# BAIRE SPACES, $k$-SPACES, AND SOME PROPERLY HEREDITARY PROPERTIES

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A topological property is *properly hereditary property* if whenever every proper subspace has the property, the whole space has the property. In this note, we will study some topological properties that are preserved by proper subspaces; in fact, we will study the following topological properties: Baire spaces, second category, sequentially compact, hemicompact, $\delta$-normal, and spaces having dispersion points. Also, we will solve some open problems raised by Al-Bsoul (2003) and Arenas (1996) and conclude this note by some open problems.

## 1. Introduction

In 1996, Professor M. L. Puertas suggested the following question which was studied by Arenas in his paper [2]:

(*) *if every proper subspace has the property, the whole space has the property*.

Any topological property satisfying the property (*) is called *properly hereditary property*. Moreover, if every proper closed (resp., open, $F_\sigma$, $G_\delta$, etc.) subspace has the property, then the whole space has the property, and we call such a property *properly closed* (resp., *open*, $F_\sigma$, $G_\delta$, etc.) *hereditary*.

In [2], Arenas studied some topological properties having the property (*). In fact, Arenas [2] proved that the topological properties $T_i$, $i = 0,1,2,3$, are properly hereditary properties. He also studied countability axioms and metrizability. Nevertheless, for Arenas, the property (*) makes sense only for the topological properties that are hereditary to all subsets. On the other hand, as we saw in [1], there are some topological properties which are properly hereditary properties but are not hereditary properties, for instance, paracompactness. Thus, we may ask the converse.

*Question 1.1.* Is there a topological property $P$ which is hereditary but is not properly hereditary?

It is obvious to see that if a topological property is properly hereditary, it is not necessary that it is properly open (closed) hereditary. So, it is natural to redefine the following (compare with the definition in [2]).


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*Definition 1.2.* A topological property is called strong (resp., closed, open, $F_\sigma$, $G_\delta$, etc.) hereditary property provided that it is (resp., closed, open, $F_\sigma$, $G_\delta$, etc.) hereditary property and properly (resp., closed, open, $F_\sigma$, $G_\delta$, etc.) hereditary property.

Hence, $T_i$ is a strong hereditary property for $i = 0, 1/4, 1/2, 1, 1(1/3), 2, 2(1/2), 3$, but $T_1, T_{1(1/3)}, T_2, T_{2(1/2)}, T_3$ are neither strong open hereditary properties nor strong closed hereditary properties.

In [1], Al-Bsoul studied some topological properties as some nonfamiliar separation axioms beside some strong separation axioms, and also some covering properties.

Reading [1, 2], the reader may conclude that all classical topological properties are properly hereditary properties. This is not the case; however, in this note we are going to give two topological properties which are not properly (resp., closed, open) hereditary properties. Moreover, we will positively solve two open problems raised in [1, 2]. Also, we will prove that some topological properties, such as sequential compactness, hemicompactness, and $\delta$-normality, are properly hereditary properties.

## 2. Baire spaces and $k$-spaces

In this section we will positively solve the following open problem raised in [1].

*Question 2.1* [1]. Are Baire spaces and second category properly hereditary properties?

THEOREM 2.2. *If every proper subspace of $X$ is a Baire subspace, then $X$ is a Baire space.*

*Proof.* Let $\{G_n : n \in \mathbb{N}\}$ be a countable family of open dense subsets of $X$. Pick $x_0 \in X \setminus (\bigcap_{n\in\mathbb{N}} G_n)$. If such $x_0$ does not exist, then $\bigcap_{n\in\mathbb{N}} G_n = X$, and the proof is finished.

Let $Y = X \setminus \{x_0\}$, then $G_n \setminus \{x_0\} \neq \Phi$ for all $n \in \mathbb{N}$. If $\overline{G_i \setminus \{x_0\}}^Y \neq Y$ for some $i \in \mathbb{N}$, then there exist $t \in Y \setminus (\overline{G_i \setminus \{x_0\}}^Y)$ and an open set $V$ in $Y$ such that $t \in V \subseteq Y \setminus (\overline{G_i \setminus \{x_0\}}^Y)$. So, there exists an open set $U$ in $X$ such that $V = U \cap Y$. Now, $U \cap G_i \neq \Phi$ and $U = V \cup \{x_0\}$, so $V \cap G_i = \Phi$, and hence $U \cap G_i = \{x_0\}$, which is a contradiction. Thus $G_n \setminus \{x_0\}$ is an open dense set in $Y$ for all $n \in \mathbb{N}$. Therefore, $(\bigcap_{n\in\mathbb{N}} G_n)$ is dense in $X$. □

COROLLARY 2.3. *Let $X$ be a $T_1$-space. If every proper open subspace of $X$ is a Baire subspace, then $X$ is a Baire space.*

COROLLARY 2.4. *Second category is properly hereditary property.*

*Proof.* Let $X$ be a space such that every proper subspace of $X$ is a second category, and let $\{G_n : n \in \mathbb{N}\}$ be a countable family of open dense subsets of $X$. Suppose that $\bigcap_{n\in\mathbb{N}} G_n = \Phi$. Pick $x_0 \in X$, and let $Y = X \setminus \{x_0\}$, then $G_n \setminus \{x_0\}$ is an open dense subset of $Y$ for all $n \in \mathbb{N}$. Therefore, $\bigcap_{n\in\mathbb{N}} G_n \neq \Phi$. □

Also, we will solve the following open problem for $k$-spaces positively.

*Question 2.5* [2]. Is being a $k$-space a properly (closed) hereditary property?

Recall that a Hausdorff space $X$ is called a *$k$-space* if and only if the following condition holds:

$$A \subseteq X \text{ is open if } A \cap K \text{ is open in } K \text{ for each compact set } K \text{ in } X. \tag{2.1}$$

In [2, Example 2.2], we see that every proper closed subspace of $X$ is a $k$-subspace, but $X$ itself is not a $k$-space. Also, [1, Example 1.1], shows that every proper open subspace of $X$ is a $k$-subspace, but $X$ itself is not a $k$-space. But we have the following theorem.

Theorem 2.6. *If every proper subspace of $X$ is a $k$-subspace, then $X$ is a $k$-space.*

*Proof.* Let $U$ be an open subset in $X$ and let $K$ be any compact subset in $X$. If $U \subseteq K$ or $K \subseteq U$, we are done. Assume that $U \backslash K \neq \Phi$ and $K \backslash U \neq \Phi$. Fix a point $x_0 \in U \backslash K$, then $U \cap K = (U \backslash \{x_0\}) \cap K$ is open in $K$.

Conversely, Assume that $U \cap K$ is open in $K$ for each compact set $K$ in $X$. If $U = X$ or $U = \Phi$, we are done. Assume that $U$ is a nonempty proper subset of $X$, so choose $y_0 \in X \backslash U$, and let $Y = X \backslash \{y_0\}$, so $U \cap K_Y$ is open in $K_Y$ for each compact set $K_Y$ in $Y$, so $U$ is open in $Y$ and hence in $X$. □

Corollary 2.7. *For Hausdorff spaces, if every proper closed (resp., open) subspace of $X$ is a $k$-subspace, then $X$ is a $k$-space.*

## 3. More topological properties satisfying (*)

Several topological properties will be studied in this section. Let us start with some types of compactness.

Theorem 3.1. *$\sigma$-compactness is a properly hereditary property.*

Theorem 3.2. *Sequential compactness is a properly hereditary property.*

*Proof.* Let $\{x_n\}$ be a sequence in $X$. If $|\{x_n : n \in \mathbb{N}\}| < \aleph_0$, then it has a constant subsequence. If the value $x_{n_0}$ is repeated infinitely many times, that is, the set $\{n : x_n = x_{n_0}\}$ is infinite, then this sequence has a constant subsequence. If $x_1$ is repeated only finitely many times, assume that $x_n \neq x_1$ for all $n > n_0$, hence the subsequence $x_{n_j} = x_{n_0+j}$ has a convergent subsequence in $X \backslash \{x_1\}$ and hence in $X$. □

Since the following topological property (hemicompact or denumerable at infinity) is not familiar, we give its definition.

*Definition 3.3.* A Hausdorff space $X$ is said to be hemicompact or denumerable at infinity if and only if there is a sequence $K_1, K_2, \ldots$ of compact subsets of $X$ such that if $K$ is any compact subset of $X$, then $K \subseteq K_n$ for some $n$.

Theorem 3.4. *If every proper subspace of $X$ is a hemicompact subspace, then $X$ is a hemicompact space.*

*Proof.* Let $x \neq y$ in $X$, so there exists an open set $V$ in $X$ such that $x \notin \overline{V}$. Since $Y = X \backslash V$ and $\overline{V}$ are hemicompact, there exist compact subsets $C_1, C_2, \ldots$ of $Y$, such that for every compact subset $K_Y$ of $Y$ there exists $C_{n_0}$ such that $K_Y \subseteq C_{n_0}$, and there exist compact subsets $D_1, D_2, \ldots$ of $\overline{V}$, such that for every compact subset $K_{\overline{V}}$ of $\overline{V}$ there exists $D_{m_0}$ such that $K_{\overline{V}} \subseteq D_{m_0}$. Let $W = \{0\} \cup \mathbb{N}$, and define a sequence of compact subsets $K_{m,n}$ of $X$ as follows: $K_{m,n} = C_m \cup D_n$, $K_{m,0} = C_m$, and $K_{0,n} = D_n$ for all $(m, n) \in W \times W \backslash \{(0,0)\}$.

Now, if $K$ is any compact subset of $X$, then $K \cap Y$ is compact in $Y$ and $K \cap \overline{V}$ is compact in $\overline{V}$. Thus, there exist $C_{m_0}$ and $D_{n_0}$ such that $K \cap Y \subset C_{m_0}$ and $K \cap \overline{V} \subset D_{n_0}$, hence $K = (K \cap Y) \cup (K \cap \overline{V}) \subset C_{m_0} \cup D_{n_0} = K_{m_0,n_0}$. □

It is easy to construct examples to show that hemicompactness is neither a properly open hereditary property nor a properly closed hereditary property. Recall that a space $X$ is said to be $\delta$-*normal* if whenever $A$ and $B$ are disjoint closed subsets of $X$, there exist two disjoint $G_\delta$-sets $H$ and $K$ such that $A \subseteq H$ and $B \subseteq K$.

Example 1.1 in [1], shows that if every closed proper subspace of $X$ is a $\delta$-normal $T_1$ subspace, then $X$ need not be a $\delta$-normal space. Moreover, [2, Example 2.2] shows that if every open proper subspace of $X$ is a $\delta$-normal $T_1$-subspace, then $X$ need not be a $\delta$-normal space. Fortunately, we have the following result.

Theorem 3.5. *If every proper subspace of $X$ is a $\delta$-normal $T_1$-subspace, then $X$ is a $\delta$-normal space.*

*Proof.* Let $A$ and $B$ be two nonempty disjoint closed subsets of $X$. The case $X = A \cup B$ is evident. Let $y_0 \in X\backslash(A \cup B)$, and let $Y = X\backslash\{y_0\}$, so there exist two disjoint $G_\delta$-sets $H_0$, $K_0$ in $Y$ such that $A \subseteq H_0$ and $B \subseteq K_0$, thus $H_0 = \bigcap_{i=1}^{\infty} H_i$ and $K_0 = \bigcap_{i=1}^{\infty} K_i$, where $H_i$ and $K_i$ are open in $Y$ for all $i \in \mathbb{N}$. Hence, there exist open sets $U_i$ and $V_i$ in $X$, such that $H_i = U_i \cap Y$ and $K_i = V_i \cap Y$ for all $i \in \mathbb{N}$. Thus, $U = \bigcap_{i=1}^{\infty} U_i$ and $V = \bigcap_{i=1}^{\infty} V_i$ are $G_\delta$-sets. If $U \cap V = \Phi$, the proof is completed, otherwise, $U \cap V = \{y_0\}$. Take $z_0 \in X\backslash(A \cup B \cup \{y_0\})$, and let $Z = X\backslash\{z_0\}$, so there exist two disjoint $G_\delta$-sets $H^0$, $K^0$ in $Z$ such that $A \subseteq H^0$ and $B \subseteq K^0$, thus $H^0 = \bigcap_{i=1}^{\infty} H^i$ and $K^0 = \bigcap_{i=1}^{\infty} K^i$, where $H^i$ and $K^i$ are open in $Z$ for all $i \in \mathbb{N}$. Hence, there exist open sets $U^i$ and $V^i$ in $X$, such that $H^i = U^i \cap Z$ and $K^i = V^i \cap Z$ for all $i \in \mathbb{N}$. Let $L_i = U_i \cap U^i$ and $M_i = V_i \cap V^i$, and define the required $G_\delta$-sets $L = \bigcap_{i=1}^{\infty} L_i$ and $M = \bigcap_{i=1}^{\infty} M_i$. □

It is easy to prove the following.

Theorem 3.6. *Pseudocompactness is a properly hereditary property.*

Recall that a space $X$ is *extremely disconnected* if and only if the closure of every open set is open. This topological property is not a properly (resp., closed, open) hereditary property, as we will see in the next example.

*Example 3.7.* Let $X = \{u,x,y\}$ and let $\tau = \{\{u\},\{y\},\{u,y\},\Phi,X\}$, hence every proper subspace of $X$ is an extremely disconnected subspace, but $X$ itself is not.

Theorem 3.8. *The topological property "extremely disconnected" is a properly hereditary property in the class of $T_1$-spaces.*

*Remark 3.9.* If every proper open subspace of $X$ is an extremely disconnected $T_1$-subspace, then $X$ is not necessarily an extremely disconnected space.

## 4. Open problems

In this section, we will study a special kind of topological properties, namely, *having a dispersion point*. For this, let us state its definition. Recall that a connected space $X$ is said to have a *dispersion point* $p$ if and only if $X\backslash\{p\}$ is totally disconnected. Thus, no space $X$ of cardinality greater than 2 has the property that every subspace of $X$ has a dispersion point because, if $x, y \in X\backslash\{p\}$, where $p$ is the dispersion point of $X$, then the subspace $\{x, y\}$ has the discrete topology, and hence has no dispersion point. Thus, it is natural to ask the following question.

*Question 4.1.* Is there a connected space for which every proper open (closed) subspace of $X$ has a dispersion point?

Indeed, take [2, Example 2.2] under consideration for the case of proper open subspaces, and take [1, Example 1.1] for the case of proper closed subspaces. Thus, one may ask the following.

*Question 4.2.* Is the property of having a dispersion point a properly open (closed) hereditary property?

The next example is a modification of [1, Example 1.1], which shows that the answer of Question 4.2 is negative in general.

*Example 4.3.* Let $X = \mathbb{Z} \cup \{\pi, \sqrt{2}\}$, and topologize $X$ as

$$\tau = \{A \cup \{\pi\} : A \subseteq \mathbb{Z}\} \cup \{X, \Phi\}. \tag{4.1}$$

So, $\pi$ is a dispersion point of every nonempty proper open subspace of $X$, and $\sqrt{2}$ is a dispersion point of every nonempty proper closed subspace of $X$. But $X$ itself has no dispersion points.

Note that the space in Example 4.3 is not a $T_1$-space, but it is easy to see that there is no $T_1$-space for which every proper closed (open) subspace has a dispersion point. A topological space $X$ is called $T_{1/2}$ if and only if every singleton is open or closed. Also, a topological space $X$ is called $T_{1/4}$ if and only if for every finite subset $F$ of $X$ and for every $y \in X \backslash F$, there exists a set $A$ containing $F$ and not containing $y$ which is either open or closed in $X$. Hence we can restate the previous questions as follows.

*Question 4.4.* Is there a $T_{1/4}$ ($T_{1/2}$)-space for which every proper open (closed) subspace of $X$ has a dispersion point? If so, does $X$ have a dispersion point?

Questions 1.1 and 4.4 are still open.

For the following two questions, the reader may consult [3].

*Question 4.5.* Is maximal compactness (minimal Hausdorffness, realcompactness, absolutely countably compactness) a properly (resp., closed, open, $F_\sigma$, $G_\delta$, etc.) hereditary property?

*Question 4.6.* Is $H$-closedness a properly (resp., closed, open, $F_\sigma$, $G_\delta$, etc.) hereditary property?

## References

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