

# A FINITE-DIMENSIONAL INTEGRABLE SYSTEM ASSOCIATED WITH A POLYNOMIAL EIGENVALUE PROBLEM

TAIXI XU, WEIHUA MU, AND ZHIJUN QIAO

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M. Antonowicz and A. P. Fordy (1988) introduced the second-order polynomial eigenvalue problem  $L\phi = (\partial^2 + \sum_{i=1}^n v_i \lambda^i)\phi = \alpha\phi$  ( $\partial = \partial/\partial x$ ,  $\alpha = \text{constant}$ ) and discussed its multi-Hamiltonian structures. For  $n = 1$  and  $n = 2$ , the associated finite-dimensional integrable Hamiltonian systems (FDIHS) have been discussed by Xu and Mu (1990) using the nonlinearization method and Bargmann constraints. In this paper, we consider the general case, that is,  $n$  is arbitrary, provide the constrained Hamiltonian systems associated with the above-mentioned second-order polynomial eigenvalue problem, and prove them to be completely integrable.

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## 1. Introduction

In classical mechanics one describes the equation of motion by Hamiltonian systems of the form [2]

$$\frac{dq_j}{dt} = \frac{\partial H}{\partial p_j} \quad \frac{dp_j}{dt} = -\frac{\partial H}{\partial q_j} \quad (j = 1, \dots, n), \quad (1.1)$$

where  $q = (q_1, \dots, q_n) \in \mathbb{R}^n$ ,  $p = (p_1, \dots, p_n) \in \mathbb{R}^n$ , and  $H = H(q, p)$  is a smooth function on an open domain  $\Omega$  of  $\mathbb{R}^{2n}$ .

It is customary to introduce the “Poisson bracket”  $\{F, G\}$  for two functions  $F, G \in C^1(\Omega)$  by [7]

$$\{F, G\} = \sum_{j=1}^n \left( \frac{\partial F}{\partial q_j} \frac{\partial G}{\partial p_j} - \frac{\partial F}{\partial p_j} \frac{\partial G}{\partial q_j} \right) = \langle F_q, G_p \rangle - \langle F_p, G_q \rangle. \quad (1.2)$$

$F, G$  are called in involution if  $\{F, G\} = 0$ .

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Using a notation borrowed from differential geometry we associate with (1.1) the “vector field” or first-order differential operator

$$X_H = \sum_{j=1}^n \left( H_{p_j} \frac{\partial}{\partial q_j} - H_{q_j} \frac{\partial}{\partial p_j} \right). \quad (1.3)$$

*Definition 1.1.* A nonconstant function  $F \in C^1(\Omega)$  is called an integral of  $X_H$  if

$$X_H F = \{F, H\} = 0. \quad (1.4)$$

*Definition 1.2.* A Hamiltonian vector field  $X_H$  in  $\Omega \subset \mathbb{R}^{2n}$  is called “integrable” if it possesses  $n$  integrals  $F_j \in C^1(\Omega)$  satisfying the following conditions

- (i)  $\{F_j, H\} = 0$ ,
- (ii)  $\{F_j, F_k\} = 0$ ,
- (iii) the gradients  $dF_j$  are linearly independent in  $\Omega$ .

The first condition expresses that the  $F_j$  are integrals; the second one means that any two such integrals commute. The third condition is a requirement for nondegeneracy, which we will have to relax frequently.

Since Cao introduced the nonlinearization method to search for finite-dimensional completely integrable Hamiltonian systems [3–6] associated with soliton equations, numerous such systems have been obtained by many mathematicians [8–12, 14, 15].

In this paper, we consider the integrable systems associated with the polynomial eigenvalue problem

$$L\phi = (\partial^2 + \sum_{i=1}^n v_i \lambda^i) \phi = \alpha \phi (\partial = \partial/\partial x, \alpha = \text{constant}). \quad (1.5)$$

When  $n = 1$ , denote  $v_1$  by  $v$ , (1.5) becomes

$$\phi_{xx} + \lambda v \phi = \alpha \phi. \quad (1.6)$$

Equation (1.6) is associated with the Harry-Dym (HD) equation

$$v_t = \left( \frac{1}{2\sqrt{v}} \right)_{xxx} + \left( \frac{2\alpha}{\sqrt{v}} \right)_x. \quad (1.7)$$

*Remark 1.3.* Actually, spectral problem (1.6) generates the Camassa-Holm (CH) equation in its negative-order hierarchy, whereas it produces the HD equation (1.7) in its positive-order hierarchy [9]. Both hierarchies are integrable. When  $n = 2$ ,  $v_1 = u$ ,  $v_2 = v$ , (1.5) becomes

$$\phi_{xx} + (\lambda u + \lambda^2 v) \phi = \alpha \phi. \quad (1.8)$$

Equation (1.8) is associated with the following coupled Harry-Dym (CHD) equation:

$$\begin{aligned} u_t &= \left( \frac{1}{2\sqrt{v}} \right)_{xxx} - 2\alpha \left( \frac{1}{\sqrt{v}} \right)_x, \\ v_t &= 2u \frac{1}{\sqrt{v}} + u_x \frac{1}{\sqrt{v}}. \end{aligned} \quad (1.9)$$

We have already obtained the integrable Hamiltonian systems associated with Harry-Dym and coupled Harry-Dym equations [13]. For general positive integer  $n$ , the associated integrable system is given in this paper. In the next section, we give the Hamiltonian system associated with the polynomial eigenvalue problem, and in Section 3, we obtain the involutive integrals and prove they are linearly independent.

## 2. The Hamiltonian system

Consider the evolution equation

$$\phi_{t_m} = -\frac{1}{2}B_x^{(m)}\phi + B^{(m)}\phi_x, \quad (2.1)$$

where

$$B^{(m)} = \sum_{j=0}^{m-1} b_j \lambda^{m-j}, \quad b_{-j} = 0 \quad (j = 1, 2, \dots). \quad (2.2)$$

From the solvability condition of (1.5) and (2.1), the hierarchy of evolution equations of potentials  $v = (v_1, \dots, v_n)^T$  can be written as

$$v_i = \begin{bmatrix} 0 & \cdots & 0 & J_0 \\ \vdots & \cdot & \cdot & J_1 \\ 0 & \cdot & \cdot & \vdots \\ J_0 & J_1 & \cdots & J_{n-1} \end{bmatrix} \begin{bmatrix} b_{m-n} \\ b_{m-n-1} \\ \vdots \\ b_{m-1} \end{bmatrix}. \quad (2.3)$$

Also, from the solvability condition, it is found that  $b_k$  satisfies

$$J_0 b_j + J_1 b_{j+1} + \cdots + J_n b_{j+n} = 0, \quad (2.4)$$

or

$$K(b_j, b_{j+1}, \dots, b_{j+n-1})^T = J(b_{j+1}, b_{j+2}, \dots, b_{j+n})^T, \quad (2.5)$$

or

$$KG_j = JG_{j+1}, \quad (2.6)$$

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where

$$K = \begin{bmatrix} 0 & \cdots & 0 & J_0 \\ \vdots & \cdot & \cdot & J_1 \\ 0 & \cdot & \cdot & \vdots \\ J_0 & J_1 & \cdots & J_{n-1} \end{bmatrix}, \quad J = \begin{bmatrix} 0 & \cdots & 0 & J_0 & 0 \\ \vdots & \cdot & \cdot & J_1 & 0 \\ 0 & \cdot & \cdot & \vdots & \vdots \\ J_0 & J_1 & \cdots & J_{n-2} & 0 \\ 0 & 0 & \cdots & 0 & -J_n \end{bmatrix} \quad (2.7)$$

are the Lenard pair of operators

$$G_j = (b_j, b_{j+1}, \dots, b_{j+n-1}), \quad (2.8)$$

$$J_0 = \frac{1}{2}\partial^3 - 2\alpha\partial, \quad J_i = v_i\partial + \partial v_i \quad (i = 1, 2, \dots, n).$$

It is evident that if  $\phi$  is a solution of (1.5), then

$$\sum_{i=0}^n \lambda^i J_i \phi^2 = 0. \quad (2.9)$$

Rewrite it as

$$\sum_{i=0}^n \lambda^i J_i P = 0. \quad (2.10)$$

Letting  $P = \sum_{j=0}^{\infty} P_j \lambda^{-j}$ ; we find that  $P_j$  satisfies the same relationship (2.4) as  $b_j$  does. Multiplying both sides of (2.10) by  $P$  and integrating it once, we get

$$P_{xx}P - \frac{1}{2}P_x^2 - 2\alpha P^2 + 2 \sum_{i=1}^n \lambda^i v_i P^2 = C(\lambda). \quad (2.11)$$

Set  $P_0 = V_n^{-1/2}$ ,  $C(\lambda) = \lambda^n$ . By substituting  $P = \sum_{j=0}^{\infty} P_j \lambda^{-j}$  into (2.11), we find that  $P_j = b_j$ ,

$$b_{k+n}b_0^{-1} = -\frac{1}{4} \sum_{j=0}^k b_{jxx}b_{k-j} + \frac{1}{8} \sum_{j=0}^k b_{jx}b_{k-j,x} + \frac{\alpha}{2} \sum_{j=0}^k b_j b_{k-j} \quad (2.12)$$

$$- \frac{1}{2} \sum_{i=0}^{n-1} \sum_{j=0}^{k+i} b_j b_{k+i-j} - \frac{1}{2} v_n \sum_{j=1}^{k+n-1} b_j b_{k+n-j} \quad (k = 1, 2, \dots).$$

**PROPOSITION 2.1.** *Let  $\lambda_j$  be an eigenvalue of (1.5) and  $\phi_j$  an eigenfunction corresponding to  $\lambda_j$ . Then*

$$\text{grad } \lambda_j = \left( \frac{\delta \lambda_j}{\delta v_1}, \dots, \frac{\delta \lambda_j}{\delta v_1} \right)^T = (\lambda_j \phi_j^2, \lambda_j^2 \phi_j^2, \dots, \lambda_j^n \phi_j^2)^T, \quad (2.13)$$

$$K \text{ grad } \lambda_j = \lambda_j J \text{ grad } \lambda_j.$$

Define the Lenard sequence recursively:  $G_0 = (b_0, \dots, b_{n-1})^T$ ,  $KG_{j-1} = JG_j$  ( $j = 1, 2, \dots$ ),  $X_j = JG_j$  ( $j = 0, 1, 2, \dots$ ) are the CHD vector fields.

Let

$$G_0 = \sum_{j=1}^N \text{grad} \lambda_j. \quad (2.14)$$

Then

$$b_j = \langle \Lambda^{j+1} \phi, \phi \rangle \quad (j = 0, 1, 2, \dots, n-1), \quad (2.15)$$

where  $\langle \cdot, \cdot \rangle$  is the standard inner-product in  $\mathbb{R}^N$ ,  $\Lambda = \text{diag}(\lambda_1, \dots, \lambda_N)$ .

From (2.4), (2.14), and (2.15), we have

$$\sum_{j=0}^{k-1} J_{n-j} \langle \Lambda^{k-j} \phi, \phi \rangle = 0 \quad (k = 1, 2, \dots, n), \quad (2.16)$$

which yields

$$v_{n-k} = \langle \Lambda \phi, \phi \rangle^{-1} \partial^{-1} \langle \Lambda \phi, \phi \rangle [J_n \langle \Lambda^k \phi, \phi \rangle + \dots + J_{n-k+2} \langle \Lambda^2 \phi, \phi \rangle]. \quad (2.17)$$

By making use of the recursion formula of  $v_k$ , we have the following proposition.

**PROPOSITION 2.2.** *The constraint between the potentials and the eigenfunctions (1.5) is of the form*

$$v_n = \langle \Lambda \phi, \phi \rangle^{-2}, \quad (2.18)$$

$$v_{n-k} = \sum_{j=1}^k a_j \langle \Lambda \phi, \phi \rangle^{-(j+2)} \sum_{l_1 + \dots + l_j = k-j} \langle \Lambda^{l_1+2} \phi, \phi \rangle \dots \langle \Lambda^{l_j+2} \phi, \phi \rangle \quad (k = 1, 2, \dots, n-1), \quad (2.19)$$

where  $a_j = (-1)^j(j+1)$ ,  $a_0 = 1$ , and  $\phi_j, \lambda_j$  satisfy (1.5).

We now consider the following system instead of (1.5):

$$\phi_{jxx} + \sum_{i=1}^n v_i \lambda_j^i \phi_j = \alpha \phi_j \quad (j = 1, 2, \dots, N), \quad (2.20)$$

where  $\lambda_j \neq \lambda_k$  when  $j \neq k$ . Let

$$q = (q_1, q_2, \dots, q_N)^T = (\phi_1, \phi_2, \dots, \phi_N)^T; \quad (2.21)$$

then (2.20) can be condensed as

$$q_{xx} + \sum_{i=1}^n v_i \Lambda^i q = \alpha q. \quad (2.22)$$

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By using the identity

$$\sum_{i=1}^k \langle \Lambda^{k-j} q, q \rangle \sum_{j=1}^i a_j \gamma_{i,j} = \sum_{i=1}^k a_j \sum_{i=0}^{k-j} \langle \Lambda^i q, q \rangle \gamma_{k-i,j} \quad (2.23)$$

and substituting (2.18), (2.19) into (2.22), we get

$$p_x = \alpha q - \sum_{i=0}^{n-1} a_j \langle \Lambda q, q \rangle^{i+2} \cdot \sum_{l_1+\dots+l_{i+1}} \langle \Lambda^{l_1+2} q, q \rangle \cdots \langle \Lambda^{l_{i+2}} q, q \rangle \Lambda^{l_{i+1}+2} q, \quad (2.24)$$

$$q_x = p,$$

which can be written in canonical Hamiltonian system

$$q_x = \frac{\partial H_0}{\partial p}, \quad p_x = -\frac{\partial H_0}{\partial q}, \quad (2.25)$$

where  $p = (p_1, \dots, p_N)^T = (q_{1x}, \dots, q_{Nx})^T$ ,

$$H_0 = \frac{1}{2} \langle p, p \rangle - \frac{\alpha}{2} \langle q, q \rangle + \frac{1}{2} \sum_{i=0}^{n-2} \bar{b}_i \langle \Lambda q, q \rangle^{-(i+2)} \cdot \sum_{l_1+\dots+l_{i+1}=n-2-i} \langle \Lambda^{l_1+2} q, q \rangle \cdots \langle \Lambda^{l_{i+1}+2} q, q \rangle$$

$$\bar{b}_i = \frac{a_i}{i+1} = (-1)^i, \quad i = 0, 1, \dots \quad (2.26)$$

### 3. Involutivity and integrability

Consider the constraint of (2.12)

$$F_{n+k} = \left[ \frac{b_{k+n}}{b_0} + \frac{1}{4} \sum_{j=0}^k b_{jxx} b^{k-j} - \frac{1}{8} \sum_{j=0}^k b_{jx} b_{k-j,x} \right. \\ \left. + \frac{1}{2} \sum_{i=1}^{n-1} v_i \sum_{j=0}^{k+i} b_j b_{k+i-j} - \frac{\alpha}{2} \sum_{j=0}^k b_j^{k-j} + \frac{1}{2} v_n \sum_{j=1}^{k+n-1} b_j b_{k+n-j} \right] \Big|_A, \quad (3.1)$$

where subscript  $A$  means to substitute  $b_j = \langle \Lambda^{j+1} q, q \rangle$  into (2.12). So

$$F_{n+k} = \frac{1}{2} \sum_{j=0}^{k-1} \langle \Lambda^{j+1} q, p_x \rangle \langle \Lambda^{k-j-1} q, q \rangle + \frac{1}{2} \sum_{j=0}^k \langle \Lambda^{j+1} p, p \rangle \langle \Lambda^{k-j+1} q, q \rangle \\ + \frac{1}{2} \langle \Lambda^{k+1} q, p_x \rangle \langle \Lambda q, q \rangle - \frac{1}{2} \langle \Lambda^{j+1} q, p \rangle \langle \Lambda^{k-j+1} p, q \rangle \\ - \frac{\alpha}{2} \sum_{j=0}^k \langle \Lambda^{j+1} q, q \rangle \langle \Lambda^{k-j+1} q, q \rangle + \frac{1}{2} \sum_{i=0}^{n-1} v_i \sum_{j=0}^{k+i} \langle \Lambda^{j+1} q, q \rangle \langle \Lambda^{k+i-j+1} q, q \rangle \\ + \frac{1}{2} v_n \sum_{j=1}^{k+n-1} \langle \Lambda^{j+1} q, p \rangle \langle \Lambda^{k+i-j+1} q, q \rangle + \langle \Lambda q, q \rangle^{-1} \langle \Lambda^{k+n+1} q, q \rangle. \quad (3.2)$$

Through direct calculations from (2.18), (2.19), and (2.24), we have

$$\begin{aligned}
 F_{n+k} &= \frac{1}{2} \sum_{i=0}^{n-1} \bar{b}_i \langle \Lambda q, q \rangle^{-(i+1)} \cdot \sum_{l_1+\dots+l_{i+1}=n-1-i} \langle \Lambda^{l_1+2} q, q \rangle \cdots \langle \Lambda^{l_i+2} q, q \rangle \langle \Lambda^{i+1+k+2} q, q \rangle \\
 &+ \frac{1}{2} \sum_{j=0}^k [\langle \Lambda^{j+1} p, p \rangle \langle \Lambda^{k-j+1} q, q \rangle - \langle \Lambda^{j+1} q, q \rangle \langle \Lambda^{k-j+1} p, p \rangle], \quad k = 1, 2, \dots
 \end{aligned} \tag{3.3}$$

Set

$$\begin{aligned}
 G_k &= \frac{1}{2} \sum_{j=0}^k [\langle \Lambda^{j+1} p, p \rangle \langle \Lambda^{j+1} q, q \rangle - \langle \Lambda^{j+1} q, p \rangle \langle \Lambda^{k-j+1} p, q \rangle], \\
 Q_k &= F_{k+n} - G_k.
 \end{aligned} \tag{3.4}$$

It is known (see Cao [3]) that  $G_k$  are in involution. Using the identity

$$\begin{aligned}
 &\sum_{i=0}^l \langle \Lambda^{l+k+j-i} p, p \rangle \langle \Lambda^i q, q \rangle + \sum_{i=0}^k \langle \Lambda^i p, p \rangle \langle \Lambda^{l+k+j-i} q, q \rangle \\
 &= \sum_{i=0}^{l+k+j} \langle \Lambda^{l+k+j-i} p, p \rangle \langle \Lambda^i q, q \rangle - \sum_{i=l+1}^{l+j-1} \langle \Lambda^{l+k+j-i} p, p \rangle \langle \Lambda^i q, q \rangle,
 \end{aligned} \tag{3.5}$$

we can show by straightforward calculations that

$$\{Q_k, G_l\} + \{G_k, Q_l\} + \{Q_k, Q_l\} = 0. \tag{3.6}$$

So

$$\{F_{k+n}, F_{l+n}\} = 0. \tag{3.7}$$

Since all  $\lambda_k$  are distinct, the Vandermonde determinant of  $\lambda_1, \lambda_2, \dots, \lambda_N$  is not zero. Then it is easy to see that

$$\text{grad} F_{n+k} = \left( \frac{\partial F_{n+k}}{\partial q_1}, \dots, \frac{\partial F_{n+k}}{\partial q_N}, \frac{\partial F_{n+k}}{\partial p_1}, \dots, \frac{\partial F_{n+k}}{\partial p_N} \right), \quad k = 1, 2, \dots, \tag{3.8}$$

are functionally independent. So we have the following proposition.

**PROPOSITION 3.1.** *The Hamiltonian system  $(\mathbb{R}^{2N}, dp \wedge dq, H_0)$  is completely integrable in the sense of Liouville.*

Consider the systems obtained from (2.1)

$$\phi_{jt_m} = -\frac{1}{2} R_x^{(m)} \phi_j + R^{(m)} \phi_{jx} \quad (j = 1, 2, \dots, N). \tag{3.9}$$

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Substituting  $b - j = \langle \Lambda^{j+1} q, q \rangle$  ( $j = 1, 2, \dots$ ) into it, we have

$$\begin{aligned} q_{t_m} &= \sum_{j=0}^{m-1} [\langle \Lambda^{j+1} q, q \rangle \Lambda^{m-j} p - \langle \Lambda^{j+1} q, p \rangle \Lambda^{m-j} q], \\ p_{t_m} &= \frac{\partial}{\partial x} q_{t_m} = \sum_{j=0}^{m-1} [\langle \Lambda^{j+1} q, p \rangle \Lambda^{m-j} p - \langle \Lambda^{j+1} p, p \rangle \Lambda^{m-j} q] \\ &\quad + \sum_{j=0}^{m-1} [\langle \Lambda^{j+1} q, q \rangle \Lambda^{m-j} p_x - \langle \Lambda^{j+1} q, p_x \rangle \Lambda^{m-j} q]. \end{aligned} \quad (3.10)$$

Through direct calculation from (2.24), (3.10) can be written in canonical Hamiltonian system

$$q_{t_m} = \frac{\partial F_{n+m-1}}{\partial p}, \quad p_{t_m} = -\frac{\partial F_{n+m-1}}{\partial q}. \quad (3.11)$$

**PROPOSITION 3.2.** *The Hamiltonian systems in the last equation are completely integrable in the sense of Liouville, and if  $(p, q)$  satisfies (2.25) and (3.11), then  $v$  given by (2.18) and (2.19) is a solution of CHD equation.*

*Proof.* Since  $F_k$  are in involution, the systems (3.11) ( $m = 1, 2, \dots$ ) are completely integrable. Observe that (2.3) is deduced from the solvability condition of (2.22) and (3.9); (2.25) and (3.11) are obtained by substituting (2.18) and (2.19) into (2.22) and (3.9), respectively. It is easy to see that if  $(q, p)$  satisfies both (2.25) and (3.11), then  $v$  given by (2.18) and (2.19) is a solution of CHD equation.  $\square$

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Taixi Xu: Department of Mathematics, Southern Polytechnic State University,  
 1100 South Marietta Parkway, Marietta, GA 30060, USA  
*E-mail address:* txu@spsu.edu

Weihua Mu: Department of Mathematics, Shijiazhuang Railway Institute, Hebei 050043, China  
*E-mail address:* slx@sjzri.edu.cn

Zhijun Qiao: Department of Mathematics, University of Texas – Pan American,  
 1201 W. University Drive Edinburg, TX 78541, USA  
*E-mail address:* qiao@utpa.edu