SPACELIKE HYPERSURFACES IN DE SITTER SPACE WITH CONSTANT HIGHER-ORDER MEAN CURVATURE

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The authors apply the generalized Minkowski formula to set up a spherical theorem. It is shown that a compact connected hypersurface with positive constant higher-order mean curvature H_r for some fixed r, $1 \leq r \leq n$, immersed in the de Sitter space S_1^{n+1} must be a sphere.

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1. Introduction

The classical Liebmann theorem states that a connected compact surface with constant Gauss curvature or constant mean curvature in $\mathbb{R}^3$ is a sphere. The natural generalizations of the Gauss curvature and mean curvature are the rth mean curvature H_r, $r = 1,\dots,n$, which are defined as the rth elementary symmetric polynomial in the principal curvatures of M. Later many authors [1, 4, 5, 7, 8] have generalized Liebmann theorem to the cases of hypersurfaces with constant higher-order mean curvature in the Euclidian space, hyperbolic space, the sphere, and so on. A significant result due to Ros [8] is that a compact hypersurface with the rth constant mean curvature H_r, for some $r = 1,\dots,n$, embedded into the Euclidian space must be a sphere.

The purpose of this note is to prove a spherical theorem of the Liebmann type for the compact spacelike hypersurface immersed in the de Sitter space by setting up a generalized Minkowski formula. The main result is the following.

Theorem 1.1. *Let M be a compact connected hypersurface immersed in the de Sitter space S_1^{n+1}. If for some fixed r, $1 \leq r \leq n$, the rth mean curvature H_r is a positive constant on M, then M is isometric to a sphere.*

For the cases of the constant mean curvature and constant scalar curvature, that is, $r = 1,2$, the theorem was founded by Montiel [4] and Cheng and Ishikawa [1], respectively.

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2. Preliminaries

Let $\mathbb{R}_1^{n+2}$ be the real vector space $\mathbb{R}^{n+2}$ endowed with the Lorentzian metric $\langle \cdot, \cdot \rangle$ given by

$$\langle x, y \rangle = -x_0 y_0 + \sum_{i=1}^{n+2} x_i y_i \tag{2.1}$$

for $x, y \in \mathbb{R}^{n+2}$. The de Sitter space $S_1^{n+1}(c)$ can be defined as the following hyperquadratic:

$$S_1^{n+1}(c) = \left\{ x \in \mathbb{R}_1^{n+2} \mid |x|^2 = \frac{1}{c}, \ \frac{1}{c} > 0 \right\}. \tag{2.2}$$

In this way, the de Sitter space inherits from $\langle \cdot, \cdot \rangle$ a metric which makes it an indefinite Riemannian manifold of constant sectional curvature c. If $x \in S_1^{n+1}(c)$, we can put

$$T_x S_1^{n+1}(c) = \{ v \in \mathbb{R}_1^{n+2} \mid \langle v, x \rangle = 0 \}. \tag{2.3}$$

Let $\psi : M \to S_1^{n+1}$ be a connected spacelike hypersurface immersed in the de Sitter space with the sectional curvature 1. Following O'Neill [6], the unit normal vector field N for ψ can be viewed as the Gauss map of M:

$$N : M \longrightarrow \{ x \in \mathbb{R}_1^{n+2} \mid |x|^2 = -1 \}. \tag{2.4}$$

Let $S_r : \mathbb{R}^n \to \mathbb{R}$, $r = 1, \ldots, n$, be the normalized rth elementary symmetric function in the variables $y_1, \ldots, y_n$. For $r = 1, \ldots, n$, we denote by C_r the connected component of the set $\{ y \in \mathbb{R}^n \mid S_r(y) > 0 \}$ containing the vector $y = (1, \ldots, 1)$. Notice that every vector $(y_1, \ldots, y_n)$ with all its components greater than zero lies in each C_r. We derive the following two lemmas, which will be needed for the proof of the theorem.

Lemma 2.1 [3]. (i) *If* $r \geq k$, *then* $C_r \subset C_k$; (ii) *for* $y \in C_r$,

$$S_r^{1/r} \leq S_{r-1}^{1/r-1} \leq \cdots \leq S_2^{1/2} \leq S_1. \tag{2.5}$$

Lemma 2.2 (Minkowski formula). *Let* $\psi : M \to S_1^{n+1} \subset \mathbb{R}_1^{n+2}$ *be a connected spacelike hypersurface immersed in de Sitter space* S_1^{n+1}. *For the* r*th mean curvature* H_r *of* ψ, $r = 0, 1, \ldots, n-1$,

$$\int_M \left(H_r \langle \psi, a \rangle + H_{r+1} \langle N, a \rangle \right) dV = 0, \tag{2.6}$$

where $H_0 = 1$ *and* $a \in \mathbb{R}_1^{n+1}$ *is an arbitrary fixed vector and* N *is the unit normal vector of* M.

Proof. The argument is based on the approach of geodesic parallel hypersurfaces in [5]. Let k_r and e_i, $i = 1, \ldots, n$, be the principal curvatures and the principal directions at a point $p \in M$. The rth mean curvature of ψ is defined by the identity

$$P_n(t) = (1 + tk_1) \cdots (1 + tk_n) = 1 + \binom{n}{1} H_1 t + \cdots + \binom{n}{n} H_n t^n \tag{2.7}$$

for all $t \in \mathbb{R}$. Thus $H_1 = H$ is the mean curvature, $H_2 = (n^2H^2 - S)/n(n-1)$, where S is the square length of the second fundamental form and H_n is the Gauss-Kronecker curvature of M immersed in S_1^{n+1}. Let us consider a family of geodesic parallel hypersurfaces ψ_t given by

$$\psi_t(p) = \exp_{\psi(p)}(-tN(p)) = \cosh t \cdot \psi(p) + \sinh t \cdot N(p). \tag{2.8}$$

Then the unit normal vector field of ψ_t with $|N_t|^2 = -1$ can be written as

$$N_t(p) = -\sinh t \cdot \psi(p) - \cosh t \cdot N(p). \tag{2.9}$$

Because we have

$$\begin{aligned} \psi_{t_*}(e_i) &= (\cosh t - k_i \sinh t)(e_i), \\ N_{t_*}(e_i) &= (-\sinh t + k_i \cosh t)(e_i); \end{aligned} \tag{2.10}$$

for the principal directions $\{e_i\}$, $i = 1,\dots,n$ and $|t| < \varepsilon$, the second fundamental form of ψ_t can be expressed as

$$\begin{aligned} \sigma_t(\psi_{t_*}(e_i), \psi_{t_*}(e_j)) &= -\langle N_{t_*}(e_i), \psi_{t_*}(e_j)\rangle \\ &= (\sinh t - k_i \cosh t)\langle e_i, \psi_{t_*}(e_j)\rangle \\ &= \frac{\sinh t - k_i \cosh t}{\cosh t - k_i \sinh t}\langle \psi_{t_*}(e_i), \psi_{t_*}(e_j)\rangle. \end{aligned} \tag{2.11}$$

Then the mean curvature $H(t)$ of ψ can be expressed as

$$\begin{aligned} H(t) &= \frac{1}{n}\sum_{i=1}^{n} k_i(t) = \frac{1}{n}\sum_{i=1}^{n} \frac{\tanh t - k_i}{1 - k_i \tanh t} \\ &= \frac{1}{nP_n(-\tanh t)} \sum_{i=1}^{n} (\tanh t - k_i) \prod_{j \neq i} (1 - k_j \tanh t). \end{aligned} \tag{2.12}$$

But

$$\prod_{j \neq i} (1 - k_j \tanh t) = nP_n(-\tanh t) - \cosh t \sinh t P_n'(-\tanh t). \tag{2.13}$$

Then we get

$$H(t) = \tanh t + \frac{P_n'(-\tanh t)}{nP_n(-\tanh t)}. \tag{2.14}$$

By the way, we must point out that the formula (7′) in [5] is incorrect because the second term in the right-hand side of the expression of $H(t)$ should be $P_n'(\tanh t)/nP_n(\tanh t)$. The volume element dV_t for immersion ψ_t can be given by

$$\begin{aligned} dV_t &= (\cosh t - k_1 \sinh t)\cdots(\operatorname{conh} t - k_n \sinh t)\,dV \\ &= -\operatorname{conh}^n t P_n(-\tanh t)\,dV, \end{aligned} \tag{2.15}$$

where dV is the volume element of ψ. It is an easy computation that

$$\triangle(\langle\psi,a\rangle + H\langle N,a\rangle) = 0, \tag{2.16}$$

where N is a unit normal field of ψ and $a \in \mathbb{R}_1^{n+2}$ an arbitrary fixed vector (cf. [4, page 914]). Integrating both sides of (2.16) over the hypersurface M and applying Stoke's theorem, we get

$$\int_M (\langle\psi,a\rangle + H_1\langle N,a\rangle)dV = 0. \tag{2.17}$$

For $\psi_t, |t| < \varepsilon$, we obtain

$$\int_M \Big(\langle\psi_t,a\rangle + H(t)\langle N_t,a\rangle\Big)dV_t = 0. \tag{2.18}$$

Substituting (2.14) and (2.15) into (2.18), we get

$$\begin{aligned}
&\int_M \langle\psi_t,a\rangle + H(t)\langle N_t,a\rangle dV_t \\
&\quad = \frac{1}{n}\cosh^{n-1} t\int_M ((nP_n(-\tanh t) - \sinh t\cosh tP_n^{'}(-\tanh t))\langle\psi,a\rangle \\
&\qquad\qquad - \cosh^2 tP_n^{'}(-\tanh t)\langle N,a\rangle)dV = 0.
\end{aligned} \tag{2.19}$$

By using the expression

$$\begin{aligned}
&nP_n(-\tanh t) - \sinh t\cosh tP_n^{'}(-\tanh t) \\
&\quad = n + (n-1)\binom{n}{1}H_1(-\tanh t) + \cdots + n\binom{n}{n-1}H_n(-\tanh t)^{n-1},
\end{aligned} \tag{2.20}$$

we obtain

$$\begin{aligned}
&\int_M \{(nP_n(-\tanh t) - \sinh t\cosh tP_n^{'}(-\tan t))\langle\psi,a\rangle - \operatorname{conh}^2 tP_n^{'}(-\tanh t)\langle N,a\rangle\}dV \\
&\quad = \sum_{r=1}^{n}(n-r-1)\binom{n}{r-1}(-\tanh t)^{r-1}, \\
&\int_M (H_{r-1}\langle\psi_t,a\rangle + H_r\langle N_t,a\rangle)dV = 0.
\end{aligned} \tag{2.21}$$

The left-hand side in the equality is a polynomial in the variable $\tanh t$. Therefore, all its coefficients are null. This completes the proof of Lemma 2.2. □

3. Proof of Theorem 1.1

Here we work for the immersed hypersurfaces in S_1^{n+1} instead of embedded hypersurfaces because we can only use algebraic inequalities and the integral formula above to complete the proof. Let some H_r be a positive constant. Multiplying (2.17) by H_r and then abstracting from (2.6), we obtain that

$$\int_M (H_1H_r - H_{r+1})\langle N,a\rangle dV = 0. \tag{3.1}$$

We know from Newton inequality [2] that $H_{r-1}H_{r+1} \leq H_r^2$, where the equality implies that $k_1 = \cdots = k_n$. Hence

$$H_{r-1}(H_1H_r - H_{r+1}) \geq H_r(H_1H_{r-1} - H_r). \tag{3.2}$$

It derives from Lemma 2.1 that

$$0 \leq H_r^{1/r} \leq H_{r-1}^{1/r-1} \leq \cdots \leq H_2^{1/2} \leq H_1. \tag{3.3}$$

Thus we conclude that

$$H_{r-1}(H_1H_r - H_{r+1}) \geq H_r(H_1H_{r_1} - H_r) \geq 0, \tag{3.4}$$

and if $r \geq 2$, the equalities happen only at umbilical points of M. We choose a constant vector a such that $|a|^2 = -1$ and $a_0 \leq -1$. Since the normal vector N satisfies $|N|^2 = -1$, we have $\langle N,a\rangle \geq 1$ on M. It follows from (3.1) that

$$H_1H_r - H_{r+1} = 0. \tag{3.5}$$

Thus, $k_1 = \cdots = k_n$, M is totally umbilical, and M is isometric to a sphere. This ends the proof of Theorem 1.1.

If there is a convex point on M, that is, a point at which $k_i > 0$, for all $i = 1,\ldots,n$, then the constant rth mean curvature H_r is positive. By means of the same argument as that of Theorem 1.1, we derive the following.

Corollary 3.1. *Let M be a compact connected hypersurface immersed in the de Sitter space S_1^{n+1}. If for some fixed r, $1 \leq r \leq n$, the rth mean curvature H_r is constant, and there is a convex point on M, then M is isometric to a sphere.*

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