A NEW PROOF OF A LEMMA BY PHELPS

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We give a different proof of a lemma by Phelps (1960) which asserts, roughly speaking, that if two norm-one functionals f and g have their hyperplanes $f^{-1}(0)$ and $g^{-1}(0)$ sufficiently close together, then either $\|f-g\|$ or $\|f+g\|$ must be small. We also extend this result to a complex Banach space.

In 1960 in [2], Phelps proved the following lemma.

Lemma 1. *Suppose that E is a real normed linear space and that $\epsilon > 0$. If $f,g \in S^*$ are such that $f^{-1}(0) \cap \mathcal{U} \subset g^{-1}[-\epsilon/2, \epsilon/2]$, then either $\|f-g\| \leq \epsilon$ or $\|f+g\| \leq \epsilon$. (Here, $\mathcal{U}$ represents the unit ball of E and S^* is the unit sphere of E^*.)*

This lemma was then used the following year as a crucial step in the proof of the well-known Bishop-Phelps theorem [1] that every Banach space is subreflexive; in other words, every functional on a Banach space E can be approximated by a norm-attaining functional on the same space. The original proof of this lemma uses the Hahn-Banach theorem and is therefore fairly abstract.

In this note, we present an alternate proof for Lemma 1 stated above. This proof gives a geometric argument while extending the lemma to a complex Banach space. Lemma 1 is shown to be a special case when the bound of ϵ on $\|f \pm g\|$ is replaced by 5ϵ. This replacement does not affect the fundamental conclusion of Lemma 1.

We now state the extended lemma.

Lemma 2. *Let X be a complex Banach space and let ϵ be such that $0 < \epsilon < 1/2$. Let $\varphi, \psi \in X^*$, $\|\varphi\| = \|\psi\| = 1$. Suppose that for all $x \in X$ with $\|x\| \leq 1$ and $\varphi(x) = 0$, it holds that $|\psi(x)| \leq \epsilon$. Then there is some complex number α such that $|\alpha| = 1$ and $\|\varphi - \alpha\psi\| \leq 5\epsilon$.*

It will be shown that if φ and ψ are real-valued functionals on a real Banach space X, then α will in fact be either 1 or -1, thus proving the amended original result.

We now prove Lemma 2.

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Proof. Let $e \in X$ be such that $\|e\| = 1$ and $|\varphi(e)| \geq 1 - \epsilon/4$. We will first show that $|\psi(e)| \geq 1 - (5/2)\epsilon$. To see this, let $f \in X$ such that $\|f\| = 1$ and $|\psi(f)| \geq 1 - \epsilon/4$. Let $k = 1 - \epsilon/4$ and let $t = \varphi(f)/\varphi(e)$. Then $0 \leq |t| \leq 1/(1 - \epsilon/4) = 1/k \leq 8/7$ and if we take $w = (k/(k+1))(f - te)$, then $\|w\| \leq (k/(k+1))(\|f\| + |t|\|e\|) \leq (k/(k+1))(1 + 1/k) = 1$. Moreover,

$$\varphi(w) = \frac{k}{k+1}\left(\varphi(f) - \frac{\varphi(f)}{\varphi(e)}\varphi(e)\right) = 0 \tag{1}$$

so we have

$$\begin{aligned} \epsilon \geq |\psi(w)| &= \frac{k}{k+1}|\psi(f) - t\psi(e)| \\ &\geq \frac{k}{k+1}\big||\psi(f)| - |t||\psi(e)|\big| \\ &\geq \frac{k}{k+1}(|\psi(f)| - |t||\psi(e)|). \end{aligned} \tag{2}$$

Thus

$$\frac{1}{k}|\psi(e)| \geq |t||\psi(e)| \geq |\psi(f)| - \frac{k+1}{k}\epsilon \geq \left(1 - \frac{\epsilon}{4}\right) - \frac{k+1}{k}\epsilon = k - \frac{k+1}{k}\epsilon. \tag{3}$$

This gives

$$|\psi(e)| \geq k^2 - (k+1)\epsilon = \left(1 - \frac{\epsilon}{4}\right)^2 - \left(2 - \frac{\epsilon}{4}\right)\epsilon = 1 - \frac{\epsilon}{2} + \frac{\epsilon^2}{16} - 2\epsilon + \frac{\epsilon^2}{4} \geq 1 - \frac{5}{2}\epsilon \tag{4}$$

as required. Notice that, if φ and ψ are real valued, the above still holds.

Now, there exist $\beta, \gamma \in \mathbb{C}$ such that $|\beta| = |\gamma| = 1$, $\beta\varphi(e) \in [1 - \epsilon/4, 1] \subset \mathbb{R}$, and $\gamma\psi(e) \in [1 - 5\epsilon/2, 1] \subset \mathbb{R}$; and so $|\beta\varphi(e) - \gamma\psi(e)| \leq 5\epsilon/2$.

Let $x \in X$ be such that $\|x\| \leq 1$ and write $x = \lambda e + y$, where $\lambda = \varphi(x)/\varphi(e)$ and $y = x - \lambda e$. Then $|\lambda| \leq |\varphi(x)|/|\varphi(e)| \leq 1/(1 - \epsilon/4) \leq 8/7$, $\|y\| \leq \|x\| + |\lambda|\|e\| \leq 15/7$, and $\varphi(y) = \varphi(x) - (\varphi(x)/\varphi(e))\varphi(e) = 0$. So, by hypothesis, $|\psi((7/15)y)| \leq \epsilon$, that is, $|\psi(y)| \leq (15/7)\epsilon$. Then, if we take $\alpha = \gamma/\beta$, we have $|\alpha| = 1$ and

$$\begin{aligned} |\varphi(x) - \alpha\psi(x)| &= \frac{1}{|\beta|}|\beta\varphi(x) - \gamma\psi(x)| = |\beta\lambda\varphi(e) + \beta\varphi(y) - \gamma\lambda\psi(e) - \gamma\psi(y)| \\ &\leq |\lambda||\beta\varphi(e) - \gamma\psi(e)| + |\gamma||\psi(y)| \leq \frac{8}{7}\cdot\frac{5}{2}\epsilon + 1\cdot\frac{15}{7}\epsilon = 5\epsilon. \end{aligned} \tag{5}$$

But x was an arbitrary element of the unit ball of X, so we have $\|\varphi - \alpha\psi\| \leq 5\epsilon$.

Notice that if X is a real Banach space, the above argument still holds. Also, if φ and ψ are real valued, we can choose $e \in X$ such that $\varphi(e) \geq 1 - \epsilon/4$, giving $\beta = 1$, and then from the claim, either $\gamma = 1$ or $\gamma = -1$. So either $\alpha = 1$ or $\alpha = -1$, yielding Phelps' result, up to a constant. □

References

[1] E. Bishop and R. R. Phelps, *A proof that every Banach space is subreflexive*, Bulletin of the American Mathematical Society **67** (1961), 97–98.
[2] R. R. Phelps, *A representation theorem for bounded convex sets*, Proceedings of the American Mathematical Society **11** (1960), no. 6, 976–983.

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