delta_\lambda^2\|\mathcal{H}\tilde{\eta}_\lambda\|_{L^2}^2\right\}^{1/2}\left\{\sum_\lambda\delta_\lambda^2\|u_\lambda\|_{L^2}^2\right\}^{1/2} \\ &\quad\lesssim \|\partial_x\mathcal{H}\tilde{\eta}\|_{L^\infty}\left(\sum_\lambda\delta_\lambda^2\|\tilde{\eta}_\lambda\|_{L^2}^2 + \sum_\lambda\delta_\lambda^2\|u_\lambda\|_{L^2}^2\right). \end{aligned} \tag{2.59}$$

It follows from (2.55), (2.57), and (2.59) that

$$\begin{aligned} &\sum_{\lambda>2}\delta_\lambda^2\int\left(\mathcal{H}u\left(\left(\frac{ad}{c}\right)^{1/2}|D|\right)\tilde{\eta}\right)_\lambda\cdot\tilde{\eta}_\lambda\,dx \\ &\quad\lesssim \left(\|u_x\|_{L^\infty} + \|\partial_x\mathcal{H}\tilde{\eta}\|_{L^\infty}\right)\left(\sum_\lambda\delta_\lambda^2\|\tilde{\eta}_\lambda\|_{L^2}^2 + \sum_\lambda\delta_\lambda^2\|u_\lambda\|_{L^2}^2\right). \end{aligned} \tag{2.60}$$

What remains is to estimate the term $|\delta_\lambda^2\int h^{-1}(D)((u\eta)_x)_\lambda\cdot\tilde{\eta}_\lambda dx|$, with $\lambda = 1$ or 2. Let $\chi(\xi)\in C_0^\infty(\mathbb{R})$ satisfy $\chi(\xi) = 1$ for $|\xi| < 10$ and $\chi(\xi) = 0$ for $|\xi| > 20$. Since $\lambda = 1$ or 2 implies $\|\partial_x\tilde{\eta}_\lambda\|_{L^2} \lesssim \|\tilde{\eta}\|_{L^2}$ and $\chi(D)((u\eta)_x)_\lambda \cdot \tilde{\eta}_\lambda = ((u\eta)_x)_\lambda \cdot \tilde{\eta}_\lambda$, we get

$$\begin{aligned}
&\left|\delta_\lambda^2 \int h^{-1}(D)\big((u\eta)_x\big)_\lambda \cdot \tilde{\eta}_\lambda \, dx\right| \\
&\quad = \left|\delta_\lambda^2 \int h^{-1}(D)\partial_x\chi(D)(u\eta)_\lambda \cdot \tilde{\eta}_\lambda \, dx\right| \\
&\quad \lesssim \left|\int h^{-1}(D)\partial_x\chi(D)\Big[\big(u\cdot h(D)\chi(D)\tilde{\eta}\big)_\lambda \cdot \tilde{\eta}_\lambda + \big(u\cdot h(D)(\mathrm{Id}-\chi(D))\tilde{\eta}\big)_\lambda \cdot \tilde{\eta}_\lambda\Big]dx\right| \\
&\quad \lesssim \big\|\big(u\cdot h(D)\chi(D)\tilde{\eta}\big)_\lambda \cdot \tilde{\eta}_\lambda\big\|_{L^1} + \left\|\left(u\cdot\partial_x \frac{h(D)(\mathrm{Id}-\chi(D))}{iD}\tilde{\eta}\right)_\lambda \cdot \tilde{\eta}_\lambda\right\|_{L^1} \\
&\quad \lesssim \|u\|_{L^\infty}\|\tilde{\eta}\|_{L^2}^2 + \left\|\left(\partial_x u\cdot \frac{h(D)(\mathrm{Id}-\chi(D))}{iD}\tilde{\eta}\right)_\lambda \cdot \tilde{\eta}_\lambda\right\|_{L^1} \\
&\qquad + \left\|\left(u\cdot \frac{h(D)(\mathrm{Id}-\chi(D))}{iD}\tilde{\eta}\right)_\lambda \cdot \partial_x\tilde{\eta}_\lambda\right\|_{L^1} \\
&\quad \lesssim \Big(\|u\|_{L^\infty} + \|u_x\|_{L^\infty}\Big)\|\tilde{\eta}\|_{L^2}^2 \lesssim \Big(\|u\|_{L^\infty} + \|u_x v\|_{L^\infty}\Big)\sum_\lambda \delta_\lambda^2\|(\tilde{\eta}_\lambda, u_\lambda)\|_{L^2}^2.
\end{aligned} \tag{2.61}$$

A combination of (2.52), (2.60), and (2.61) with (2.49) yields

$$\left|\sum_\lambda \delta_\lambda^2 \int h^{-1}(D)\big((u\eta)_x\big)_\lambda \cdot \tilde{\eta}_\lambda \, dx\right| \lesssim \big\|(u, u_x, \mathcal{H}\tilde{\eta}, \partial_x\mathcal{H}\tilde{\eta})\big\|_{L^\infty}\sum_\lambda \delta_\lambda^2\|(\tilde{\eta}_\lambda, u_\lambda)\|_{L^2}^2, \tag{2.62}$$

and so

$$I \lesssim \int_\tau^t \big\|(u, u_x, \mathcal{H}\tilde{\eta}, \partial_x\mathcal{H}\tilde{\eta})\big\|_{L^\infty}\sum_\lambda \delta_\lambda^2\|(\tilde{\eta}_\lambda, u_\lambda)\|_{L^2}^2 \, d\sigma. \tag{2.63}$$

Hence, by (2.44), (2.46), and (2.63)

$$\begin{aligned}
&\sum_\lambda \delta_\lambda^2\big\|\big(\tilde{\eta}(t), u_\lambda(t)\big)\big\|_{L^2}^2 \\
&\quad \lesssim \sum_\lambda \delta_\lambda^2\big\|\big(\tilde{\eta}(\tau), u_\lambda(\tau)\big)\big\|_{L^2}^2 + \int_\tau^t \big\|(u, u_x, \mathcal{H}\tilde{\eta}, \partial_x\mathcal{H}\tilde{\eta})\big\|_{L^\infty}\sum_\lambda \delta_\lambda^2\|(\tilde{\eta}, u_\lambda)\|_{L^2}^2 \, d\sigma.
\end{aligned} \tag{2.64}$$

The Gronwall inequality implies

$$\sum_\lambda \delta_\lambda^2\big\|\big(\tilde{\eta}(t), u_\lambda(t)\big)\big\|_{L^2}^2 \lesssim \sum_\lambda \delta_\lambda^2\big\|\big(\tilde{\eta}(\tau), u_\lambda(\tau)\big)\big\|_{L^2}^2 \exp\Big(2\big\|(u, u_x, \mathcal{H}\tilde{\eta}, \partial_x\mathcal{H}\tilde{\eta})\big\|_{L_I^1 L^\infty}\Big). \tag{2.65}$$

We complete the proof. □

Theorem 2.4. *Fix $T > 0$ and $1 < \sigma \leq 2 - 1/p$. Let (η, u) be a smooth solution of (1.1). Then for every admissible pair (p, q),*

$$\begin{aligned} \|J^\sigma(\tilde{\eta}, u)\|_{L_T^p L^q} \lesssim (1+T)^{1/p} \left(1 + \|J^\sigma u\|_{L_T^\infty L^2}\right) \|J^{\sigma+1/p}(\tilde{\eta}, u)\|_{L_T^\infty L^2} \\ \times \left(1 + \|(u, u_x, \mathcal{H}\tilde{\eta}, \partial_x \mathcal{H}\tilde{\eta}, \tilde{\eta})\|_{L_T^1 L^\infty}^2\right). \end{aligned} \tag{2.66}$$

Proof. If (p, q) is an admissible pair, then both p and q are greater than or equal to two and different from infinity. Therefore Minkowski inequality, Littlewood-Paley square function theorem, and Mikhlin-Hörmander theorem show

$$\|J^\sigma(\tilde{\eta}, u)\|_{L_T^p L^q}^2 \lesssim \sum_\lambda \|J^\sigma(\tilde{\eta}_\lambda, u_\lambda)\|_{L_T^p L^q}^2 \lesssim \sum_\lambda \lambda^{2\sigma} \|(\tilde{\eta}_\lambda, u_\lambda)\|_{L_T^p L^q}^2. \tag{2.67}$$

By Lemma 2.2,

$$\begin{aligned} \sum_\lambda \lambda^{2\sigma} \|(\tilde{\eta}_\lambda, u_\lambda)\|_{L_T^p L^q}^2 \lesssim (1+T)^{2/p} \left(1 + \|J^\sigma u\|_{L_T^\infty L^2}\right)^2 \left(1 + \|(u, u_x, \mathcal{H}\tilde{\eta}, \partial_x \mathcal{H}\tilde{\eta}, \tilde{\eta})\|_{L_T^1 L^\infty}^2\right)^2 \\ \times \sum_\lambda \lambda^{2\sigma+2/p} \|(\tilde{\eta}_\lambda, u_\lambda)\|_{L_T^\infty L^2}^2. \end{aligned} \tag{2.68}$$

Choosing $\tau = 0$, $\delta_\lambda = \lambda^{\sigma+1/p}$, and $I = [0, T]$ in (2.64), we get

$$\begin{aligned} &\sum_\lambda \lambda^{2\sigma+2/p} \|(\tilde{\eta}_\lambda, u_\lambda)\|_{L_T^\infty L^2}^2 \\ &\quad \lesssim \sum_\lambda \lambda^{2\sigma+2/p} \|(\tilde{\eta}_\lambda(0), u_\lambda(0))\|_{L^2}^2 \\ &\qquad + \int_0^t \|(u, u_x, \mathcal{H}\tilde{\eta}, \partial_x \mathcal{H}\tilde{\eta})\|_{L^\infty} \sum_\lambda \lambda^{2\sigma+2/p} \|(\tilde{\eta}_\lambda(\tau), u_\lambda(\tau))\|_{L^2}^2 \, d\tau \\ &\quad \lesssim \left(1 + \|(u, u_x, \mathcal{H}\tilde{\eta}, \partial_x \mathcal{H}\tilde{\eta})\|_{L_T^1 L^\infty}^2\right) \|J^{\sigma+1/p}(\tilde{\eta}, u)\|_{L_T^\infty L^2}. \end{aligned} \tag{2.69}$$

Theorem 2.4 follows from (2.67), (2.68), and (2.69). □

3. The local well-posedness

3.1. Uniqueness. Let (η_1, u_1) and (η_2, u_2) be two solutions of the system (1.1). Let $\eta_j = h(D)(v_j + w_j)$ and $u_j = v_j - w_j$ $(j = 1, 2)$. Then $(v_1 - v_2, w_1 - w_2)$ satisfies the system (2.6) associated with $F = -(u_1\eta_1 - u_2\eta_2)_x$ and $G = -(u_1\partial_x u_1 - u_2\partial_x u_2)$. Multiplying the equation satisfied by $v_1 - v_2$ (resp., $w_1 - w_2$) with $v_1 - v_2$ (resp., $w_1 - w_2$) and integrating by parts easily yield

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|(v_1-v_2,w_1-w_2)\|_{L^2}^2\\
&\quad=-\int\Gamma(D)(v_1-v_2)\cdot(v_1-v_2)\,dx+\int\Gamma(D)(w_1-w_2)\cdot(w_1-w_2)\,dx\\
&\qquad-\frac{1}{2}\int\left[h^{-1}(D)(u_1\eta_1-u_2\eta_2)_x\right]\cdot(\tilde{\eta}_1-\tilde{\eta}_2)\,dx\\
&\qquad-\frac{1}{2}\int\left[\frac{1}{1+dD^2}(u_1\partial_x u_1-u_2\partial_x u_2)\right]\cdot(u_1-u_2)\,dx\\
&\quad\lesssim\|(\tilde{\eta}_1-\tilde{\eta}_2,u_1-u_2)\|_{L^2}^2+III+IV,
\end{aligned}\tag{3.1}$$

with

$$\begin{aligned}
III&=-\frac{1}{2}\int\partial_x h^{-1}(D)\left[(u_1-u_2)\eta_1+u_2(\eta_1-\eta_2)\right]\cdot(\tilde{\eta}_1-\tilde{\eta}_2)\,dx\\
&=\frac{1}{2}\int(u_1-u_2)\eta_1\cdot\partial_x h^{-1}(D)(\tilde{\eta}_1-\tilde{\eta}_2)\,dx-\frac{1}{2}\int\partial_x h^{-1}(D)(u_2(\eta_1-\eta_2))\cdot(\tilde{\eta}_1-\tilde{\eta}_2)\,dx,\\
IV&=-\frac{1}{2}\int\frac{1}{1+dD^2}\left[(u_1-u_2)\partial_x u_1+u_2(\partial_x u_1-\partial_x u_2)\right]\cdot(u_1-u_2)\,dx.
\end{aligned}\tag{3.2}$$

It is obvious that

$$|IV|\lesssim\|(\partial_x u_1,\partial_x u_2)\|_{L^\infty}\|u_1-u_2\|_{L^2}^2.\tag{3.3}$$

By (2.28), we have

$$h(\xi)=\left(\frac{ad}{c}\right)^{1/2}|\xi|+(|\xi|+1)\frac{A(\xi)}{1+|\xi|}.\tag{3.4}$$

Moreover, $A(\xi)/(1+|\xi|)\in H^1(\mathbb{R})$ and $A(\xi)\operatorname{sgn}(\xi)/(1+|\xi|)\in H^1(\mathbb{R})$ decay like $|\xi|^{-2}$ at infinity, and so they define some L^∞-multipliers. Thus,

$$\begin{aligned}
\|\eta_1\|_{L^\infty}&=\left\|\left(\left(\frac{ad}{c}\right)^{1/2}\partial_x+\frac{A(D)}{1+|D|}\partial_x-\frac{A(D)\mathcal{H}}{1+|D|}\right)\mathcal{H}\tilde{\eta}_1\right\|_{L^\infty}\\
&\lesssim\|\mathcal{H}\tilde{\eta}_1\|_{L^\infty}+\|\partial_x\mathcal{H}\tilde{\eta}_1\|_{L^\infty},
\end{aligned}\tag{3.5}$$

and so

$$\begin{aligned}
&\left|\int(u_1-u_2)\eta_1\cdot\partial_x h^{-1}(D)(\tilde{\eta}_1-\tilde{\eta}_2)\,dx\right|\\
&\qquad\lesssim\|\eta_1\|_{L^\infty}\|(\tilde{\eta}_1-\tilde{\eta}_2,u_1-u_2)\|_{L^2}^2\\
&\qquad\lesssim\left[\|\mathcal{H}\tilde{\eta}_1\|_{L^\infty}+\|\partial_x\mathcal{H}\tilde{\eta}_1\|_{L^\infty}\right]\|(\tilde{\eta}_1-\tilde{\eta}_2,u_1-u_2)\|_{L^2}^2.
\end{aligned}\tag{3.6}$$

To bound the term $\int \partial_x h^{-1}(D)(u_2(\eta_1-\eta_2))\cdot(\widetilde{\eta}_1-\widetilde{\eta}_2)dx$, we use (2.28) again and get

$$\begin{aligned}
&\left|\int \partial_x h^{-1}(D)(u_2(\eta_1-\eta_2))\cdot(\widetilde{\eta}_1-\widetilde{\eta}_2)dx\right|\\
&\quad=\left|\int \partial_x h^{-1}(D)\left[u_2\left(\left(\frac{ad}{c}\right)^{1/2}|D|+A(D)\right)(\widetilde{\eta}_1-\widetilde{\eta}_2)\right]\cdot(\widetilde{\eta}_1-\widetilde{\eta}_2)dx\right|\\
&\quad=\left|\int\left[\left(\frac{c}{ac}\right)^{1/2}\mathcal{H}+\partial_x B(D)\right]\left[u_2\left(\frac{ad}{c}\right)^{1/2}|D|(\widetilde{\eta}_1-\widetilde{\eta}_2)\right]\cdot(\widetilde{\eta}_1-\widetilde{\eta}_2)dx\right|\\
&\qquad+\left|\int \partial_x h^{-1}(D)[u_2A(D)(\widetilde{\eta}_1-\widetilde{\eta}_2)]\cdot(\widetilde{\eta}_1-\widetilde{\eta}_2)dx\right|\\
&\quad\lesssim\left|\int[u_2\partial_x\mathcal{H}(\widetilde{\eta}_1-\widetilde{\eta}_2)]\cdot\mathcal{H}(\widetilde{\eta}_1-\widetilde{\eta}_2)dx\right|\\
&\qquad+\left|\int \partial_x B(D)\mathcal{H}[u_2\partial_x\mathcal{H}(\widetilde{\eta}_1-\widetilde{\eta}_2)]\cdot(\widetilde{\eta}_1-\widetilde{\eta}_2)dx\right|\\
&\qquad+\|u_2A(D)(\widetilde{\eta}_1-\widetilde{\eta}_2)\|_{L^2}\|\widetilde{\eta}_1-\widetilde{\eta}_2\|_{L^2}\\
&\quad\lesssim\left|\int \partial_x u_2[\mathcal{H}(\widetilde{\eta}_1-\widetilde{\eta}_2)]^2dx\right|+\left|\int \partial_x^2 B(D)\mathcal{H}[u_2\mathcal{H}(\widetilde{\eta}_1-\widetilde{\eta}_2)]\cdot(\widetilde{\eta}_1-\widetilde{\eta}_2)dx\right|\\
&\qquad+\left|\int \partial_x B(D)\mathcal{H}[\partial_x u_2\mathcal{H}(\widetilde{\eta}_1-\widetilde{\eta}_2)]\cdot(\widetilde{\eta}_1-\widetilde{\eta}_2)dx\right|+\|u_2\|_{L^\infty}\|\widetilde{\eta}_1-\widetilde{\eta}_2\|_{L^2}^2\\
&\quad\lesssim\left[\|\partial_x u_2\|_{L^\infty}+\|u_2\|_{L^\infty}\right]\|\widetilde{\eta}_1-\widetilde{\eta}_2\|_{L^2}^2.
\end{aligned}\tag{3.7}$$

It follows from (3.1)–(3.7) that

$$\frac{d}{dt}\|(v_1-v_2,w_1-w_2)\|_{L^2}^2\lesssim\left[1+\|(\mathcal{H}\widetilde{\eta}_1,\partial_x\mathcal{H}\widetilde{\eta}_1,u_2,\partial_x u_2)\|_{L^\infty}\right]\|(\widetilde{\eta}_1-\widetilde{\eta}_2,u_1-u_2)\|_{L^2}^2.\tag{3.8}$$

By the Gronwall lemma,

$$\begin{aligned}
&\|(\widetilde{\eta}_1-\widetilde{\eta}_2,u_1-u_2)(t)\|_{L^2}\\
&\quad\lesssim\|(\widetilde{\eta}_1(0)-\widetilde{\eta}_2(0),u_1(0)-u_2(0))\|_{L^2}\exp\left(1+\|(\mathcal{H}\widetilde{\eta}_1,\partial_x\mathcal{H}\widetilde{\eta}_1,u_2,\partial_x u_2)\|_{L_T^1L^\infty}\right),
\end{aligned}\tag{3.9}$$

which clearly implies the uniqueness.

3.2. Existence. Without loss of generality, we assume $1/4<s<1$. Let (η,u) be a smooth solution of the system (1.1). Setting $\sigma=s+3/4$, $\delta_\lambda=\lambda^{s+1}$, and $I=[0,T]$ in Lemma 2.3, we deduce that

$$\|J^{s+1}(\widetilde{\eta}(t),u(t))\|_{L_T^\infty L^2}^2\lesssim\|J^{s+1}(\widetilde{\eta}(0),u(0))\|_{L^2}^2\exp\left(2\|(u,u_x,\mathcal{H}\widetilde{\eta},\partial_x\mathcal{H}\widetilde{\eta})\|_{L_T^1L^\infty}\right).\tag{3.10}$$

If (p,q) is an admissible pair, then $\sigma + 1/p < \sigma + 1/4 = s+1 < 2$. Therefore using Theorem 2.4 and (3.10), we get that for every admissible pair (p,q) and every $T > 0$,

$$\begin{aligned} \|J^\sigma(\tilde{\eta},u)\|_{L^p_T L^q} \lesssim{}& (1+T)^{1/p}\left(1+\|J^\sigma u\|_{L^\infty_T L^2}\right)\left(1+\|(u,u_x,\mathcal{H}\tilde{\eta},\partial_x\mathcal{H}\tilde{\eta},\tilde{\eta})\|^2_{L^1_T L^\infty}\right) \\ &\times \|J^{s+1}(\tilde{\eta}(0),u(0))\|_{L^2}\exp\left(2\|(u,u_x,\mathcal{H}\tilde{\eta},\partial_x\mathcal{H}\tilde{\eta},\tilde{\eta})\|_{L^1_T L^\infty}\right). \end{aligned} \tag{3.11}$$

Using the Sobolev embedding in the spatial variable together with the Hölder inequality in time variable, we can choose an admissible pair (p,q) such that

$$\|(u,u_x,\mathcal{H}\tilde{\eta},\partial_x\mathcal{H}\tilde{\eta},\tilde{\eta})\|_{L^1_T L^\infty} \lesssim T^{1-1/p}\|J^\sigma(\tilde{\eta},u)\|_{L^p_T L^q}. \tag{3.12}$$

A combination of (3.12) with (3.11) yields

$$\begin{aligned} &\|(u,u_x,\mathcal{H}\tilde{\eta},\partial_x\mathcal{H}\tilde{\eta},\tilde{\eta})\|_{L^1_T L^\infty} \\ &\quad\lesssim T^{1-1/p}(1+T)^{1/p}\left(1+\|J^\sigma u\|_{L^\infty_T L^2}\right)\left(1+\|(u,u_x,\mathcal{H}\tilde{\eta},\partial_x\mathcal{H}\tilde{\eta},\tilde{\eta})\|^2_{L^1_T L^\infty}\right) \\ &\quad\times \|J^{s+1}(\tilde{\eta}(0),u(0))\|_{L^2}\exp\left(2\|(u,u_x,\mathcal{H}\tilde{\eta},\partial_x\mathcal{H}\tilde{\eta},\tilde{\eta})\|_{L^1_T L^\infty}\right). \end{aligned} \tag{3.13}$$

Choosing $\delta_\lambda = \lambda^{s+1}$, $\tau = 0$, and $t = T$ in (2.64), we deduce

$$\begin{aligned} \|J^{s+1}(\tilde{\eta}(t),u(t))\|^2_{L^\infty_T L^2} \lesssim{}& \|J^{s+1}(\tilde{\eta}(0),u(0))\|^2_{L^2} \\ &+\|J^{s+1}(\tilde{\eta}(t),u(t))\|_{L^\infty_T L^2}\|(u,u_x,\mathcal{H}\tilde{\eta},\partial_x\mathcal{H}\tilde{\eta})\|_{L^1_T L^\infty}. \end{aligned} \tag{3.14}$$

Then there exists a positive constant C_0 so small that

$$\|J^{s+1}(\tilde{\eta}(t),u(t))\|_{L^\infty_T L^2} \lesssim \frac{1}{1-C_0}\|J^{s+1}(\tilde{\eta}(0),u(0))\|_{L^2} \quad \text{for } t\in[0,T] \tag{3.15}$$

providing that $\|(u,u_x,\mathcal{H}\tilde{\eta},\partial_x\mathcal{H}\tilde{\eta},\tilde{\eta})\|_{L^1_T L^\infty} \le C_0$. Let $H(T) = \|(u,u_x,\mathcal{H}\tilde{\eta},\partial_x\mathcal{H}\tilde{\eta})\|_{L^1_T L^\infty}$. Equations (3.13) and (3.15) imply that

$$\begin{aligned} H(T) \lesssim{}& \left(1+\frac{1}{1-C_0}\|J^{s+1}(\tilde{\eta}(0),u(0))\|_{L^2}\right)\|J^{s+1}(\tilde{\eta}(0),u(0))\|_{L^2} \\ &\times T^{1-1/p}(1+T)^{1/p}\left(1+H^2(T)\right)\exp\left(2H(T)\right) \end{aligned} \tag{3.16}$$

providing that $H(T) \le C_0$. For every $R > 0$, we choose a positive constant T_R such that for all $T \in [0, T_R]$,

$$\left(1+\frac{1}{1-C_0}R\right)RT^{1-1/p}(1+T)^{1/p}\left(1+C_0^2\right)\exp\left(2C_0\right) < \frac{C_0}{2}. \tag{3.17}$$

Then for $\|(\tilde{\eta}(0), u(0))\|_{H^{s+1}} \le R$, we deduce from (3.16) that, for all $T \in [0, T_R]$,

$$H(T) \le C_0 \text{ implies } H(T) \le \frac{1}{2}C_0. \tag{3.18}$$

Note that $H(0) = 0$. A straightforward continuity argument shows that $H(T) \le C_0/2$ for all $T \le T_R \lesssim \|(\tilde{\eta}(0), u(0))\|_{H^{s+1}}^{-1}$. Using (3.10), we obtain that if (η, u) is a smooth solution of the system (1.1), then it satisfies

$$\left|\left|\left(u, u_x, \mathcal{H}\tilde{\eta}, \partial_x\mathcal{H}\tilde{\eta}\right)\right|\right|_{L_T^1 L^\infty} \le C \quad \forall T \le T_R, \tag{3.19}$$

$$\left|\left|\left(\tilde{\eta}_\lambda, u_\lambda\right)\right|\right|_{L_T^\infty H^{s+1}} \lesssim \left|\left|\left(\tilde{\eta}(0), u(0)\right)\right|\right|_{H^{s+1}} \quad \forall T \le T_R. \tag{3.20}$$

The bounds (3.19) and (3.20) enable us to perform a standard compactness argument. More precisely, consider that the smooth sequence $\{f_n(x), g_n(x)\}$ satisfies $\|(\tilde{f}_n, g_n)\|_{H^{s+1}} \le R$ for some positive constant R, which converges to $(f(x), g(x))$ in $H^s(\mathbb{R}) \times H^{s+1}(\mathbb{R})$, where we denote by $\tilde{f}_n = h^{-1}(D)f_n$. Let $\{\eta_n, u_n\}$ be the solution of the system (1.1) with data $(f_n(x), g_n(x))$ which exists globally in time due to [2, Theorem 3.5]. We will prove that $\{\eta_n, u_n\}$ converges and the limit object is a solution of the system (1.1) with data $(f(x), g(x))$. Indeed, (3.20) implies that $\{\eta_n, u_n\}$ converges in weak$*$-topology of $L^\infty([0, T_R] : H^s \times H^{s+1})$ to some limit (η, u). Using (3.9), we deduce that $\{\eta_n, u_n\}$ converges strongly to (η, u) in $L^\infty([0, T_R] : H^{-1} \times L^2)$, and therefore $(u_n\eta_n)_x$ and $u_n(u_n)_x$ converge to $(u\eta)_x$ and uu_x, respectively, in a distributional sense. This proves that the limit (η, u) satisfies the system (1.1) in a distributional sense. The map $[0, T_R] \ni t \mapsto (\eta(t), u(t)) \in H^s(\mathbb{R}) \times H^{s+1}(\mathbb{R})$ is weakly continuous. Lemma 2.3, with $\delta_\lambda = \lambda^{s+1}$, implies that the map $[0, T_R] \ni t \mapsto \|\eta(t)\|_{H^s} + \|u(t)\|_{H^{s+1}}$ is continuous because $\exp(C\|(u, u_x, \mathcal{H}\tilde{\eta}, \partial_x\mathcal{H}\tilde{\eta})\|_{L_I^1 L^\infty})$ tends to one as τ tends to t if $I = [\tau, t]$. Hence $[0, T_R] \ni t \mapsto (\eta(t), u(t)) \in H^s(\mathbb{R}) \times H^{s+1}(\mathbb{R})$ is continuous.

3.3. Continuous dependence on the data. We present a proof of continuous dependence on the data based on Lemma 2.3.

Lemma 3.1. *Fix $s \in [0, 1)$. Suppose that $(\nu^n, \gamma^n) \to (\nu, \gamma)$ in H^{s+1}. Then there exists a sequence $\{\delta_\lambda\}$ of positive numbers which satisfies $2^{s+1}\delta_\lambda \le \delta_{2\lambda} \le 4\delta_\lambda$, $\lambda \le \delta_\lambda \le \lambda^2$, and $\delta_\lambda/\lambda^{s+1} \to \infty$ such that $\sup_n \sum_\lambda \delta_\lambda^2 \|(\Delta_\lambda \nu^n, \Delta_\lambda \gamma^n)\|_{L^2}^2 < +\infty$.*

Proof. For $\lambda = 2^j$, set $a_j^n = \lambda^{2(s+1)}\|(\nu_\lambda^n, \gamma_\lambda^n)\|_{L^2}^2$, $a_j = \lambda^{2(s+1)}\|(\nu_\lambda, \gamma_\lambda)\|_{L^2}^2$. The assumptions imply that $\{a_j^n\}_{j\in\mathbb{N}} \to \{a_j\}_{j\in\mathbb{N}}$ in $\ell^1(\mathbb{N})$. Then for all $k \in \mathbb{N}$, there exists N_k such that

$$N_k \ge k, \quad \sup_n \sum_{j=N_k}^{\infty} a_j^n < 2^{-2k}. \tag{3.21}$$

For a fixed $j \in \mathbb{N}$, there exists a unique $k \in \mathbb{N}$ such that $N_{k-1} \leq j < N_k$. We set $\mu_j = 2^{k(1-s)}$ and $\delta_\lambda = \lambda^{s+1}\mu_j$ for $\lambda = 2^j$. Obviously $2^{s+1}\delta_\lambda \leq \delta_{2\lambda} \leq 4\delta_\lambda$ and $\delta_\lambda/\lambda^{s+1} \to +\infty$, and

$$\begin{aligned} \sup_n \sum_\lambda \delta_\lambda^2 \|(\Delta_\lambda v^n, \Delta_\lambda \gamma^n)\|_{L^2}^2 &\leq \sum_{j=1}^{\infty} \mu_j a_j^n \leq \sum_{k=0}^{\infty} \sum_{j=N_k}^{N_{k+1}} \mu_j a_j^n \\ &\leq \sum_{k=0}^{\infty} 2^{k(1-s)} \sum_{j=N_k}^{N_{k+1}} a_j^n \leq \sum_{k=0}^{\infty} 2^{k(1-s)} 2^{-2k} < +\infty. \end{aligned} \tag{3.22}$$

Let $\{(\eta^n, u^n)\}$ be a sequence of solutions in $C([0,T]; H^{s-1} \times H^s)$ with $(\eta^n(0), u^n(0)) \to (\eta(0), u(0))$ in $H^s \times H^{s+1}$. As in the proof of the existence of solutions, we have

$$(\tilde{\eta}^n(t), u^n(t)) \longrightarrow (\tilde{\eta}(t), u(t)) \quad \text{in } C([0,T]; L^2 \times L^2). \tag{3.23}$$

Using Lemmas 3.1 and 2.3, we deduce

$$\sup_n \sup_{0 \leq t \leq T} \sum_\lambda \delta_\lambda^2 \left(\|(\tilde{\eta}_\lambda^n(t), u_\lambda^n(t))\|_{L^2}^2 + \|(\tilde{\eta}_\lambda(t), u_\lambda(t))\|_{L^2}^2 \right) < +\infty. \tag{3.24}$$

Set $\tilde{\eta}_\Lambda = \sum_{\lambda \leq \Lambda} \tilde{\eta}_\lambda$, $u_\Lambda = \sum_{\lambda \leq \Lambda} u_\lambda$. Fix $\epsilon > 0$, there exists by (3.24) a Λ such that for every $t \in [0,T]$,

$$\sup_n \left\{ \|(\tilde{\eta}_\Lambda^n(t) - \tilde{\eta}^n(t), u_\Lambda^n(t) - u^n(t))\|_{H^{s+1}} + \|(\tilde{\eta}_\Lambda(t) - \tilde{\eta}(t), u_\Lambda(t) - u(t))\|_{H^{s+1}} \right\} < \frac{\epsilon}{2}. \tag{3.25}$$

By (3.23), there exists n_0 such that for $n \geq n_0$ and $0 \leq t \leq T$,

$$\|(\tilde{\eta}_\Lambda^n(t) - \tilde{\eta}_\Lambda(t), u_\Lambda^n(t) - u_\Lambda(t))\|_{H^{s+1}} \leq (2\Lambda)^{s+1} \|(\tilde{\eta}_\Lambda^n(t) - \tilde{\eta}_\Lambda(t), u_\Lambda^n(t) - u_\Lambda(t))\|_{L^2} < \frac{\epsilon}{2}. \tag{3.26}$$

Therefore, we get for $n \geq n_0$ and $0 \leq t \leq T$ that

$$\begin{aligned} &\|(\tilde{\eta}_\Lambda^n(t) - \tilde{\eta}(t), u_\Lambda^n(t) - u(t))\|_{L_T^\infty H^{s+1}} \\ &\quad \leq \|(\tilde{\eta}_\Lambda^n(t) - \tilde{\eta}_\Lambda(t), u_\Lambda^n(t) - u_\Lambda(t))\|_{L_T^\infty H^{s+1}} \\ &\qquad + \|(\tilde{\eta}_\Lambda^n(t) - \tilde{\eta}^n(t), u_\Lambda^n(t) - u^n(t))\|_{H^{s+1}} \\ &\qquad + \|(\tilde{\eta}_\Lambda(t) - \tilde{\eta}(t), u_\Lambda(t) - u(t))\|_{L_T^\infty H^{s+1}} < \epsilon. \end{aligned} \tag{3.27}$$

We complete the proof of the continuous dependence on the data. □

Acknowledgment

This work is partially supported by the NSF of China, Grant no. 10271108.

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