APPLICATION OF AN INTEGRAL FORMULA TO CR-SUBMANIFOLDS OF COMPLEX HYPERBOLIC SPACE

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The purpose of this paper is to study n-dimensional compact CR-submanifolds of complex hyperbolic space $\mathrm{CH}^{(n+p)/2}$, and especially to characterize geodesic hypersphere in $\mathrm{CH}^{(n+1)/2}$ by an integral formula.

1. Introduction

Let $\bar{M}$ be a complex space form of constant holomorphic sectional curvature c and let M be an n-dimensional CR-submanifold of $(n-1)$ CR-dimension in $\bar{M}$. Then M has an almost contact metric structure (F,U,u,g) (see Section 2) induced from the canonical complex structure of $\bar{M}$. Hence on an n-dimensional CR-submanifold of $(n-1)$ CR-dimension, we can consider two structures, namely, almost contact structure F and a submanifold structure represented by second fundamental form A. In this point of view, many differential geometers have classified M under the conditions concerning those structures (cf. [3, 5, 8, 9, 10, 11, 12, 14, 15, 16]). In particular, Montiel and Romero [12] have classified real hypersurfaces M of complex hyperbolic space $\mathrm{CH}^{(n+1)/2}$ which satisfy the commutativity condition

(C)

$$FA = AF \tag{1.1}$$

by using the S^1-fibration $\pi : H_1^{n+2} \to \mathrm{CH}^{(n+1)/2}$ of the anti-de Sitter space H_1^{n+2} over $\mathrm{CH}^{(n+1)/2}$, and obtained Theorem 4.1 stated in Section 2. We notice that among the model spaces in Theorem 4.1, the geodesic hypersphere is only compact.

In this paper, we will investigate n-dimensional compact CR-submanifold of $(n-1)$ CR-dimension in complex hyperbolic space and provide a characterization of the geodesic hypersphere, which is equivalent to condition (C), by using the following integral formula established by Yano [17, 18]:

$$\int_M \operatorname{div}\{(\nabla_X X)-(\operatorname{div}X)X\} * 1 = \int_M \left\{ \operatorname{Ric}(X,X)+\frac{1}{2}\|\mathscr{L}_X g\|^2 - \|\nabla X\|^2 - (\operatorname{div}X)^2 \right\} * 1 = 0, \tag{1.2}$$

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where X is an arbitrary vector field tangent to M. Our results of the paper are complex hyperbolic versions of those in [6, 15].

2. Preliminaries

Let M be an n-dimensional CR-submanifold of $(n-1)$ CR-dimension isometrically immersed in a complex space form $\bar{M}^{(n+p)/2}(c)$. Denoting by $(J,\bar{g})$ the Kähler structure of $\bar{M}^{(n+p)/2}(c)$, it follows by definition (cf. [5, 6, 8, 9, 13, 16]) that the maximal J-invariant subspace

$$\mathcal{D}_x := T_xM \cap JT_xM \tag{2.1}$$

of the tangent space T_xM of M at each point x in M has constant dimension $(n-1)$. So there exists a unit vector field U_1 tangent to M such that

$$\mathcal{D}_x^{\perp} = \operatorname{Span}\{U_1\}, \quad \forall x \in M, \tag{2.2}$$

where $\mathcal{D}_x^{\perp}$ denotes the subspace of T_xM complementary orthogonal to $\mathcal{D}_x$. Moreover, the vector field ξ_1 defined by

$$\xi_1 := JU_1 \tag{2.3}$$

is normal to M and satisfies

$$JTM \subset TM \oplus \operatorname{Span}\{\xi_1\}. \tag{2.4}$$

Hence we have, for any tangent vector field X and for a local orthonormal basis $\{\xi_1, \xi_\alpha\}_{\alpha=2,\dots,p}$ of normal vectors to M, the following decomposition in tangential and normal components:

$$JX = FX + u^1(X)\xi_1, \tag{2.5}$$

$$J\xi_\alpha = -U_\alpha + P\xi_\alpha, \quad \alpha = 1,\dots,p. \tag{2.6}$$

Since the structure $(J,\bar{g})$ is Hermitian and $J^2 = -I$, we can easily see from (2.5) and (2.6) that F and P are skew-symmetric linear endomorphisms acting on T_xM and $T_xM^{\perp}$, respectively, and that

$$g(FU_\alpha, X) = -u^1(X)\bar{g}(\xi_1, P\xi_\alpha), \tag{2.7}$$

$$g(U_\alpha, U_\beta) = \delta_{\alpha\beta} - \bar{g}(P\xi_\alpha, P\xi_\beta), \tag{2.8}$$

where $T_xM^{\perp}$ denotes the normal space of M at x and g the metric on M induced from $\bar{g}$. Furthermore, we also have

$$g(U_\alpha, X) = u^1(X)\delta_{1\alpha}, \tag{2.9}$$

and consequently,

$$g(U_1, X) = u^1(X), \qquad U_\alpha = 0, \quad \alpha = 2,\dots,p. \tag{2.10}$$

Next, applying J to (2.5) and using (2.6) and (2.10), we have

$$F^2X = -X + u^1(X)U_1, \qquad u^1(X)P\xi_1 = -u^1(FX)\xi_1, \tag{2.11}$$

from which, taking account of the skew-symmetry of P and (2.7),

$$u^1(FX) = 0, \qquad FU_1 = 0, \qquad P\xi_1 = 0. \tag{2.12}$$

Thus (2.6) may be written in the form

$$J\xi_1 = -U_1, \qquad J\xi_\alpha = P\xi_\alpha, \quad \alpha = 2,\dots,p. \tag{2.13}$$

These equations tell us that (F,g,U_1,u^1) defines an almost contact metric structure on M (cf. [5, 6, 8, 9, 16]), and consequently, $n = 2m+1$ for some integer m.

We denote by $\bar{\nabla}$ and ∇ the Levi-Civita connection on $\bar{M}^{(n+p)/2}(c)$ and M, respectively. Then the Gauss and Weingarten formulas are given by

$$\bar{\nabla}_X Y = \nabla_X Y + h(X,Y), \tag{2.14}$$

$$\bar{\nabla}_X \xi_\alpha = -A_\alpha X + \nabla^\perp_X \xi_\alpha, \quad \alpha = 1,\dots,p, \tag{2.15}$$

for any vector fields X, Y tangent to M. Here $\nabla^\perp$ denotes the normal connection induced from $\bar{\nabla}$ in the normal bundle $TM^\perp$ of M, and h and A_α the second fundamental form and the shape operator corresponding to ξ_α, respectively. It is clear that h and A_α are related by

$$h(X,Y) = \sum_{\alpha=1}^{p} g(A_\alpha X, Y)\xi_\alpha. \tag{2.16}$$

We put

$$\nabla^\perp_X \xi_\alpha = \sum_{\beta=1}^{p} s_{\alpha\beta}(X)\xi_\beta. \tag{2.17}$$

Then $(s_{\alpha\beta})$ is the skew-symmetric matrix of connection forms of $\nabla^\perp$.

Now, using (2.14), (2.15), and (2.17), and taking account of the Kähler condition $\bar{\nabla}J = 0$, we differentiate (2.5) and (2.6) covariantly and compare the tangential and normal parts. Then we can easily find that

$$(\nabla_X F)Y = u^1(Y)A_1X - g(A_1Y,X)U_1, \tag{2.18}$$

$$(\nabla_X u^1)(Y) = g(FA_1X,Y), \tag{2.19}$$

$$\nabla_X U_1 = FA_1X, \tag{2.20}$$

$$g(A_\alpha U_1, X) = -\sum_{\beta=2}^{p} s_{1\beta}(X)\bar{g}(P\xi_\beta, \xi_\alpha), \quad \alpha = 2,\dots,p, \tag{2.21}$$

for any X, Y tangent to M.

In the rest of this paper, *we suppose that the distinguished normal vector field* ξ_1 *is parallel with respect to the normal connection* $\nabla^{\perp}$. Hence (2.17) gives

$$s_{1\alpha} = 0, \quad \alpha = 2,\dots,p, \tag{2.22}$$

which, together with (2.21), yields

$$A_\alpha U_1 = 0, \quad \alpha = 2,\dots,p. \tag{2.23}$$

On the other hand, the ambient manifold $\bar{M}^{(n+p)/2}(c)$ is of constant holomorphic sectional curvature c and consequently, its Riemannian curvature tensor $\bar{R}$ satisfies

$$\bar{R}_{\bar{X}\bar{Y}}\bar{Z} = \frac{c}{4}\{\bar{g}(\bar{Y},\bar{Z})\bar{X} - \bar{g}(\bar{X},\bar{Z})\bar{Y} + \bar{g}(J\bar{Y},\bar{Z})J\bar{X} - \bar{g}(J\bar{X},\bar{Z})J\bar{Y} - 2\bar{g}(J\bar{X},\bar{Y})J\bar{Z}\} \tag{2.24}$$

for any $\bar{X}$, $\bar{Y}$, $\bar{Z}$ tangent to $\bar{M}^{(n+p)/2}(c)$ (cf. [1, 2, 4, 19]). So, the equations of Gauss and Codazzi imply that

$$\begin{aligned} R_{XY}Z = {}& \frac{c}{4}\{g(Y,Z)X - g(X,Z)Y + g(FY,Z)FX - g(FX,Z)FY - 2g(FX,Y)FZ\} \\ &+ \sum_\alpha \{g(A_\alpha Y,Z)A_\alpha X - g(A_\alpha X,Z)A_\alpha Y\}, \end{aligned} \tag{2.25}$$

$$(\nabla_X A_1)Y - (\nabla_Y A_1)X = \frac{c}{4}\{g(X,U_1)FY - g(Y,U_1)FX - 2g(FX,Y)U_1\}, \tag{2.26}$$

for any X, Y, Z tangent to M with the aid of (2.22), where R denotes the Riemannian curvature tensor of M. Moreover, (2.11) and (2.25) yield

$$\operatorname{Ric}(X,Y) = \frac{c}{4}\{(n+2)g(X,Y) - 3u^1(X)u^1(Y)\} + \sum_\alpha \{(\operatorname{tr}A_\alpha)g(A_\alpha X,Y) - g(A_\alpha^2 X,Y)\}, \tag{2.27}$$

$$\rho = \frac{c}{4}(n+3)(n-1) + n^2\|\mu\|^2 - \sum_\alpha \operatorname{tr}{A_\alpha}^2, \tag{2.28}$$

where Ric and ρ denote the Ricci tensor and the scalar curvature, respectively, and

$$\mu = \frac{1}{n}\sum_\alpha (\operatorname{tr}A_\alpha)\xi_\alpha \tag{2.29}$$

is the mean curvature vector (cf. [1, 2, 4, 19]).

3. Codimension reduction of CR-submanifolds of $\mathrm{CH}^{(n+p)/2}$

Let M be an n-dimensional CR-submanifold of $(n-1)$ CR-dimension in a complex hyperbolic space $\mathrm{CH}^{(n+p)/2}$ with constant holomorphic sectional curvature $c = -4$.

Applying the integral formula (1.2) to the vector field U_1, we have

$$\int_M \left\{\operatorname{Ric}(U_1,U_1) + \frac{1}{2}\|\mathcal{L}_{U_1}g\|^2 - \|\nabla U_1\|^2 - (\operatorname{div}U_1)^2\right\} * 1 = 0. \tag{3.1}$$

Now we take an orthonormal basis $\{U_1, e_a, e_{a^*}\}_{a=1,\dots,(n-1)/2}$ of tangent vectors to M such that

$$e_{a^*} := Fe_a, \quad a = 1,\dots,\frac{n-1}{2}. \tag{3.2}$$

Then it follows from (2.11) and (2.20) that

$$\operatorname{div} U_1 = \operatorname{tr}(FA_1) = \sum_{a=1}^{(n-1)/2} \{g(FA_1e_a, e_a) + g(FA_1e_{a^*}, e_{a^*})\} = 0. \tag{3.3}$$

Also, using (2.20), we have

$$\|\nabla U_1\|^2 = g(FA_1U_1, FA_1U_1) + \sum_{a=1}^{(n-1)/2} \{g(FA_1e_a, FA_1e_a) + g(FA_1e_{a^*}, FA_1e_{a^*})\}, \tag{3.4}$$

from which, together with (2.11) and (2.12), we can easily obtain

$$\|\nabla U_1\|^2 = \operatorname{tr} A_1^2 - \|A_1U_1\|^2. \tag{3.5}$$

Furthermore, (2.20) yields

$$(\mathcal{L}_{U_1}g)(X,Y) = g(\nabla_X U_1, Y) + g(\nabla_Y U_1, X) = g((FA_1 - A_1F)X, Y), \tag{3.6}$$

and consequently,

$$\|\mathcal{L}_{U_1}g\|^2 = \|FA_1 - A_1F\|^2. \tag{3.7}$$

On the other hand, (2.27) and (2.28) with $c = -4$ yield

$$\operatorname{Ric}(U_1, U_1) = -(n-1) + u^1(A_1U_1)(\operatorname{tr} A_1) - \|A_1U_1\|^2, \tag{3.8}$$

$$\operatorname{tr}(A_1^2) = -\rho - (n+3)(n-1) + n^2\|\mu\|^2 - \sum_{\alpha=2}^{p} \operatorname{tr} A_\alpha{}^2. \tag{3.9}$$

Substituting (3.3), (3.5), (3.7), (3.8), and (3.9) into (3.1), we have

$$\begin{aligned}\int_M \Big\{ &\frac{1}{2}\|FA_1 - A_1F\|^2 + \operatorname{Ric}(U_1, U_1) + \rho - n^2\|\mu\|^2 \\ &+ \|A_1U_1\|^2 + (n+3)(n-1) + \sum_{\alpha=2}^{p} \operatorname{tr} A_\alpha{}^2 \Big\} * 1 = 0,\end{aligned} \tag{3.10}$$

or equivalently,

$$\int_M \left\{ \frac{1}{2}\|FA_1 - A_1F\|^2 + u^1(A_1U_1)(\operatorname{tr} A_1) - \operatorname{tr} A_1^2 - (n-1) \right\} * 1 = 0. \tag{3.11}$$

Thus we have the following lemma.

Lemma 3.1. *Let M be an n-dimensional compact orientable* CR-*submanifold of* $(n-1)$ CR-*dimension in a complex hyperbolic space* $\mathrm{CH}^{(n+p)/2}$. *If the distinguished normal vector field ξ_1 is parallel with respect to the normal connection and if the inequality*

$$\operatorname{Ric}(U_1,U_1)+\rho-n^2\|\mu\|^2+\left\|A_1U_1\right\|^2+(n+3)(n-1)\geq 0 \tag{3.12}$$

holds on M, then

$$A_1F=FA_1 \tag{3.13}$$

and $A_\alpha=0$ for $\alpha=2,\dots,p$.

Corollary 3.2. *Let M be a compact orientable real hypersurface of* $\mathrm{CH}^{(n+1)/2}$ *over which the inequality*

$$\operatorname{Ric}(U_1,U_1)+\rho-n^2\|\mu\|^2+\left\|A_1U_1\right\|^2+(n+3)(n-1)\geq 0 \tag{3.14}$$

holds. Then M satisfies the commutativity condition (C).

Combining Lemma 3.1 and the codimension reduction theorem proved in [7, Theorem 3.2, page 126], we have the following theorem.

Theorem 3.3. *Let M be an n-dimensional compact orientable* CR-*submanifold of* $(n-1)$ CR-*dimension in a complex hyperbolic space* $\mathrm{CH}^{(n+p)/2}$. *If the distinguished normal vector field ξ_1 is parallel with respect to the normal connection and if the inequality*

$$\operatorname{Ric}(U_1,U_1)+\rho-n^2\|\mu\|^2+\left\|A_1U_1\right\|^2+(n+3)(n-1)\geq 0 \tag{3.15}$$

holds on M, then there exists a totally geodesic complex hyperbolic space $\mathrm{CH}^{(n+1)/2}$ *immersed in* $\mathrm{CH}^{(n+p)/2}$ *such that* $M\subset\mathrm{CH}^{(n+1)/2}$. *Moreover M satisfies the commutativity condition (C) as a real hypersurface of* $\mathrm{CH}^{(n+1)/2}$.

Proof. Let

$$N_0(x):=\{\eta\in T_xM^\perp \mid A_\eta=0\} \tag{3.16}$$

and let $H_0(x)$ be the maximal holomorphic subspace of $N_0(x)$, that is,

$$H_0(x)=N_0(x)\cap JN_0(x). \tag{3.17}$$

Then, by means of Lemma 3.1,

$$H_0(x)=N_0(x)=\operatorname{Span}\{\xi_2,\dots,\xi_p\}. \tag{3.18}$$

Hence, the orthogonal complement $H_1(x)$ of $H_0(x)$ in $TM^\perp$ is $\operatorname{Span}\{\xi_1\}$ and so, $H_1(x)$ is invariant under the parallel translation with respect to the normal connection and $\dim H_1(x)=1$ at any point $x\in M$. Thus, applying the codimension reduction theorem in [4] proved by Kawamoto, we verify that there exists a totally geodesic complex hyperbolic space $\mathrm{CH}^{(n+1)/2}$ immersed in $\mathrm{CH}^{(n+p)/2}$ such that $M\subset\mathrm{CH}^{(n+1)/2}$. Therefore, M can be regarded as a real hypersurface of $CH^{(n+1)/2}$ which is totally geodesic in $CH^{(n+p)/2}$. Tentatively, we denote $CH^{(n+1)/2}$ by M', and by i_1 we denote the immersion of M into M', and by i_2 the totally geodesic immersion of M' into $CH^{(n+p)/2}$. Then it is clear from (2.14) that

$$\nabla'_{i_1X} i_1 Y = i_1 \nabla_X Y + h'(X,Y) = i_1 \nabla_X Y + g(A'X,Y)\xi', \tag{3.19}$$

where ∇' is the induced connection on M' from that of $CH^{(n+p)/2}$, h' the second fundamental form of M in M', and A' the corresponding shape operator to a unit normal vector field ξ' to M in M'. Since $i = i_2 \circ i_1$ and M' is totally geodesic in $CH^{(n+p)/2}$, we can easily see that (2.15) and (3.19) imply that

$$\xi_1 = i_2\xi', \qquad A_1 = A'. \tag{3.20}$$

Since M' is a holomorphic submanifold of $CH^{(n+p)/2}$, for any X in TM,

$$Ji_2X = i_2J'X \tag{3.21}$$

is valid, where J' is the induced Kähler structure on M'. Thus it follows from (2.5) that

$$\begin{aligned} JiX &= Ji_2 \circ i_1X = i_2J'i_1X = i_2\big(i_1F'X + u'(X)\xi'\big) \\ &= iF'X + u'(X)i_2\xi' = iF'X + u'(X)\xi_1 \end{aligned} \tag{3.22}$$

for any vector field X tangent to M. Comparing this equation with (2.5), we have $F = F'$ and $u^1 = u'$, which, together with Lemma 3.1, implies that

$$A'F' = F'A'. \tag{3.23}$$

□

4. An integral formula on the model space $M^h_{2p+1,2q+1}(r)$

We first explain the model hypersurfaces of complex hyperbolic space due to Montiel and Romero for later use (for the details, see [12]).

Consider the complex $(n+3)/2$-space $C_1^{(n+3)/2}$ endowed with the pseudo-Euclidean metric g_0 given by

$$g_0 = -dz_0 d\bar{z}_0 + \sum_{j=1}^{m} dz_j d\bar{z}_j, \qquad \left(m+1 := \frac{n+3}{2}\right), \tag{4.1}$$

where $\bar{z}_k$ denotes the complex conjugate of z_k.

On $C_1^{(n+3)/2}$, we define

$$F(z,w) = -z_0\bar{w}_0 + \sum_{k=1}^{m} z_k\bar{w}_k. \tag{4.2}$$

Put

$$H_1^{n+2} = \left\{z = (z_0, z_1, \dots, z_m) \in C_1^{(n+3)/2} : \langle z,z\rangle = -1\right\}, \tag{4.3}$$

where $\langle\cdot,\cdot\rangle$ denotes the inner product on $C_1^{(n+3)/2}$ induced from g_0. Then it is known that H_1^{n+2}, together with the induced metric, is a pseudo-Riemannian manifold of constant sectional curvature -1, which is known as an anti-de Sitter space. Moreover, H_1^{n+2} is a principal S^1-bundle over $\mathrm{CH}^{(n+1)/2}$ with projection $\pi : H_1^{n+2} \to \mathrm{CH}^{(n+1)/2}$ which is a Riemannian submersion with fundamental tensor J and time-like totally geodesic fibers.

Given p, q integers with $2p+2q=n-1$ and $r \in R$ with $0<r<1$, we denote by $M_{2p+1,2q+1}(r)$ the Lorentz hypersurface of H_1^{n+2} defined by the equations

$$-|z_0|^2+\sum_{k=1}^{m}|z_k|^2=-1, \qquad r\left(-|z_0|^2+\sum_{k=1}^{p}|z_k|^2\right)=-\sum_{k=p+1}^{m}|z_k|^2, \tag{4.4}$$

where $z=(z_0,z_1,\dots,z_m)\in C_1^{(n+3)/2}$. In fact, $M_{2p+1,2q+1}(r)$ is isometric to the product

$$H_1^{2p+1}\left(\frac{1}{r-1}\right)\times S^{2q+1}\left(\frac{r}{1-r}\right), \tag{4.5}$$

where $1/(r-1)$ and $r/(1-r)$ denote the squares of the radii and each factor is embedded in H_1^{n+2} in a totally umbilical way. Since $M_{2p+1,2q+1}(r)$ is S^1-invariant, $M^h_{2p+1,2q+1}(r):=\pi(M_{2p+1,2q+1}(r))$ is a real hypersurface of $\mathrm{CH}^{(n+1)/2}$ which is complete and satisfies the condition (C).

As already mentioned in Section 1, Montiel and Romero [12] have classified real hypersurfaces M of $\mathrm{CH}^{(n+1)/2}$ which satisfy the condition (C) and obtained the following classification theorem.

Theorem 4.1. *Let M be a complete real hypersurface of $\mathrm{CH}^{(n+1)/2}$ which satisfies the condition (C). Then there exist the following possibilities.*

(1) *M has three constant principal curvatures $\tanh\theta$, $\coth\theta$, $2\coth 2\theta$ with multiplicities $2p$, $2q$, 1, respectively, $2p+2q=n-1$. Moreover, M is congruent to $M^h_{2p+1,2q+1}$ $(\tanh^2\theta)$.*

(2) *M has two constant principal curvatures λ_1,λ_2 with multiplicities $n-1$ and 1, respectively. (i) If $\lambda_1>1$, then $\lambda_1=\coth\theta$, $\lambda_2=2\coth 2\theta$ with $\theta>0$, and M is congruent to a geodesic hypersphere $M^h_{1,n}(\tanh^2\theta)$. (ii) If $\lambda_1<1$, then $\lambda_1=\tanh\theta$, $\lambda_2=2\coth 2\theta$ with $\theta>0$, and M is congruent to $M^h_{n,1}(\tanh^2\theta)$. (iii) If $\lambda_1=1$, then $\lambda_2=2$ and M is congruent to a horosphere.*

Combining Corollary 3.2 and Theorem 4.1, we have the following theorem.

Theorem 4.2. *Let M be a compact orientable real hypersurface of $\mathrm{CH}^{(n+1)/2}$ over which the inequality*

$$\operatorname{Ric}(U_1,U_1)+\rho-n^2\|\mu\|^2+\|A_1U_1\|^2+(n+3)(n-1)\geq 0 \tag{4.6}$$

holds. Then M is congruent to a geodesic hypersphere $M^h_{1,n}(r)$ in $\mathrm{CH}^{(n+1)/2}$.

Combining Theorems 3.3 and 4.2, we have the following theorem.

THEOREM 4.3. *Let M be an n-dimensional compact orientable* CR-*submanifold of* $(n-1)$ CR-*dimension in a complex hyperbolic space* $\mathrm{CH}^{(n+p)/2}$. *If the distinguished normal vector field ξ_1 is parallel with respect to the normal connection and if the inequality*

$$\operatorname{Ric}(U_1, U_1) + \rho - n^2\|\mu\|^2 + \left\|A_1 U_1\right\|^2 + (n+3)(n-1) \geq 0 \tag{4.7}$$

holds on M, then M is congruent to a geodesic hypersphere $M_{1,n}^h(\tanh^2\theta)$ in $\mathrm{CH}^{(n+1)/2}$.

Remark 4.4. As already shown in (3.10) and (3.11), the equality

$$\begin{aligned} &\operatorname{Ric}(U_1, U_1) + \rho - n^2\|\mu\|^2 + \left\|A_1 U_1\right\|^2 + (n+3)(n-1) \\ &\quad = u^1(A_1 U_1)(\operatorname{tr} A_1) - \operatorname{tr} A_1^2 - (n-1) \end{aligned} \tag{4.8}$$

holds on M. On the other hand, the geodesic hypersphere $M_{1,n}^h(\tanh^2\theta)$ in Theorem 4.1 has constant principal curvatures $\coth\theta$ and $2\coth 2\theta$ with multiplicities $n-1$ and 1, respectively. Hence we can easily verify the equality

$$u^1(A_1 U_1)(\operatorname{tr} A_1) - \operatorname{tr} A_1^2 - (n-1) = 0, \tag{4.9}$$

and consequently,

$$\operatorname{Ric}(U_1, U_1) + \rho - n^2\|\mu\|^2 + \left\|A_1 U_1\right\|^2 + (n+3)(n-1) = 0 \tag{4.10}$$

on $M_{1,n}^h(\tanh^2\theta)$.

Remark 4.5. If we put $V := \nabla_{U_1} U_1 - (\operatorname{div} U_1) U_1$, then it easily follows from (2.11) that $V = FA_1 U_1$. Taking account of (3.3), (3.5), (3.7), and (3.8), we obtain

$$\operatorname{div} V = \frac{1}{2}\left\|FA_1 - A_1 F\right\|^2 + u^1(A_1 U_1)(\operatorname{tr} A_1) - \operatorname{tr} A_1^2 - (n-1). \tag{4.11}$$

Hence if the commutativity condition (C) holds on M, then the vector field V is zero since U_1 is a principal vector of A_1, and consequently,

$$u^1(A_1 U_1)(\operatorname{tr} A_1) - \operatorname{tr} A_1^2 - (n-1) = 0. \tag{4.12}$$

Thus, on n-dimensional CR-submanifold M of $(n-1)$ CR-dimension in a complex hyperbolic space $\mathrm{CH}^{(n+p)/2}$ over which the commutativity condition C holds, the function $u^1(A_1 U_1)$ cannot be zero at any point of M. A real hypersurface of a complex hyperbolic space $\mathrm{CH}^{(n+p)/2}$ satisfying the commutativity condition (C) cannot be minimal.

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