COMPLETIONS OF NON-T_2 FILTER SPACES

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The well-known completions of T_2 Cauchy spaces and T_2 filter spaces are extended to the completions of non-T_2 filter spaces, and a completion functor on the category of all filter spaces is described.

1. Introduction

The categorical topologists Bentley et al. [1] have shown that the category FIL of filter spaces is isomorphic to the category of filter merotopic spaces which were introduced by Katětov [3]. The category CHY of Cauchy spaces is also known to be a bireflective, finally dense subcategory of FIL [7]. So the category FIL is an important category which deserves special discussion. A completion theory for filter spaces was introduced in [4], where a completion functor was defined on the subcategory T_2FIL of T_2 filter spaces. This completion theory was later applied to completion of filter semigroups [9]. Several other types of completions and their properties were also studied by Minkler et al. [5] and Császár [2]. In this paper, a completion theory is developed for filter spaces without the T_2 restriction on the spaces. Also, a completion functor is defined on a subcategory of FIL, which is constructed by taking all the filter spaces as objects and morphisms as certain special type of continuous maps which we call *s-maps*.

2. Preliminaries

For basic definitions and terminologies related to filters, the reader is referred to [11], though a few of the definitions will be mentioned here. Let X be a set and let $\mathbf{F}(X)$ be the set of all filters on X. If $\mathcal{F},\mathcal{G} \in \mathbf{F}(X)$ and $F \cap G \neq \phi$ for all $F \in \mathcal{F}$, $G \in \mathcal{G}$, then $\mathcal{F} \vee \mathcal{G}$ denotes the filter generated by $\{F \cap G : F \in \mathcal{F} \text{ and } G \in \mathcal{G}\}$. If there exist $F \in \mathcal{F}$ and $G \in \mathcal{G}$ such that $F \cap G = \phi$, we say that $\mathcal{F} \vee \mathcal{G}$ *fails to exist*. For each $x \in X$, we denote by $\dot{x}$ the ultrafilter generated by $\{x\}$. If $\zeta \subset \mathbf{F}(X)$ satisfies the conditions

(c_1) $\dot{x} \in \zeta$, for all $x \in X$,

(c_2) $\mathcal{F} \in \zeta$, $\mathcal{G} \geq \mathcal{F}$ imply that $\mathcal{G} \in \zeta$, then the pair (X,ζ) is called a *filter space*.

The two filters $\mathcal{F},\mathcal{G} \in \mathbf{F}(X)$ are said to be *ζ-linked* if there exist a finite number of filters $\mathcal{H}_1,\mathcal{H}_2,\ldots,\mathcal{H}_n \in \zeta$ such that $\mathcal{F} \vee \mathcal{H}_1$, $\mathcal{H}_1 \vee \mathcal{H}_2,\ldots,\mathcal{H}_{n-1} \vee \mathcal{H}_n$, $\mathcal{H}_n \vee \mathcal{G}$ all exist. In

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particular, if $\mathcal{F},\mathcal{G} \in \zeta$, we write $\mathcal{F} \sim_\zeta \mathcal{G}$ if and only if $\mathcal{F}$, $\mathcal{G}$ are ζ-linked. Note that "$\sim_\zeta$" is an equivalence relation on ζ. For $\mathcal{F} \in \zeta$, let $[\mathcal{F}]_\zeta$ denote the equivalence class containing $\mathcal{F}$. There is a *preconvergence* structure p_ζ associated with ζ in a natural way: $\mathcal{F} \xrightarrow{p_\zeta} x$ if and only if $\mathcal{F} \sim_\zeta \dot{x}$. Also, there is a convergence structure q_ζ associated with ζ defined by $\mathcal{F} \xrightarrow{q_\zeta} x$ if and only if $\mathcal{F} \cap \dot{x} \in \zeta$. Note that for any filter space (X,ζ), $p_\zeta \leq q_\zeta$ (see [4]).

A filter space (X,ζ) is a *c-filter space* if, in addition,

(c_3) $\mathcal{F} \in \zeta$, $\mathcal{F} \sim_\zeta \dot{x}$ imply that $\mathcal{F} \cap \dot{x} \in \zeta$, and it is called a *Cauchy space* if

(c_4) whenever $\mathcal{F},\mathcal{G} \in \zeta$ and $\mathcal{F} \sim_\zeta \mathcal{G}$, then $\mathcal{F} \cap \mathcal{G} \in \zeta$.

Lemma 2.1. *For a filter space (X,ζ), $p_\zeta = q_\zeta$ if and only if (X,ζ) is a c-filter space.*

A convergence structure q on a set X is said to be *compatible* (resp., *c-filter compatible, Cauchy compatible*) if there exists a filter structure (resp., c-filter structure, Cauchy structure) ζ on X such that $q = q_\zeta$. As shown above, given a filter space (X,ζ), we can always associate a convergence q_ζ. However, every convergence structure on X is not c-filter compatible.

Example 2.2. Let $X = \{a,b,c\}$ and let q be the convergence structure on X defined by $\dot{a} \to^q a$, $\dot{b} \to^q b$, $\dot{t} \to^q t$, $\dot{a} \cap \dot{b} \to^q b$, $\dot{b} \cap \dot{t} \to^q t$, and all other filters fail to converge. If possible, let ζ be the filter structure on X such that $q = q_\zeta$. Let $\mathcal{F} = \dot{b} \cap \dot{t}$, so $\mathcal{F} \xrightarrow{q_\zeta} t$. However, $\mathcal{F} \cap \dot{b} = \dot{b} \cap \dot{t} \in \zeta \Rightarrow \mathcal{F} \xrightarrow{q_\zeta} b$, but $\mathcal{F} \not\to^q b$. So q is not compatible.

The following lemma states the necessary and sufficient conditions for such compatibilities of a convergence structure on X.

Lemma 2.3. *A convergence structure q on X is*

(a) *compatible if and only if $\mathcal{F}$ is q-convergent and $\dot{x} \geq \mathcal{F} \Rightarrow \mathcal{F} \to^q x$;*

(b) *c-filter compatible if and only if either $q(x) = q(y)$ or $q(x) \cap q(y) = \phi$;*

(c) *Cauchy compatible if and only if $\mathcal{F},\mathcal{G} \in q(x) \Rightarrow \mathcal{F} \cap \mathcal{G} \in q(x)$ and for all $x,y \in X$, $q(x) = q(y)$ or $q(x) \cap q(y) = \phi$.*

Proof. The proof of (b) is similar to [4, Proposition 1.3] and the proof of (c) is well known. So we will prove only (a). Let $q = q_\zeta$, where ζ is a filter structure on X. Let $\mathcal{F}$ be q-convergent, $\mathcal{F} \to^q y$ (say), and $\dot{x} \geq \mathcal{F}$. So $\mathcal{F} \to^{q_\zeta} y$, that is, $\mathcal{F} \cap \dot{y} \in \zeta$. So $\mathcal{F} = \mathcal{F} \cap \dot{x} \in \zeta \Rightarrow \mathcal{F} \to^q x$. Conversely, let q satisfy the given condition and let ζ_q be the set of all q-convergent filters. We show that $q = q_{\zeta_q}$. If $\mathcal{F} \to^q x$, then $\mathcal{F} \cap \dot{x} \in \zeta_q \Rightarrow \mathcal{F} \to^{q_{\zeta_q}} x$. Also, if $\mathcal{F} \to^{q_{\zeta_q}} x$, then $\mathcal{F} \cap \dot{x} \in \zeta_q \Rightarrow \mathcal{F} \cap \dot{x}$ is q-convergent and since $\dot{x} \geq \mathcal{F} \cap \dot{x}$, $\mathcal{F} \cap \dot{x} \to^q x$. So $\mathcal{F} \to^q x$. □

Compatibilities of a preconvergence structure on a filter space is defined in a similar way. The conditions for compatibility, c-filter compatibility, and Cauchy compatibility of a preconvergence structure p were established in [4].

Note that if q is a convergence structure on X and there is a filter structure ζ on X such that $q = p_\zeta$, then (X,ζ) is a c-filter space and $q = q_\zeta$. In particular, the following lemma holds when q is a pretopology.

Lemma 2.4. *A pretopology σ on X is compatible and $\sigma = p_\zeta$ for some filter structure ζ on X if and only if ζ is a c-filter structure.*

However, if the pretopology $\sigma = q_\zeta$ for some filter structure ζ on X, then as illustrated in the following example, the above lemma may not hold in general.

Example 2.5. Let $X = \mathbb{R}$, the set of real numbers, $\mathcal{F} = \{(0,1/n) \mid n \in N\}$, and $\mathcal{L} = \{\text{all complements of countable sets}\}$. It is clear that $\zeta = \{\dot{x} \mid x \in X\} \cap \{\Gamma \mid \Gamma \geq \mathcal{F} \cap \dot{0}\} \cap \{\Psi \mid \Psi \geq \mathcal{L} \cap \dot{1}\}$ is a filter structure on $\mathbb{R}$. For each $x \in X$, let $V_\sigma(x) = \cap\{\mathcal{G} \in \mathbf{F}(X) \mid \mathcal{G} \to^\sigma x\}$ denote the neighborhood filter of x. We define σ as follows: $V_\sigma(x) = \dot{x}$, for all $x \neq 0,1$ and $V_\sigma(0) = \mathcal{F} \cap \dot{0}$, $V_\sigma(1) = \mathcal{L} \cap \dot{1}$. It is clear that σ is a pretopology and $\sigma = q_\zeta$. However, ζ is not a c-filter structure because $\dot{0} \to^{p_\zeta} 1$ and $\dot{1} \to^{p_\zeta} 0$, but $\dot{0} \cap \dot{1} \not\to^{p_\zeta} 1$, which implies $\dot{0} \cap \dot{1} \notin \zeta$. So $p_\zeta \neq q_\zeta$ and hence, by Lemma 2.1, ζ is not a c-filter structure on X.

A filter space (X,ζ) is said to be

(i) T_2 (resp., w-T_2) if and only if $\mathcal{F} \sim \dot{x}$, $\mathcal{F} \sim \dot{y} \Rightarrow x = y$ (resp., $\mathcal{F} \cap \dot{x}, \mathcal{F} \cap \dot{y} \in \zeta \Rightarrow x = y$),

(ii) *complete* (resp., *w-complete*) if and only if for each $\mathcal{F} \in \zeta$, p_ζ converges (resp., for each $\mathcal{F} \in \zeta$, q_ζ converges),

(iii) *regular* (resp., w-regular) if and only if $\text{cl}_{p_\zeta}\mathcal{F} \in \zeta$ whenever $\mathcal{F} \in \zeta$ (resp., $\text{cl}_{q_\zeta}\mathcal{F} \in \zeta$ whenever $\mathcal{F} \in \zeta$),

(iv) *totally bounded* if each ultrafilter on X is in ζ.

Note that if a filter space is T_2 (resp., regular, complete), then it is w-T_2 (resp., w-regular, w-complete). The proof of the following lemma is immediate from the definitions.

Lemma 2.6. *The following are true for any filter space* (X,ζ)*:*

(I) (X,ζ) *is* T_2 *(resp.,* w-T_2*) if and only if* (X,p_ζ) *(resp.,* (X,q_ζ)*) is* T_2*;*

(II) *if* (X,ζ) *is regular (resp., w-regular), then* (X,p_ζ) *(resp.,* (X,q_ζ)*) is regular;*

(III) *if* (X,p_ζ) *(resp.,* (X,q_ζ)*) is regular and* (X,ζ) *is complete (resp., w-complete), then* (X,ζ) *is regular (resp., w-regular).*

Lemma 2.7. *Any regular filter space is a c-filter space.*

Lemma 2.8. *A filter space* (X,ζ) *is totally bounded and complete if and only if* (X,p_ζ) *is compact. If a filter space* (X,ζ) *is totally bounded and w-complete, then* (X,q_ζ) *is compact. Also* (X,q_ζ) *is compact implies* (X,ζ) *is totally bounded. However, if* (X,q_ζ) *is compact, then* (X,ζ) *is not necessarily w-complete.*

A map $f : (X,\zeta) \to (Y,\kappa)$ between two filter spaces is called *continuous* if $f(\mathcal{F}) \in \kappa$ whenever $\mathcal{F} \in \zeta$. We denote by FIL the category of all filter spaces and continuous maps as morphisms. Let CFIL and CHY be the full subcategories of FIL whose objects are c-filter spaces and Cauchy spaces, respectively. In [4], a completion of objects in T_2 FIL and a completion functor on T_2 CFIL and its bireflective subcategory C_3 FIL were constructed. The completion functor and the completion subcategory constructed in [4] deal with T_2 filter spaces. The underlying reason for this is the existence of unique limits for convergent filters which are also preserved by the continuous map. In this paper, we partially overcome that limitation by using a special type of continuous map called *s-map* which will be introduced later.

The map $f : (X,\zeta) \to (Y,\kappa)$ between two filter spaces is a *homeomorphism* if f is bijective and both f and f^{-1} are continuous maps. In this case, (X,ζ) and (Y,κ) are called *homeomorphic filter spaces*. Note that the underlying preconvergence spaces (X,p_ζ) and (Y,p_κ) are also homeomorphic. A map $f:(X,\zeta)\to(Y,\kappa)$ is an *embedding* of (X,ζ) into (Y,κ) if $f:(X,\zeta)\to(f(X),\kappa_{f(X)})$ is a homeomorphism, where $\kappa_{f(X)}$ is a subspace structure on $f(X)$.

In this paper, we construct completions for a filter space without the T_2 restriction on the space. Results obtained in this paper also generalise the completion theory developed for non-T_2 Cauchy spaces obtained in [8] where the author introduced the morphisms on the category CHY as s-map. The corresponding extension of the notion of s-maps to filter spaces can be used to form a completion subcategory of FIL.

Definition 2.9. A continuous map between two filter spaces $f:(X,\zeta)\to(Y,\kappa)$ is said to be an *s-map* if it satisfies the following condition: $\mathcal{F}\in\zeta$, p_ζ converges to at most one point in X implies $f(\mathcal{F})\in\kappa$, p_κ converges to at most one point in Y.

There are several examples of s-maps. Any continuous map is an s-map if the codomain of the map is a T_2 filter space. The identity map on a filter space and the embedding map φ for a stable completion is also an s-map. Note that it follows from the definition of s-map that composition of two s-maps is an s-map. The class of all filter spaces with the s-maps as morphisms forms a category which we denote by FIL$'$. Since every continuous map is not necessarily an s-map, FIL$'$ is not a full subcategory of FIL.

3. Completion and extension theorems for filter spaces

Throughout this section, (X,ζ) denotes a filter space (not necessarily T_2). A *completion* of a filter space (X,ζ) is a pair $((Y,\kappa),\phi)$ consisting of a complete filter space (Y,κ) and an embedding $\phi:(X,\zeta)\to(Y,\kappa)$ such that $\mathrm{cl}_{p_\kappa}(\phi(X))=Y$. The completion $((Y,\kappa),\phi)$ is called a *weak completion* if (Y,κ) is w-complete and $\mathrm{cl}_{q_\kappa}(\phi(X))=Y$. A completion $((Y,\kappa),\varphi)$ of a filter space (X,ζ) is said to be a $\mathcal{P}$*-completion* if (Y,κ) has the property $\mathcal{P}$ whenever (X,ζ) has the property $\mathcal{P}$. For instance, a completion which is T_2 will be called a T_2 completion. The results related to completions in standard form and extension theorems are established here.

When (X,ζ) is a T_2 filter space, completion and extension theorems were established in [4] and later a few other classes of completions were constructed in [2, 5]. In this section, we will construct non-T_2 completion and non-T_2 weak completion of a filter space and establish some extension theorems.

We construct a completion $((\tilde{X},\zeta),j)$ for a filter space (X,ζ). Let

$$\eta_\zeta=\{[\mathcal{F}]\mid\mathcal{F}\in\zeta,\ \mathcal{F}\not\to_\zeta\dot{x},\ \forall x\in X\}.\tag{3.1}$$

For an arbitrary filter space (X,ζ), we define the following:

(i) $\tilde{X}=\eta_\zeta\cup X$,

(ii) $j:X\to\tilde{X}$ is the inclusion map,

(iii) $\tilde{\zeta}=\{\mathcal{A}\in\mathbf{F}(\tilde{X})\mid\exists\mathcal{F}\in\zeta$, non-$p_\zeta$ convergent such that $\mathcal{A}\geq j(\mathcal{F})\cap[\dot{\mathcal{F}}]\}\cup j(\zeta)$.

Definition 3.1. A completion $((Y,\kappa),\varphi)$ of a filter space (X,ζ) is said to be in *standard form* if $Y=\tilde{X}$, $\varphi=j$ and for each non-p_ζ convergent filter $\mathcal{F}\in\zeta$, $j(\mathcal{F})\xrightarrow{q_\kappa}[\mathcal{F}]$.

A similar notion of completion was introduced in [10] for T_2 Cauchy spaces. We extend that notion to completion of non-T_2 filter spaces. In [4], we defined an equivalence relation between T_2 completions. For a T_2 filter space (Z,δ), a completion $\varkappa_1 = ((Y_1,\kappa_1),\varphi_1)$ is said to be *finer than* a completion $\varkappa_2 = ((Y_2,\kappa_2),\varphi_2)$, written $\varkappa_1 \geq \varkappa_2$ if there exists a continuous map $h:(Y_1,\kappa_1) \to (Y_2,\kappa_2)$ such that $h \circ \phi_1 = \phi_2$. The two completions $\varkappa_1$ and $\varkappa_2$ are *equivalent* if $\varkappa_1 \geq \varkappa_2$ and $\varkappa_2 \geq \varkappa_1$. Observe that in case of equivalence the two completion spaces are isomorphic in the category $T_2\,$FIL of T_2 filter spaces.

We can define an equivalence relation for non-T_2 completions of filter spaces in the same way, but it is not a categorical equivalence in the sense of Preuss [6], since in this case the map h is not necessarily a unique homeomorphism. This motivates the introduction of a weakly stable completion.

Definition 3.2. A completion $((Y,\kappa),\varphi)$ of a filter space (X,ζ) is *weakly stable* if whenever $z \in Y \backslash \phi(X)$ and $\phi(\mathcal{F}) \xrightarrow{p_\kappa} z$, for some $\mathcal{F} \in \zeta$, it follows that z is the unique limit of $\phi(\mathcal{F})$ in Y.

Remark 3.3. (I) Stable completion of a T_2 filter space was defined in [4]. Note that any stable completion of a T_2 filter space is always weakly stable. Also, if (X,ζ) is a T_2 c-filter space, then a weakly stable completion is stable. If (X,ζ) is a Cauchy space, then every weakly stable completion is stable.

(II) A weakly stable completion $\varkappa_1 = ((Y_1,\kappa_1),\varphi_1)$ is said to be *finer than* another weakly stable completion $\varkappa_2 = ((Y_2,\kappa_2),\varphi_2)$ if there is a continuous map $h:(Y_1,\kappa_1) \to (Y_2,\kappa_2)$ such that $h \circ \phi_1 = \phi_2$. Note that the map h is a unique homeomorphism when the two weakly stable completions are equivalent.

In the following result, we show that the Wyler completion of a non-T_2 filter space has a property similar to the universal property of the T_2 completions.

Theorem 3.4. *$((\tilde{X},\tilde{\zeta}),j)$ is the finest weakly stable completion of the filter space (X,ζ) in standard form.*

Proof. It is clear that $\tilde{\zeta}$ is a filter structure on $\tilde{X}$ and the inclusion map j is an embedding. To show that $(\tilde{X},\tilde{\zeta})$ is complete, let $\mathcal{A} \in \tilde{\zeta}$. If $\mathcal{A} \geq j(\mathcal{F})$ for some $\mathcal{F} \in \zeta$, then $\mathcal{F} \xrightarrow{p_\zeta} x$ implies $\mathcal{A} \xrightarrow{p_{\tilde{\zeta}}} x$, and $\mathcal{F}$ is nonconvergent implies $j(\mathcal{F}) \cap [\dot{\mathcal{F}}] \in \tilde{\zeta}$ which shows that $\mathcal{A} \xrightarrow{p_{\tilde{c}}} [\mathcal{F}]$. If $\mathcal{A} \geq j(\mathcal{F}) \cap [\dot{\mathcal{F}}]$ for some nonconvergent $\mathcal{F} \in \zeta$, then as we have already shown $\mathcal{A} \xrightarrow{p_{\tilde{\zeta}}} [\mathcal{F}]$. So $(\tilde{X},\tilde{\zeta})$ is complete. Also, if $[\mathcal{F}] \in \tilde{X} \setminus j(X)$, then $\mathcal{F}$ is nonconvergent and so $j(\mathcal{F}) \xrightarrow{p_{\tilde{\zeta}}} [\mathcal{F}]$ which implies that $[\mathcal{F}] \in \mathrm{cl}_{p_{\tilde{\zeta}}} j(X)$. This proves that $((\tilde{X},\tilde{\zeta}),j)$ is a completion of (X,ζ). Since for each non-p_ζ convergent filter $\mathcal{F} \in \zeta$, $j(\mathcal{F}) \cap [\dot{\mathcal{F}}] \in \zeta$, we have $j(\mathcal{F}) \sim_{p_{\tilde{\zeta}}} [\dot{\mathcal{F}}]$. So it follows that $j(\mathcal{F}) \xrightarrow{p_{\tilde{\zeta}}} [\mathcal{F}]$ and this completion is in standard form.

Next we show that it is a weakly stable completion. Let $[\mathcal{F}] \in \tilde{X} \backslash j(X)$, then $\mathcal{F} \in \zeta$ is non-p_ζ convergent and $j(\mathcal{F}) \xrightarrow{p_{\tilde{\zeta}}} [\mathcal{F}]$. If there exists $[\mathcal{G}] \in \tilde{X} \backslash j(X)$ such that $j(\mathcal{F}) \xrightarrow{p_{\tilde{\zeta}}} [\mathcal{G}]$, then $j(\mathcal{F}) \sim_{\tilde{\zeta}} [\dot{\mathcal{G}}]$. So there exist a finite number of filters $\Lambda_1, \Lambda_2, \ldots, \Lambda_n \in \zeta$ such that $j(\mathcal{F}) \vee \Lambda_1$, $\Lambda_1 \vee \Lambda_2, \ldots, \Lambda_n \vee [\dot{\mathcal{G}}]$ exist. If $\Lambda_1 \geq j(\mathcal{H})$ for some p_ζ convergent filter

$\mathcal{H} \in \zeta$, then $j(\mathcal{F}) \vee \Lambda_1$ exists. This implies that $j(\mathcal{F}) \vee j(\mathcal{H})$ exists from which it follows that $\mathcal{F} \vee \mathcal{H}$ exists. This leads to a contradiction since $\mathcal{F}$ is non-p_ζ convergent. So $\Lambda_1 \geq j(\mathcal{L}_1) \cap [\dot{\mathcal{L}}_1]$, for some non-$p_c$ convergent filter $\mathcal{L}_1 \in \zeta$. So $j(\mathcal{F}) \vee \Lambda_1$ implies that $j(\mathcal{F}) \vee (j(\mathcal{L}_1) \cap [\dot{\mathcal{L}}_1])$ exists from which it follows that either $j(\mathcal{F}) \vee j(\mathcal{L}_1)$ or $j(\mathcal{F}) \vee [\dot{\mathcal{L}}_1]$ exists. Since the latter is an impossibility, $j(\mathcal{F}) \vee j(\mathcal{L}_1)$ exists. This implies $\mathcal{F} \vee \mathcal{L}_1$ exists, that is, $\mathcal{F} \sim_\zeta \mathcal{L}_1$ or $[\mathcal{F}] = [\mathcal{L}_1]$. Since $\Lambda_1 \vee \Lambda_2$ exists and $\Lambda_1 \geq j(\mathcal{L}_1) \cap [\dot{\mathcal{L}}_1]$, it follows that $(j(\mathcal{L}_1) \cap [\dot{\mathcal{L}}_1]) \vee \Lambda_2$ exists, that is, $(j(\mathcal{L}_1) \cap [\dot{\mathcal{F}}]) \vee \Lambda_2$ exists. Following the same type of argument as above, we can show that there is no p_ζ convergent filter $\mathcal{H} \in \zeta$ such that $\Lambda_2 \geq j(\mathcal{H})$. So $\Lambda_2 \geq j(\mathcal{L}_2) \cap [\dot{\mathcal{L}}_2]$, for some non-$p_\zeta$ convergent filter $\mathcal{L}_2 \in \zeta$. So $(j(\mathcal{L}_1) \cap [\dot{\mathcal{F}}]) \vee (j(\mathcal{L}_2) \cap [\dot{\mathcal{L}}_2])$ exists. This implies at least one of $j(\mathcal{L}_1) \vee j(\mathcal{L}_2)$, $j(\mathcal{L}_1) \vee [\dot{\mathcal{L}}_2]$, $[\dot{\mathcal{F}}] \vee j(\mathcal{L}_2)$, $[\dot{\mathcal{F}}] \vee [\dot{\mathcal{L}}_2]$ exists. Since $[\dot{\mathcal{F}}], [\dot{\mathcal{L}}_2] \notin j(X)$, neither $j(\mathcal{L}_1) \vee [\dot{\mathcal{L}}_2]$ nor $[\dot{\mathcal{F}}] \vee j(\mathcal{L}_2)$ exists. However, $[\dot{\mathcal{F}}] \vee [\dot{\mathcal{L}}_2]$ implies that $[\dot{\mathcal{F}}] = [\dot{\mathcal{L}}_2]$ and $j(\mathcal{L}_1) \vee j(\mathcal{L}_2)$ implies that $\mathcal{L}_1 \vee \mathcal{L}_2$ exists, so that $[\mathcal{L}_2] = [\mathcal{L}_1] = [\mathcal{F}]$. Repeating a similar argument, we can show that each $\Lambda_i \geq j(\mathcal{L}_i) \cap [\dot{\mathcal{L}}_i]$, for some non-$p_\zeta$ convergent filter $\mathcal{L}_i \in \zeta$ and $[\mathcal{L}_i] = [\mathcal{F}]$, for $1 \leq i \leq n$. Also, $\Lambda_n \geq j(\mathcal{L}_n) \cap [\mathcal{L}_n]$, for some non-$p_\zeta$ convergent filter $\mathcal{L}_n \in \zeta$ and $(j(\mathcal{L}_n) \cap [\mathcal{L}_n]) \vee [\dot{\mathcal{G}}]$ implies that $[\mathcal{L}_n] = [G]$. Hence, $[\mathcal{F}] = [\mathcal{G}]$. This proves that $((\tilde{X}, \tilde{\zeta}), j)$ is a weakly stable completion.

Let $((\tilde{X}, \beta), j)$ be another weakly stable completion of (X, ζ) in standard form and let $\Lambda \in \tilde{\zeta}$. If $\Lambda \geq j(\mathcal{F})$, where $\mathcal{F}$ is p_ζ convergent, then $\Lambda \in \beta$, since $j(\mathcal{F}) \in \beta$. Next, let $\Lambda \geq j(\mathcal{F}) \cap [\dot{\mathcal{F}}]$, where $\mathcal{F}$ is non-p_ζ convergent. Then $j(\mathcal{F}) \to^{q_\beta} [\mathcal{F}]$ since $((\tilde{X}, \beta), j)$ is in standard form. So $j(\mathcal{F}) \cap [\dot{\mathcal{F}}] \in \beta$ which implies $\Lambda \in \beta$. So $((\tilde{X}, \zeta), j)$ is the finest weakly stable completion in standard form. □

We will refer to the completion $((\tilde{X}, \tilde{\zeta}), j)$ as the *Wyler completion* of (X, ζ). Obviously, the mapping j in $((\tilde{X}, \tilde{\zeta}), j)$ is an s-map. Note that if (X, ζ) is a c-filter space, then $(\tilde{X}, \tilde{\zeta})$ is a c-filter space. Also, if (X, ζ) is T_2, then $((\tilde{X}, \tilde{\zeta}), j)$ is a T_2 completion of (X, ζ). If we identify each $x \in X$ with the equivalence class $[\dot{x}]$ of all filters which are p_ζ convergent to x, then the Wyler completion coincides with $((X^*, \zeta^*), j)$ in [4]. We will refer to the latter completion as the T_2 Wyler completion of (X, ζ).

Proposition 3.5. *Any weakly stable completion $((Y, \kappa), \varphi)$ of a filter space (X, ζ) is equivalent to one in standard form.*

Proof. Define $h : Y \to \tilde{X}$ by

$$h(y) = \begin{cases} x, & \text{if } y = \phi(x), \\ [\mathcal{F}], & \text{if } y \in Y \setminus \phi(X),\ \varphi(\mathcal{F}) \xrightarrow{p_\kappa} y. \end{cases} \tag{3.2}$$

Note that h is well defined and bijective, because $((Y, \kappa), \varphi)$ is a weakly stable completion. Let ζ_κ be the quotient filter structure on $\tilde{X}$. Since $j = h\varphi$, both j and j^{-1} are continuous maps and since (Y, κ) is complete and $(\tilde{X}, \zeta_\kappa)$ is the quotient space, the latter is also complete. If $[\mathcal{F}] \in \tilde{X} \setminus j(X)$, then $\varphi(\mathcal{F}) \xrightarrow{p_\kappa} y$, for some $y \in Y$. This implies that $j(\mathcal{F}) = h \circ \varphi(\mathcal{F}) \xrightarrow{p_{\tilde{\zeta}}} h(y) = [\mathcal{F}]$. Hence, $\mathrm{cl}_{p_{\tilde{\zeta}}} j(X) = \tilde{X}$. So $(\tilde{X}, \zeta_\kappa)$ is a completion of (X, ζ) and by the same argument one can also show that it is in standard form. This proves that $((\tilde{X}, \zeta_\kappa), j)$

is a completion of (X,ζ) in standard form. It remains to show that h is a homeomorphism. Since the category FIL is a topological category, h is an injective quotient map implies h is a monomorphism which is also an extremal epimorphism. Therefore, by [6, Proposition 0.2.7], h is an isomorphism. This completes the proof of Proposition 3.5 □

In view of Proposition 3.5, we may therefore assume without loss of generality that all weakly stable completions of a filter space (X,ζ) are in standard form. If (X,ζ) is T_2, then any T_2 completion is always stable. Hence, we have the following corollary.

Corollary 3.6. *Any T_2 completion of a filter space is equivalent to one in standard form.*

Given a T_2 filter space (X,ζ), let $X^* = \{[\mathcal{F}] \mid \mathcal{F} \in \zeta\}$, let $\zeta^* = \{\mathcal{A} \mid \mathcal{A} \geq j(\mathcal{F})$, $\mathcal{F} \in \zeta$ with $\mathcal{F}\, p_\zeta$ convergent, or $\mathcal{A} \geq j(\mathcal{F}) \cap [\dot{\mathcal{F}}], \mathcal{F} \in \zeta$ with $\mathcal{F}$ non-p_ζ convergent$\}$, and let $j : X \to X^*$ be defined by $j(x) = [\dot{x}]$ for all $x \in X$.

Proposition 3.7. *A w-T_2 filter space (X,ζ) has a w-T_2 completion if and only if (X,ζ) is a T_2 c-filter space.*

Proof. $(\Rightarrow)$ Let $((Y,\kappa),\varphi)$ be a w-T_2 completion of (X,ζ). Let $\mathcal{F} \in \zeta$ and $\mathcal{F} \sim_\zeta \dot{x}$. By Proposition 3.5, it follows that $\varphi(\mathcal{F}) \xrightarrow{q_k} \varphi(\dot{x})$, that is, $\varphi(\mathcal{F}) \cap \varphi(\dot{x}) \in \kappa$. Since φ is an embedding, $\mathcal{F} \cap \dot{x} \in \zeta$ which shows that (X,ζ) is a c-filter space.

$(\Leftarrow)$ Let (X,ζ) be a T_2 c-filter space. Let X^*, ζ^*, and j be as above. Note that since (X,ζ) is c-filter space, $p_\zeta = q_\zeta$ and (X^*,ζ^*) is also a c-filter space. So $((X^*,\zeta^*), j)$ is a w-completion of (X,ζ). Hence, it remains to show that (X^*,ζ^*) is T_2. Let $\dot{y}_1 \cap \dot{y}_2 \in \zeta^*$. If $y_1, y_2 \in X$, then $y_1 = y_2$ since (X,ζ) is T_2. If at least one of y_1 or y_2 is in $X^* \setminus X$, then by the definition of ζ^* it follows that $\dot{y}_1 \cap \dot{y}_2 \in \zeta^*$ only when $y_1 = y_2$. This proves Proposition 3.7. □

Following a similar argument as in [8, Proposition 3.13], we can show that the Wyler completion is the finest completion in CFIL$'$. But it is not the finest completion in FIL. In fact, in [4] it was shown that there is no such finest completion whenever $\tilde{X} \setminus j(X)$ is infinite. However, the following proposition states that we can uniquely extend any s-map on a non-T_2 c-filter space to its Wyler completion.

Proposition 3.8. *If $f : (X,\zeta) \to (Y,\beta)$ is an s-map and (Y,β) is a c-filter space, then there is a unique extension $f^* : (\tilde{X},\tilde{\zeta}) \to (\tilde{Y},\tilde{\beta})$ which is also an s-map and $f^* \circ j_X = j_Y \circ f$, where j_X and j_Y are the corresponding embedding maps.*

Proof. Define $f^* : (\tilde{X},\tilde{\zeta}) \to (\tilde{Y},\tilde{\beta})$ as follows:

$$f^*(x) = f(x),$$
$$f^*([\mathcal{F}]) = \begin{cases} [f(\mathcal{F})], & \text{if } f(\mathcal{F}) \text{ is nonconvergent,} \\ y, & \text{if } f(\mathcal{F}) \text{ converges to } y. \end{cases} \tag{3.3}$$

The mapping f^* so defined is a well-defined map because if $[\mathcal{F}] = [\mathcal{G}]$, then $f(\mathcal{F}) \sim_\beta f(\mathcal{G})$, so either both $f(\mathcal{F})$ and $f(\mathcal{G})$ are nonconvergent or convergent. If both are nonconvergent, then obviously $f^*([\mathcal{F}]) = f^*([\mathcal{G}])$. If $f(\mathcal{F}) \xrightarrow{q_\beta} y_1$ and $f(\mathcal{G}) \xrightarrow{q_\beta} y_2$, then

$f(\mathcal{F}) \sim_\beta \dot{y}_1, \dot{y}_2$. This is a contradiction, since $\mathcal{F}$ is non-p_ζ-convergent and f is an s-map. So in either case $f^*([\mathcal{F}]) = f^*([\mathcal{G}])$. Also, it can be easily verified that $f^* \circ j_X = j_Y \circ f$.

Next we show that f^* is an s-map. Let $\mathcal{A} \in \zeta$. If $\mathcal{A} \geq j_X(\mathcal{F})$, then $f^*(\mathcal{A}) \geq f^* \circ j_X(\mathcal{F}) = j_Y \circ f(\mathcal{F}) \in \beta$. If $\mathcal{A} \geq j_X(\mathcal{F}) \cap [\mathcal{F}]$, where $\mathcal{F}$ is non-p_ζ convergent, then $f^*(\mathcal{A}) \geq (j_Y \circ f(\mathcal{F})) \cap f^*([\mathcal{F}])$. If $f(\mathcal{F})$ is nonconvergent in Y, then $(j_Y \circ f(\mathcal{F})) \cap [f(\dot{\mathcal{F}})] \in \beta$. If $f(\mathcal{F})$ converges to $y \in Y$, then since (Y,β) is a c-filter space $f(\mathcal{F}) \cap \dot{y} \in \beta$, so it follows that $(j_Y \circ f(\mathcal{F})) \cap \dot{y} \in \beta$. Therefore, f^* is a continuous map. To show that it is an s-map, it suffices to show that if $\mathcal{A} \in \zeta$ converges to only one point, then $f^*(\mathcal{A})$ converges to only one point. If $\mathcal{A} \geq j_X(\mathcal{F})$, then $j_Y \circ f(\mathcal{F}) = f^* \circ j_X(\mathcal{F})$ converges to only one point, because f is an s-map. If $\mathcal{A} \geq j_X(\mathcal{F}) \cap [\dot{\mathcal{F}}]$, then $\mathcal{F}$ is non-p_ζ convergent which implies that $f(\mathcal{F})$ converges to at most one point. Therefore, $f^*(j_X(\mathcal{F}) \cap [\dot{\mathcal{F}}]) = (f^* \circ j_X(\mathcal{F})) \cap f^*([\dot{\mathcal{F}}]) = (j_Y \circ f(\mathcal{F})) \cap [f(\dot{\mathcal{F}})]$ or $(j_Y \circ f(\mathcal{F})) \cap \dot{y}$ according as $f(\mathcal{F})$ is nonconvergent or $f(\mathcal{F})$ converges to y. But in either case $f^*(\mathcal{A})$ converges to only one point in Y.

Finally, we show that f^* is a unique extension. Let $\overline{f} : (\tilde{X},\tilde{\zeta}) \to (\tilde{Y},\beta)$ be another s-map such that $\overline{f} \circ j_X(x) = j_Y \circ f(\mathcal{F})$. It is obvious that $\overline{f} \circ j_X(x) = f^* \circ j_X(x)$, for all $x \in X$. Let $[\mathcal{F}] \in X^* \setminus j_X(X)$. Since $\mathcal{F} \in \zeta$ is non-p_ζ convergent, it follows that $j_X(\mathcal{F}) \xrightarrow{p_{\zeta^*}} [\mathcal{F}]$. Since $f^*, \overline{f}$ are s-maps, it follows that $f^* \circ j_X(\mathcal{F}) = \overline{f} \circ j_X(\mathcal{F}) = j_Y \circ f\ (\mathcal{F})$ converges to $f^*([\mathcal{F}])$ and $\overline{f}([\mathcal{F}])$. But $\mathcal{F}$ is nonconvergent and f, j_Y are s-maps imply that $j_Y \circ f(\mathcal{F})$ can converge to at most one point in $\tilde{Y}$. Hence, $f^* = \overline{f}$. This proves Proposition 3.8. □

The unique mapping f^* in Proposition 3.8 is called the *s-extension* of f.

Remark 3.9. (I) If $f : (X,\zeta) \to (Y,\kappa)$ is an s-map, where (Y,κ) is a complete c-filter space, then there exists a unique s-extension $f^* : (\tilde{X},\tilde{\zeta}) \to (Y,\kappa)$ such that $f = f^* \circ j_X$. If (Y,κ) is a regular filter space, then (X^*,ζ^*) also has the same extension property (by Lemma 2.7). In either case, the s-extension f^* is defined by $f^*(x) = f(x)$, for each $x \in X$ and $f^*([\mathcal{F}]) = y$, where $f(\mathcal{F}) \xrightarrow{p_\kappa} y$.

(II) If (X,ζ) is a T_2 filter space, then its T_2 Wyler completion has the extension property. Recall that if the codomain of an s-map is a T_2 space, then the s-map is simply a continuous map. If $f : (X,\zeta) \to (Y,\kappa)$ is a continuous map, where (Y,κ) is a complete T_2 c-filter space [4] or a complete T_3 filter space, then there exists a unique extension $f^* : (\tilde{X},\tilde{\zeta}) \to (Y,\kappa)$.

Since the composition of s-maps is an s-map and the identity map is an s-map, the class of all c-filter spaces with s-maps as morphisms forms a subcategory of FIL. We denote this category by CFIL$'$. Let CFIL$'^*$ be the subcategory of CFIL$'$ consisting of the complete objects of CFIL$'$. Let W : CFIL$'$ → CFIL$'^*$ be defined for objects by $W(X,\zeta) = (\tilde{X},\tilde{\zeta})$ and for morphisms by $W(f) = f^*$. Then W is a covariant functor on CFIL$'$ and is called the *Wyler completion functor*.

Note that a morphism $f : (X,\zeta) \to (Y,\kappa)$ in the category CFIL$'$ is an epimorphism if for each $z \in Y \setminus f(X)$, there exists a non-q_ζ convergent filter $\mathcal{F} \in \zeta$ such that $f(\mathcal{F}) \xrightarrow{q_\kappa} y$. For example, the embedding map j in the Wyler completion is an epimorphism. Since Wyler completion is the finest completion in the category CFIL$'$, we have the following corollary.

Corollary 3.10. CFIL$'^*$ *is an epireflective subcategory of* CFIL$'$.

However, CFIL$'$ is not a topological category, since it is not closed under initial structures. But it should be noted that T_2CFIL for which we could construct a completion functor [4] also fails to be a topological category. The construction of a completion functor for a subcategory of FIL which is a topological category may need further investigation. Also, constructions of regular stable completions of filter spaces and the corresponding completion categories may lead to generalisation of the existing T_3 completions.

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