# SUBSPACE GAPS AND WEYL'S THEOREM FOR AN ELEMENTARY OPERATOR

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A range-kernal orthogonality property is established for the elementary operators $\mathscr{E}(X) = \sum_{i=1}^{n} A_i X B_i$ and $\mathscr{E}_*(X) = \sum_{i=1}^{n} A_i^* X B_i^*$, where $\mathbf{A} = (A_1, A_2, \dots, A_n)$ and $\mathbf{B} = (B_1, B_2, \dots, B_n)$ are $n$-tuples of mutually commuting scalar operators (in the sense of Dunford) in the algebra $B(H)$ of operators on a Hilbert space $H$. It is proved that the operator $\mathscr{E}$ satisfies Weyl's theorem in the case in which $\mathbf{A}$ and $\mathbf{B}$ are $n$-tuples of mutually commuting generalized scalar operators.

## 1. Introduction

For a Banach space operator $T$, $T \in B(\mathscr{X})$, the kernel $T^{-1}(0)$ and the range $T(\mathscr{X})$ are said to have a $k$-gap for some real number $k \geq 1$, denoted $T^{-1}(0) \perp_k T(\mathscr{X})$, if

$$y \in T^{-1}(0) \Longrightarrow \|y\| \leq k \operatorname{dist}(y, T(\mathscr{X})) \tag{1.1}$$

[8, Definition, page 94]. Recall from [10, page 93] that a subspace $\mathcal{M}$ of the Banach space $\mathscr{X}$ is *orthogonal* to a subspace $\mathcal{N}$ of $\mathscr{X}$ if $\|m\| \leq \|m+n\|$ for all $m \in \mathcal{M}$ and $n \in \mathcal{N}$. This definition of orthogonality coincides with the usual definition of orthogonality in the case in which $\mathscr{X} = H$ is a Hilbert space. A 1-gap between $T^{-1}(0)$ and $T(\mathscr{X})$ corresponds to the *range-kernel orthogonality* for the operator $T$ (see [1, 2, 8, 14]). The following implications are straightforward to see

$$T^{-1}(0) \perp_k T(\mathscr{X}) \Longrightarrow T^{-1}(0) \cap \overline{T(\mathscr{X})} = \{0\} \Longrightarrow T^{-1}(0) \cap T(\mathscr{X}) = \{0\} \Longrightarrow \operatorname{asc}(T) \leq 1, \tag{1.2}$$

where $\overline{T(\mathscr{X})}$ denotes the closure of $T(\mathscr{X})$ and $\operatorname{asc}(T)$ denotes the *ascent of* $T$. A $k$-gap between $T^{-1}(0)$ and $T(\mathscr{X})$ does not imply that $T(\mathscr{X})$ is closed, or even when $T(\mathscr{X})$ is closed that $\mathscr{X} = T^{-1}(0) \oplus T(\mathscr{X})$ (see, e.g., [1, 2, 23]).

The classical Putnam-Fuglede commutativity theorem says that if $A$ and $B$ are normal Hilbert space operators, $A$ and $B \in B(H)$, and if $\delta_{AB} \in B(B(H))$ is the *generalized derivation* $\delta_{AB}(X) := AX - XB$, then $\delta_{AB}^{-1}(0) = \delta_{A^*B^*}^{-1}(0)$. Extant literature contains various


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generalizations of the Putnam-Fuglede theorem, amongst them the two $n$-tuples $\mathbf{A} = (A_1, A_2, \dots, A_n)$ and $\mathbf{B} = (B_1, B_2, \dots, B_n)$ of mutually commuting normal (Hilbert space) operators $A_i$ and $B_i$, $1 \le i \le n$. Let $\mathscr{E} \in B(B(H))$ be the *elementary operator*

$$\mathscr{E}(X) := \sum_{i=1}^{n} A_i X B_i. \tag{1.3}$$

If $\operatorname{asc}(\mathscr{E}) \le 1$, then $\mathscr{E}^{-1}(0) = \mathscr{E}_*^{-1}(0)$, where $\mathscr{E}_*(X) := \sum_{i=1}^{n} A_i^* X B_i^*$ (see [21, 22, 24]). The conclusion $\mathscr{E}^{-1}(0) \subseteq \mathscr{E}_*^{-1}(0)$ fails if $\operatorname{asc}(\mathscr{E}) > 1$ [21]; moreover, in such a case it may happen that $\mathscr{E}^{-1}(0) \cap \mathscr{E}(B(H)) \neq \{0\}$ [22]. For 2-tuples $\mathbf{A}$ and $\mathbf{B}$ of mutually commuting normal operators it is always the case that $\operatorname{asc}(\mathscr{E}) \le 1$ (see [14] or [8]).

This paper considers $n$-tuples $\mathbf{A}$ and $\mathbf{B}$ of mutually commuting *scalar operators* (in the sense of Dunford and Schwartz [10]) $A_i$ and $B_i$, $1 \le i \le n$, to prove that the operator $\mathscr{E}_\mu := (\mathscr{E} - \mu I) \in B(B(H))$ satisfies: (i) there exists a complex number $\lambda = \alpha \exp i\theta$, $\alpha > 0$ and $0 \le \theta < 2\pi$, such that if $\mathscr{E}_\lambda^{-1}(0) \neq \{0\}$, then $\mathscr{E}_\lambda^{-1}(0) \perp_k \mathscr{E}_\lambda(B(H))$ and $\mathscr{E}_{*\lambda}^{-1}(0) \perp_k \mathscr{E}_{*\lambda}(B(H))$, where $\mathscr{E}_{*\lambda} = (\mathscr{E}_* - \overline{\lambda} I)$. Furthermore, if the operators $A_i$ and $B_i$ in the $n$-tuples $\mathbf{A}$ and $\mathbf{B}$ are normal, then (ii) $\mathscr{E}_\lambda^{-1}(0) = \mathscr{E}_{*\lambda}^{-1}(0)$. This compares with the fact that the operator $\mathscr{E}$ may fail to satisfy the $k$-gap property of (i) or the Putnam-Fuglede-theorem-type commutativity property of (ii). However, if we restrict the length $n$ of the $n$-tuples $\mathbf{A}$ and $\mathbf{B}$ to $n = 1$ (resp., $n = 2$), then both (i) and (ii) hold for all complex numbers $\lambda$ [7, 9] (resp., $\lambda = 0$ and $\lambda = \alpha \exp i\theta$ for some real number $\alpha > 0$; see [7] and Theorem 2.4 infra). Our proof of (i) and (ii) makes explicit the relationship between the existence of a $k$-gap between the kernel and the range of the operator $\mathscr{E}_\lambda$, and the Putnam-Fuglede commutativity property for $n$-tuples $\mathbf{A}$ and $\mathbf{B}$ consisting of mutually commuting normal operators. Letting the $n$-tuples $\mathbf{A}$ and $\mathbf{B}$ consist of mutually commuting *generalized scalar operators* (in the sense of Colojoară and Foiaş [5]), it is proved that (i) a sufficient condition for $\mathscr{E}_\lambda(B(\mathscr{X}))$ to be closed is that the complex number $\lambda$ is isolated in the spectrum of $\mathscr{E}$; (ii) $f(\mathscr{E})$ and $f(\mathscr{E}^*)$ satisfy Weyl's theorem for every analytic function $f$ defined on a neighborhood of the spectrum of $\mathscr{E}$, and the conjugate operator $\mathscr{E}^*$ satisfies $a$-Weyl's theorem. These results will be proved in Sections 2 and 3, but before that, we explain our notation and terminology.

The ascent of $T \in B(\mathscr{X})$, $\operatorname{asc}(T)$, is the least nonnegative integer $n$ such that $T^{-n}(0) = T^{-(n+1)}(0)$ and the descent of $T$, $\operatorname{dsc}(T)$, is the least nonnegative integer $n$ such that $T^n(\mathscr{X}) = T^{n+1}(\mathscr{X})$. We say that $T - \lambda$ is of finite ascent (resp., finite descent) if $\operatorname{asc}(T - \lambda I) < \infty$ (resp., $\operatorname{dsc}(T - \lambda I) < \infty$). The *numerical range of* $T$ is the closed convex set

$$W(B(\mathscr{X}), T) = \{f(T) : f \in B(\mathscr{X})^*, \|f\| = \|f(I)\| = 1\} \tag{1.4}$$

of the set $\mathbb{C}$ of complex numbers (see [3]). A *spectral operator* (in the sense of Dunford) is an operator with a countable additive resolution of the identity defined on the Borel sets of $\mathbb{C}$; a spectral operator $T$ is said to be *scalar type* if it satisfies $T = \int \lambda E(d\lambda)$, where $E$ is the resolution of the identity for $T$ [11, page 1938]. If $\mathbf{A} = (A_1, A_2, \dots, A_n)$ is an $n$-tuple of mutually commuting scalar operators in $B(H)$, then there exists an invertible self-adjoint operator $S$ such that $S^{-1} A_i S = M_i$ is a normal operator for all $i = 1, 2, \dots, n$ [10, page 1947]. $T \in B(\mathscr{X})$ is a *generalized scalar operator* if there exists a continuous algebra homomorphism $\Phi : C^\infty \to B(\mathcal{X})$ for which $\Phi(1) = I$ and $\Phi(Z) = T$, where $C^\infty(\mathbb{C})$ is the Fréchet algebra of all infinitely differentiable functions on $\mathbb{C}$ (endowed with its usual topology of uniform convergence on compact sets for the functions and their partial derivatives) and $Z$ is the identity function on $\mathbb{C}$ (see [5, 16]). We will denote the *spectrum* and the *isolated points of the spectrum* of $T$ by $\sigma(T)$ and $\operatorname{iso}\sigma(T)$, respectively. The closed unit disc in $\mathbb{C}$ will be denoted by $\overline{\mathbf{D}}$, and $\partial\mathbf{D}$ will denote the boundary of the unit disc $\mathbf{D}$. The operator of *left multiplication by $T$* (*right multiplication by $T$*) will be denoted by $L_T$ (resp., $R_T$). It is clear that $[L_S, R_T] = 0$ for all $S, T \in B(\mathcal{X})$, where $[L_S, R_T]$ denotes the commutator $L_S R_T - R_T L_S$. We will henceforth shorten $(T - \lambda I)$ to $(T - \lambda)$.

An operator $T \in B(\mathcal{X})$ is said to be Fredholm, $T \in \Phi(\mathcal{X})$, if $T(\mathcal{X})$ is closed and both the *deficiency indices* $\alpha(T) = \dim(T^{-1}(0))$ and $\beta(T) = \dim(\mathcal{X}/T(\mathcal{X}))$ are finite, and then the *index* of $T$, $\operatorname{ind}(T)$, is defined to be $\operatorname{ind}(T) = \alpha(T) - \beta(T)$. The operator $T$ is *Weyl* if it is Fredholm of index zero. The (Fredholm) essential spectrum $\sigma_e(T)$ and the Weyl spectrum $\sigma_w(T)$ of $T$ are the sets

$$\begin{aligned} \sigma_e(T) &= \{\lambda \in \mathbb{C} : T - \lambda \text{ is not Fredholm}\}, \\ \sigma_w(T) &= \{\lambda \in \mathbb{C} : T - \lambda \text{ is not Weyl}\}. \end{aligned} \tag{1.5}$$

Let $\pi_0(T)$ denote the set of *Riesz points* of $T$ (i.e., the set of $\lambda \in \mathbb{C}$ such that $T - \lambda$ is Fredholm of finite ascent and descent [4]), and let $\pi_{00}(T)$ denote the set of isolated eigenvalues of $T$ of finite geometric multiplicity. Also, let $\pi_{a0}(T)$ be the set of $\lambda \in \mathbb{C}$ such that $\lambda$ is an isolated point of $\sigma_a(T)$ and $0 < \dim\ker(T - \lambda) < \infty$, where $\sigma_a(T)$ denotes the approximate point spectrum of the operator $T$. Clearly, $\pi_0(T) \subseteq \pi_{00}(T) \subseteq \pi_{a0}(T)$. We say that *Weyl's theorem holds for $T$* if

$$\sigma(T) \setminus \sigma_w(T) = \pi_{00}(T), \tag{1.6}$$

and *a-Weyl's theorem holds for $T$* if

$$\sigma_{ea}(T) = \sigma_a(T) \setminus \pi_{a0}(T), \tag{1.7}$$

where $\sigma_{ea}(T)$ denotes *the essential approximate point spectrum* (i.e., $\sigma_{ea}(T) = \cap\{\sigma_a(T + K) : K \in K(\mathcal{X})\}$ with $K(\mathcal{X})$ denoting the ideal of compact operators on $\mathcal{X}$). If we let $\Phi_+(\mathcal{X}) = \{T \in B(\mathcal{X}) : \alpha(T) < \infty \text{ and } T(\mathcal{X}) \text{ is closed}\}$ denote the semigroup of *upper semi-Fredholm operators* in $B(\mathcal{X})$ and let $\Phi_+^-(\mathcal{X}) = \{T \in \Phi_+(\mathcal{X}) : \operatorname{ind}(T) \le 0\}$, then $\sigma_{ea}(T)$ is the complement in $\mathbb{C}$ of all those $\lambda$ for which $(T - \lambda) \in \Phi_+^-(\mathcal{X})$. The concept of $a$-Weyl's theorem was introduced by Rakočević: $a$-Weyl's theorem for $T$ implies Weyl's theorem for $T$, but the converse is generally false [20].

An operator $T \in B(\mathcal{X})$ has *the single-valued extension property* (SVEP) at $\lambda_0 \in \mathbb{C}$ if for every open disc $\mathcal{D}_{\lambda_0}$ centered at $\lambda_0$, the only analytic function $f : \mathcal{D}_{\lambda_0} \to \mathcal{X}$ which satisfies

$$(T - \lambda) f(\lambda) = 0 \quad \forall \lambda \in \mathcal{D}_{\lambda_0} \tag{1.8}$$

is the function $f \equiv 0$. Trivially, every operator $T$ has SVEP at points of the resolvent $\rho(T) = \mathbb{C} \setminus \sigma(T)$; also $T$ has SVEP at $\lambda \in \operatorname{iso}\sigma(T)$. *We say that $T$ has SVEP if it has SVEP at every $\lambda \in \mathbb{C}$.* The *quasinilpotent part* $H_0(T)$ and the *analytic core* $K(T)$ of $T \in B(\mathcal{X})$ are defined, respectively, by

$$\begin{aligned} H_0(T) &= \Big\{x \in \mathscr{X} : \lim_{n\longrightarrow\infty} \|T^n(x)\|^{1/n} = 0\Big\}, \\ K(T) &= \{x \in \mathscr{X} : \exists \text{a sequence } \{x_n\} \subset \mathscr{X},\ \delta > 0 \\ &\qquad \text{for which } x = x_0,\ T(x_{n+1}) = x_n,\ \|x_n\| \le \delta^n \|x\|\ \forall n = 1,2,\dots\}. \end{aligned} \tag{1.9}$$

We note that $H_0(T-\lambda)$ and $K(T-\lambda)$ are (generally) nonclosed hyperinvariant subspaces of $T-\lambda$ such that $(T-\lambda)^{-q}(0) \subseteq H_0(T-\lambda)$ for all $q = 0,1,2,\dots$ and $(T-\lambda)K(T-\lambda) = K(T-\lambda)$ [18]. If $T$ has SVEP at $\lambda$, then $H_0(T-\lambda)$ and $K(T-\lambda)$ are closed. The operator $T \in B(\mathscr{X})$ is said to be *semiregular* if $T(\mathscr{X})$ is closed and $T^{-1}(0) \subset T^{\infty}(\mathscr{X}) = \cap_{n\in\mathbf{N}} T^n(\mathscr{X})$; $T$ admits a *generalized Kato decomposition*, (GKD), if there exists a pair of $T$-invariant closed subspaces $(\mathcal{M},\mathcal{N})$ such that $\mathscr{X} = \mathcal{M} \oplus \mathcal{N}$, the restriction $T|_{\mathcal{M}}$ is quasinilpotent and $T|_{\mathcal{N}}$ is semiregular. An operator $T \in B(\mathscr{X})$ has a (GKD) at every $\lambda \in \operatorname{iso}\sigma(T)$, namely $\mathscr{X} = H_0(T-\lambda) \oplus K(T-\lambda)$. We say that $T-\lambda$ is of *Kato type* if $(T-\lambda)|_{\mathcal{M}}$ is nilpotent in the *GKD* for $T-\lambda$. Fredholm operators are Kato type [13, Theorem 4], and operators $T \in B(\mathscr{X})$ satisfying the following property:

$\mathbf{H}(p)$ $H_0(T-\lambda) = (T-\lambda)^{-p}(0)$,

for some integer $p \ge 1$, are Kato type at isolated points of $\sigma(T)$ (but not every Kato type operator $T$ satisfies property $\mathbf{H}(p)$).

## 2. $k$-gap and the Putnam-Fuglede theorem

Let, as before, $\mathbf{A} = (A_1, A_2, \dots, A_n)$ and $\mathbf{B} = (B_1, B_2, \dots, B_n)$ be $n$-tuples of mutually commuting scalar operators, and let $\mathscr{E}_\lambda$ and $\mathscr{E}_{*\lambda}$ denote the elementary operators $\mathscr{E}_\lambda(X) = \sum_{i=1}^n A_i X B_i - \lambda X$ and $\mathscr{E}_{*\lambda}(X) = \sum_{i=1}^n A_i^* X B_i^* - \bar{\lambda} X$. We say in the following that the $n$-tuple $\mathbf{A}$ is *normally constituted* if $A_i$ is normal for all $1 \le i \le n$.

The following theorem is the main result of this section.

Theorem 2.1. (i) *There exists a real number $\alpha > 0$ such that if $(0 \ne) X \in \mathscr{E}_\alpha^{-1}(0)$, then $\|X\| \le k\|\mathscr{E}_\alpha(Y) + X\|$ and $\|X\| \le k\|\mathscr{E}_{*\alpha}(Y) + X\|$ for some real number $k \ge 1$ and all $Y \in B(H)$.*

*Furthermore, if* $\mathbf{A}$ *and* $\mathbf{B}$ *are normally constituted, then*

(ii) $\mathscr{E}_\alpha^{-1}(0) = \mathscr{E}_{*\alpha}^{-1}(0)$.

*Proof.* There exist invertible self-adjoint operators $T_1, T_2 \in B(H)$ and normal operators $M_i, N_i \in B(H)$, $1 \le i \le n$, such that $M_i = T_1^{-1} A_i T_1$, $N_i = T_2^{-1} B_i T_2$, and $[M_i, M_j] = 0 = [N_i, N_j]$ for all $1 \le i, j \le n$. Define scalars $\alpha_1$ and $\alpha_2$ by $\alpha_1 = \|\sum_{i=1}^n M_i^* M_i\|^{1/2}$ and $\alpha_2 = \|\sum_{i=1}^n N_i^* N_i\|^{1/2}$, define the scalar $\alpha$ by $\alpha = \sqrt{\alpha_1\alpha_2}$, and define the operators $C_i$ and $D_i$, $1 \le i \le n$, by $C_i = (1/\sqrt{\alpha_1})M_i$ and $D_i = (1/\sqrt{\alpha_2})N_i$. Then $(C_1, C_2, \dots, C_n)$ and $(D_1, D_2, \dots, D_n)$ are $n$-tuples of mutually commuting normal operators. Let $E(X) = \sum_{i=1}^n C_i X D_i$. Set

$$U = [C_1 \quad C_2 \quad \cdots \quad C_n], \qquad V = [D_1 \quad D_2 \quad \cdots \quad D_n]^t, \tag{2.1}$$

where $[\cdots]^t$ denotes the transpose of the row matrix $[\cdots]$. Then $U$ and $V$ are contractions. Representing $E(X)$ by

$$E(X) := U(X \bigotimes I_n)V, \tag{2.2}$$

where $I_n$ denotes the identity of $\mathbf{M}_n(\mathbb{C})$, it then follows that $E$ is a contraction. Hence,

$$W(B(B(H)),E) \subseteq \overline{\mathbf{D}}. \tag{2.3}$$

For $\mu \in \mathbb{C}$, let $E_\mu$ denote the operator $E_\mu = E - \mu$. Then

$$W(B(B(H)),E_1) \subseteq \{\lambda \in \mathbb{C} : |\lambda + 1| \leq 1\}. \tag{2.4}$$

In particular, $0 \in \partial W(B(B(H)),E_1)$. Notice that if 0 is an eigenvalue of $\mathscr{E}_\alpha$, then 0 is an eigenvalue of $E_1$. It follows from Sinclair [23, Proposition 1] that

$$\|X\| \leq \|E_1(Y) + X\| \tag{2.5}$$

for all $X \in E_1^{-1}(0)$ and $Y \in B(H)$. In particular, $\operatorname{asc}(E_1) \leq 1$, which by a result of Shulman [21] implies that $E_1^{-1}(0) \subseteq E_{*1}^{-1}(0)$ (where $E_{*1} \in B(B(H))$ is the operator $E_{*1} = E_* - 1 : X \to \sum_{i=1}^n C_i^* X D_i^* - X$). Representing $E_*$ by

$$E_*(X) = U_1(X \bigotimes I_n)V_1, \tag{2.6}$$

where

$$U_1 = [C_1^* \quad C_2^* \quad \cdots \quad C_n^*], \qquad V_1 = [D_1^* \quad D_2^* \quad \cdots \quad D_n^*]^t, \tag{2.7}$$

it follows that $E_*$ is a contraction and 0 is an eigenvalue of $E_{*1}$ in $\partial W(B(B(H)),E_{*1})$. Hence,

$$\|X\| \leq \|E_{*1}(Y) + X\| \tag{2.8}$$

for all $X \in E_{*1}^{-1}(0)$ and $Y \in B(H)$, which implies that $E_{*1}^{-1}(0) \subseteq E_1^{-1}(0)$. Hence, $E_1^{-1}(0) = E_{*1}^{-1}(0)$. The proof of (ii) is now a consequence of the observation that

$$X \in E_1^{-1}(0) \Longleftrightarrow \sum_{i=1}^n M_i X N_i - \alpha X = 0 \Longleftrightarrow X \in E_{*1}^{-1}(0) \Longleftrightarrow \sum_{i=1}^n M_i^* X N_i^* - \alpha X = 0. \tag{2.9}$$

To prove (i), we let $\|T_1\|\|T_1^{-1}\|\|T_2\|\|T_2^{-1}\| = k$. Since

$$\begin{aligned}
\alpha\|X\| &\leq \alpha\|E_1(Y) + X\| \\
&= \left\| T_1^{-1}\left\{ \sum_{i=1}^n A_i(T_1 Y T_2^{-1})B_i - \alpha T_1 Y T_2^{-1} + \alpha T_1 X T_2^{-1} \right\} T_2 \right\| \\
&\Longrightarrow \|\alpha T_1 X T_2^{-1}\| \leq \alpha\|T_1\|\|T_2^{-1}\|\|X\| \\
&\leq k\|\mathscr{E}_\alpha(T_1 Y T_2^{-1}) + \alpha T_1 X T_2^{-1}\|
\end{aligned} \tag{2.10}$$

for all $X \in E_1^{-1}(0)$ and $Y \in B(H)$ (equivalently, all $T_1XT_2^{-1} \in \mathscr{E}_\alpha^{-1}(0)$ and $T_1YT_2^{-1} \in B(H)$), it follows that $\mathscr{E}_\alpha^{-1}(0)\perp_k\mathscr{E}_\alpha(B(H))$. A similar argument, applied this time to $\|X\| \le \|E_{*1}(Y)+X\|$, implies that $\mathscr{E}_{*\alpha}^{-1}(0)\perp_k\mathscr{E}_{*\alpha}(B(H))$. This completes the proof of the theorem. □

The following corollary is an immediate consequence of the fact that $\operatorname{asc}(\mathscr{E}_\alpha) \le 1$.

Corollary 2.2. *The range of $\mathscr{E}_\alpha$ is closed if and only if $\mathscr{E}_\alpha^{-1}(0)+\mathscr{E}_\alpha(B(H))$ is closed.*

For the proof see [16, Proposition 4.10.4].

The point $\alpha$ of Theorem 2.1 is not unique. Since $0 \in \partial W(B(B(H)),E_\mu)$ for every $\mu \in \mathbb{C}$ such that $|\mu| = 1$, the argument of the proof of Theorem 2.1 implies the following theorem.

Theorem 2.3. (i) *There exists a complex number $\lambda = \alpha\exp i\theta$, $\alpha > 0$ and $0 \le \theta < 2\pi$, such that if $(0 \ne)X \in \mathscr{E}_\lambda^{-1}(0)$, then $\|X\| \le k\|\mathscr{E}_\lambda(Y)+X\|$ and $\|X\| \le k\|\mathscr{E}_{*\lambda}(Y)+X\|$ for some real number $k \ge 1$ and all $Y \in B(H)$.*

*Furthermore, if* **A** *and* **B** *are normally constituted, then*

(ii) $\mathscr{E}_\lambda^{-1}(0) = \mathscr{E}_{*\lambda}^{-1}(0)$ *for all $\lambda$ as in part (i).*

Let the Hilbert space $H$ be separable, and let $\mathscr{C}_p$ denote the von Neumann-Schatten $p$-class, $1 \le p < \infty$, with norm $\|\cdot\|_p$. Then Theorem 2.3 has the following $\mathscr{C}_p$ version.

Theorem 2.4. (i) *There exists a complex number $\lambda = \alpha\exp i\theta$, $\alpha > 0$ and $0 \le \theta < 2\pi$, such that if $(0 \ne)X \in \mathscr{E}_\lambda^{-1}(0)\cap\mathscr{C}_p$, then $\|X\|_p \le k\|\mathscr{E}_\lambda(Y)+X\|_p$ and $\|X\|_p \le k\|\mathscr{E}_{*\lambda}(Y)+X\|_p$ for some real number $k \ge 1$ and all $Y \in \mathscr{C}_p$.*

*Furthermore, if* **A** *and* **B** *are normally constituted, then*

(ii) $\mathscr{E}_\lambda^{-1}(0)\cap\mathscr{C}_p = \mathscr{E}_{*\lambda}^{-1}(0)\cap\mathscr{C}_p$ *for all $\lambda$ as in part (i).*

*Proof.* Define the real numbers $\alpha_i$, $i = 1,2$, as in the proof of Theorem 2.1, define the normal operators $C_i$ and $D_i$ by $C_i = (1/\sqrt{\alpha_1 n^{1/2p}})M_i$ and $D_i = (1/\sqrt{\alpha_2 n^{1/2p}})N_i$. Let $\alpha = \sqrt{\alpha_1\alpha_2 n^{1/p}}$. Then $E \in B(\mathscr{C}_p)$ is a contraction. Now argue as in the proof of Theorem 2.1. □

As we will see in the following section, $H_0(\mathscr{E}_\lambda) = \mathscr{E}_\lambda^{-p}(0)$ for all $\lambda \in \mathbb{C}$ and some integer $p \ge 1$ (i.e., $\mathscr{E}_\lambda$ satisfies property $\mathbf{H}(p)$), which implies that $\operatorname{asc}(\mathscr{E}_\lambda) \ge 1$ for all $\lambda \in \mathbb{C}$. (Here, as also elsewhere, the statement $\operatorname{asc}(T) \ge 1$ is to be taken to subsume the hypothesis that $T$ is not injective.) However, if the $n$-tuples **A** and **B** are of length $n = 1$, then $\operatorname{asc}(\mathscr{E}_\lambda) \le 1$ for all $\lambda \in \mathbb{C}$ and for a number of classes of not necessarily scalar or normal operators $A_1$ and $B_1$ (see [7, 9]). If $n = 2$ and $B_1 = A_2 = I$, then $\operatorname{asc}(\mathscr{E}_\lambda) \le 1$ (once again for $A_1$ and $B_2$ belonging to a number of classes of operators more general than the class of scalar operators [7]). Again, if $n = 2$, then $\operatorname{asc}(\mathscr{E}_\lambda) \le 1$ for $\lambda = 0$ and $\lambda = \alpha\exp i\theta$, as follows from Theorem 2.1 and the following argument. Define the normal operators $M_i$ and $N_i$, $i = 1,2$, as in the proof of Theorem 2.1. Then $[M_1,M_2] = [N_1,N_2] = 0$. Define $\phi \in B(B(H))$ by $\phi(X) = M_1XN_1 + M_2XN_2$. Then $\phi^{-1}(0)\perp_k\phi(B(H))$ (see [14] or [8]), which implies that $\operatorname{asc}(\phi) = \operatorname{asc}(\mathscr{E}) \le 1$. The following corollary, which generalizes [14, Theorem 2], is now obvious.

Corollary 2.5. *If $\mathbf{A} = (A_1,A_2)$ and $\mathbf{B} = (B_1,B_2)$ are 2-tuples of commuting scalar operators ($\in B(H)$), if $\mathscr{E} \in B(B(H))$ is defined by $\mathscr{E}(X) = A_1XB_1 + A_2XB_2$ and if the complex* *number* $\lambda$ *is as in Theorem 2.3, then* $\operatorname{asc}(\mathscr{E}_\mu) \leq 1$, *and* $\mathscr{E}_\mu^{-1}(0) \perp_k \mathscr{E}_\mu(B(H))$ *for* $\mu = 0, \lambda$. *Furthermore, if* **A** *and* **B** *are normally constituted, then* $\mathscr{E}_\mu^{-1}(0) = \mathscr{E}_{*\mu}^{-1}(0)$ *for* $\mu = 0, \lambda$.

*Perturbation by quasinilpotents.* Recall that every *spectral operator* $T \in B(\mathscr{X})$ is the sum $T = S + Q$ of a scalar type operator $S$ and a quasinilpotent operator $Q$ such that $[S, Q] = 0$ [11]. Let $\mathbf{A} = (J_1, J_2)$ and $\mathbf{B} = (K_1, K_2)$ be tuples of operators in $B(H)$ such that $J_i = A_i + Q_i$ and $K_i = B_i + R_i$, $i = 1, 2$, for some scalar operators $A_i$, $B_i$ and quasinilpotent operators $Q_i$, $R_i$. If we define $\mathbf{E} \in B(B(H))$ by $\mathbf{E}(X) = J_1 X K_1 + J_2 X K_2$, then $\mathbf{E}(X) = \mathscr{E}(X) + \phi(X)$, where $\mathscr{E}(X)$ is defined as in Corollary 2.5 and $\phi(X) = A_1 X R_1 + A_2 X R_2 + Q_1 X B_1 + Q_2 X B_2 + Q_1 X R_1 + Q_2 X R_2$. Recall that the sum of two commuting quasinilpotent operators, as well as the product of two commuting operators one of which is quasinilpotent, is quasinilpotent [5, Lemma 3.8, Chapter 4]. Representing the operator $X \to SXT$ by $X \to L_S R_T(X)$, where $(S, T)$ denotes any of the operator pairs $(A_i, R_i), (Q_i, B_i)$, or $(Q_i, R_i)$, $i = 1, 2$, and assuming that the operators in the sets $\{A_1, A_2, Q_1, Q_2\}$ and $\{R_1, R_2, B_1, B_2\}$ mutually commute, it follows that the operator $\phi$ is quasinilpotent.

Theorem 2.6. *Let the operator* **E** *be defined as above. If the operators in the sets* $\{A_1, A_2, Q_1, Q_2\}$ *and* $\{R_1, R_2, B_1, B_2\}$ *mutually commute, then* $X \in \mathbf{E}^{-1}(0) \Rightarrow X \in \mathscr{E}^{-1}(0)$.

*Proof.* Let $X \in \mathbf{E}^{-1}(0)$. The hypothesis that the operators in the sets $\{A_1, A_2, Q_1, Q_2\}$ and $\{R_1, R_2, B_1, B_2\}$ mutually commute then implies that

$$-\phi(X) = \mathbf{E}(X) = T_1 \{ M_1 (T_1^{-1} X T_2) N_1 + M_2 (T_1^{-1} X T_2) N_2 \} T_2^{-1}, \tag{2.11}$$

where the operator $\phi$ is quasinilpotent, and where the normal operators $M_i$, $N_i$, $[M_1, M_2] = 0 = [N_1, N_2]$, and the invertible operators $T_i$, $i = 1, 2$, are defined as in the proof of Theorem 2.1. Define $\Phi \in B(B(H))$ by $\Phi(Y) = M_1 Y N_1 + M_2 Y N_2$. Since the operator $\phi$ is quasinilpotent,

$$\lim_{n \to \infty} \left\| \Phi^n (T_1^{-1} X T_2) \right\|^{1/n} \leq \| T_1^{-1} \| \| T_2 \| \lim_{n \to \infty} \left\| \phi^n (X) \right\|^{1/n} = 0. \tag{2.12}$$

As earlier remarked upon, $H_0(\Phi) = \Phi^{-p}(0)$ for some integer $p \geq 1$. Since $\operatorname{asc}(\Phi) \leq 1$ (by Corollary 2.5), it follows that $\Phi(T_1^{-1} X T_2) = 0$. Hence $X \in \mathscr{E}^{-1}(0)$. □

## 3. Weyl's theorem

If $A, B \in B(\mathscr{X})$ are generalized scalar operators, then $L_A, R_B \in B(B(\mathscr{X}))$ are commuting generalized scalar operators with two commuting spectral distributions, which implies that $L_A R_B$ and $L_A + R_B$ are generalized scalar operators (see [5, Theorem 3.3, Proposition 4.2, Theorem 4.3, Chapter 4]). Let $\mathbf{A} = (A_1, A_2, \ldots, A_n)$ and $\mathbf{B} = (B_1, B_2, \ldots, B_n)$ be $n$-tuples of mutually commuting generalized scalar operators in $B(\mathscr{X})$, and let the elementary operator $\mathbf{E}_\lambda \in B(B(\mathscr{X}))$ be defined by $\mathbf{E}_\lambda(X) = \sum_{i=1}^n A_i X B_i - \lambda X$. Since $[L_{A_i}, R_{B_j}] = 0$ for all $1 \leq i$, $j \leq n$, the mutual commutativity of the $n$-tuples implies that $[L_{A_i} R_{B_i}, L_{A_j} R_{B_j}] = 0$ for all $1 \leq i$, $j \leq n$, the generalized scalar operators $L_{A_i} R_{B_i}$ and $L_{A_j} R_{B_j}$ have two commuting spectral distributions, and (hence that) $L_{A_i} R_{B_i} + L_{A_j} R_{B_j}$ is a generalized scalar operator. A finitely repeated application of this argument implies that $\mathbf{E}_\lambda$ is a generalized scalar operator for all $\lambda \in \mathbb{C}$. Thus

$$H_0(\mathbf{E}_\lambda) = \mathbf{E}_\lambda^{-p}(0) \tag{3.1}$$

for some integer $p \geq 1$ and all $\lambda \in \mathbb{C}$ see [5, Theorem 4.5, Chapter 4]. In particular, $\operatorname{asc}(\mathbf{E}_\lambda) \leq p < \infty$ for all $\lambda \in \mathbb{C}$ and $\mathbf{E}(= E_0)$ has SVEP.

The following proposition will be required in the proof of our main result.

Proposition 3.1. (a) *The following conditions are equivalent:*

(i) $\lambda \in \operatorname{iso}\sigma(\mathbf{E})$*;*

(ii) $\lambda$ *is a pole of order* $p$ *of the resolvent of* $\mathbf{E}$*;*

(iii) $\operatorname{dsc}(\mathbf{E}_\lambda) < \infty$*;*

(iv) $\mathbf{E}_\lambda$ *is Kato type and (in the definition of Kato type) the subspace* $\mathcal{N} \subseteq \mathbf{E}_\lambda(B(\mathcal{X}))$.

(b) *If* $\mathbf{E}^*$ *denotes the conjugate operator of* $\mathbf{E}$, *then* $\sigma_w(\mathbf{E}^*) = \sigma_w(\mathbf{E})$, $\pi_{00}(\mathbf{E}^*) = \pi_{00}(\mathbf{E}) = \pi_0(\mathbf{E}) = \pi_0(\mathbf{E}^*)$, *and* $\lambda \in \pi_{00}(\mathbf{E}) \Rightarrow \mathbf{E}_\lambda \in \Phi(B(\mathcal{X}))$, *and* $\operatorname{ind}(\mathbf{E}_\lambda) = 0$.

*Proof.* (a) (i)⇒(ii). If $\lambda \in \operatorname{iso}\sigma(\mathbf{E})$, then $B(\mathcal{X}) = H_0(\mathbf{E}_\lambda) \oplus K(\mathbf{E}_\lambda) = \mathbf{E}_\lambda^{-p}(0) \oplus K(\mathbf{E}_\lambda)$ for some integer $p \geq 1$. But then $\mathbf{E}_\lambda^{-p}(0)$ is complemented by the closed subspace $K(\mathbf{E}_\lambda) \subseteq \mathbf{E}_\lambda(B(\mathcal{X})) \Rightarrow K(\mathbf{E}_\lambda) = \mathbf{E}_\lambda^{p}(B(\mathcal{X}))$ [15, Theorem 3.4]. Hence $\lambda$ is a pole of the resolvent of $\mathbf{E}$.

(ii)⇒(iii). The implication is obvious.

(iii)⇒(iv). If $\operatorname{dsc}(\mathbf{E}_\lambda) < \infty$, then we have the following implications:

$$\begin{aligned} H_0(\mathbf{E}_\lambda) = \mathbf{E}_\lambda^{-p}(0), &\quad \forall \lambda \in \mathbb{C}, \\ &\Longrightarrow \operatorname{asc}(\mathbf{E}_\lambda) = \operatorname{dsc}(\mathbf{E}_\lambda) \leq p < \infty, \quad [16, \text{Proposition } 4.10.6], \\ &\Longrightarrow B(\mathcal{X}) = \mathbf{E}_\lambda^{-p}(0) \oplus \mathbf{E}_\lambda^{p}(B(\mathcal{X})) = \mathcal{M} \oplus \mathcal{N} \\ &\Longrightarrow \mathbf{E}_\lambda \text{ is Kato type}, \quad \mathcal{N} \subseteq \mathbf{E}_\lambda(B(\mathcal{X})). \end{aligned} \tag{3.2}$$

(iv)⇒(i). If $\mathbf{E}_\lambda$ is Kato type, then $B(\mathcal{X}) = \mathcal{M} \oplus \mathcal{N}$, where $\mathbf{E}_\lambda|_{\mathcal{M}}$ is nilpotent and $\mathbf{E}_\lambda|_{\mathcal{N}}$ is semiregular. Since $\mathbf{E}_\lambda^{-n}(0) \subseteq \mathcal{M} \subseteq H_0(\mathbf{E}_\lambda) = \mathbf{E}_\lambda^{-p}(0)$ for all nonnegative integers $n$, and the closed subspace $\mathcal{N} \subseteq \mathbf{E}_\lambda(B(\mathcal{X}))$, $\lambda \in \operatorname{iso}(\mathbf{E})$ [15, Theorem 3.2].

(b) The following implications hold:

$$\begin{aligned} \lambda \notin \sigma_w(\mathbf{E}^*) &\Longleftrightarrow \mathbf{E}_\lambda^* \in \Phi(B(\mathcal{X})^*), \quad \operatorname{ind}(\mathbf{E}_\lambda^*) = 0, \\ &\Longleftrightarrow \mathbf{E}_\lambda \in \Phi(B(\mathcal{X})), \quad \operatorname{ind}(\mathbf{E}_\lambda) = 0, \\ &\Longleftrightarrow \lambda \notin \sigma_w(\mathbf{E}). \end{aligned} \tag{3.3}$$

Hence $\sigma_w(\mathbf{E}) = \sigma_w(\mathbf{E}^*)$. Again,

$$\begin{aligned} \lambda \in \operatorname{iso}\sigma(\mathbf{E}^*) &\Longleftrightarrow \lambda \in \operatorname{iso}\sigma(\mathbf{E}) \\ &\Longleftrightarrow B(\mathcal{X}) = \mathbf{E}_\lambda^{-p}(0) \oplus \mathbf{E}_\lambda^{p}(B(\mathcal{X})) \Longleftrightarrow \lambda \in \pi_0(\mathbf{E}) \\ &\Longleftrightarrow B(\mathcal{X})^* = \mathbf{E}_\lambda^{*^{-p}}(0) \oplus \mathbf{E}_\lambda^{*^{p}}(B(\mathcal{X})^*) \\ &\Longleftrightarrow \lambda \in \pi_0(\mathbf{E}^*). \end{aligned} \tag{3.4}$$

Recall that if the ascent and the descent of an operator $T$ are finite, and either $0 < \alpha(T) < \infty$ or $0 < \beta(T) < \infty$, then $\operatorname{asc}(T) = \operatorname{dsc}(T) < \infty$ and $0 < \alpha(T) = \beta(T) < \infty$ [12, Proposition 38.6]. Hence $\pi_{00}(\mathbf{E}^*) = \pi_{00}(\mathbf{E}) = \pi_0(\mathbf{E}) = \pi_0(\mathbf{E}^*)$, and $\lambda \in \pi_{00}(\mathbf{E}) \Rightarrow \mathbf{E}_\lambda \in \Phi(B(\mathscr{X}))$ with $\operatorname{ind}(\mathbf{E}_\lambda) = 0$. □

It is evident from Proposition 3.1(a) that a sufficient condition for $\mathbf{E}_\lambda$ to have closed range is that $\lambda \in \operatorname{iso}\sigma(\mathbf{E})$. Proposition 3.1(b) implies that both $\mathbf{E}$ and $\mathbf{E}^*$ satisfy Weyl's theorem: more is true. Let $\mathbf{H}(\sigma(\mathbf{E}))$ denote the set of functions $f$ which are defined and analytic on an open neighborhood of $\sigma(\mathbf{E})$.

Theorem 3.2. (a) *$f(\mathbf{E})$ and $f(\mathbf{E}^*)$ satisfy Weyl's theorem for every $f \in \mathbf{H}(\sigma(\mathbf{E}))$.*
(b) *$\mathbf{E}^*$ satisfies a-Weyl's theorem.*

*Proof.* (a) A proof follows from [19, Theorem 3.1]. Alternatively, one argues as follows. If we let $\mathbf{E}'$ denote either of $\mathbf{E}$ or $\mathbf{E}^*$, then $\sigma(f(\mathbf{E})') = \sigma(f(\mathbf{E}'))$ and $\sigma_w(f(\mathbf{E})') = \sigma_w(f(\mathbf{E}'))$. Since $\mathbf{E}'$ is isoloid (i.e., isolated points of $\mathbf{E}'$ are eigenvalues of $\mathbf{E}'$) and Weyl's theorem holds for $\mathbf{E}'$ (by Proposition 3.1), $f(\sigma_w(\mathbf{E}')) = f(\sigma(\mathbf{E}') \setminus \pi_{00}(\mathbf{E}')) = \sigma(f(\mathbf{E}')) \setminus \pi_{00}(f(\mathbf{E}'))$ [17, lemma] and $f(\sigma_w(\mathbf{E}')) = \sigma_w(f(\mathbf{E}'))$ [6, Corollary 2.6]. (We note here that although [17, lemma] is stated for a Hilbert space, it equally holds in the setting of a Banach space.) Hence, since $f(\mathbf{E})$ satisfies property $\mathbf{H}(p)$, then [19, Theorem 3.4] implies (by Proposition 3.1) that Weyl's theorem holds for $f(\mathbf{E}')$, $\sigma(f(\mathbf{E}')) \setminus \sigma_w(f(\mathbf{E}')) = \pi_{00}(f(\mathbf{E}'))$.

(b) The operator $\mathbf{E}$ has SVEP and the operator $\mathbf{E}^*$ satisfies Weyl's theorem; hence $\sigma(\mathbf{E}^*) = \sigma_a(\mathbf{E}^*)$ [16, page 35] and $\sigma_a(\mathbf{E}^*) \setminus \sigma_w(\mathbf{E}^*) = \pi_{a0}(\mathbf{E}^*)$. We prove that $\sigma_{ea}(\mathbf{E}^*) \supseteq \sigma_w(\mathbf{E}^*)$: since $\sigma_{ea}(\mathbf{E}^*) \subseteq \sigma_w(\mathbf{E}^*)$ always, this would complete the proof. If $\lambda \notin \sigma_{ea}(\mathbf{E}^*)$, then $\mathbf{E}_\lambda^* \in \Phi_+(B(\mathscr{X})^*)$ and $\operatorname{ind}(\mathbf{E}_\lambda^*) \leq 0 \Leftrightarrow \mathbf{E}_\lambda \in \Phi_-(B(\mathscr{X}))$ and $\operatorname{ind}(\mathbf{E}_\lambda) \geq 0$, where $\Phi_-(B(\mathscr{X})) = \{T \in B(B(\mathscr{X})) : \beta(T) < \infty\}$. Since $\operatorname{asc}(\mathbf{E}_\lambda) < \infty$, $\operatorname{ind}(\mathbf{E}_\lambda) \leq 0$. Hence $\alpha(\mathbf{E}_\lambda) = \beta(\mathbf{E}_\lambda) < \infty$ and $\operatorname{asc}(\mathbf{E}_\lambda) = \operatorname{dsc}(\mathbf{E}_\lambda) < \infty$ [12, Proposition 38.6], which implies that $\lambda \notin \sigma_w(\mathbf{E}^*)$. □

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