STARLIKENESS AND CONVEXITY OF A CLASS OF ANALYTIC FUNCTIONS

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Let $\mathcal{A}$ be the class of analytic functions in the unit disk that are normalized with $f(0) = f'(0) - 1 = 0$ and let $-1 \le B < A \le 1$. In this paper we study the class $G_{\lambda,\alpha} = \{f \in \mathcal{A} : |(1-\alpha+\alpha z f''(z)/f'(z))/zf'(z)/f(z) - (1-\alpha)| < \lambda, z \in \mathcal{U}\}$, $0 \le \alpha \le 1$, and give sharp sufficient conditions that embed it into the classes $S^*[A,B] = \{f \in \mathcal{A} : zf'(z)/f(z) \prec (1+Az)/(1+Bz)\}$ and $K(\delta) = \{f \in \mathcal{A} : 1 + zf''(z)/f'(z) \prec (1-\delta)(1+z)/(1-z)+\delta\}$, where "$\prec$ " denotes the usual subordination. Also, sharp upper bound of $|a_2|$ and of the Fekete-Szegö functional $|a_3 - \mu a_2^2|$ is given for the class $G_{\lambda,\alpha}$.

1. Introduction and preliminaries

A region Ω from the complex plane $\mathbb{C}$ is called convex if for every two points $\omega_1, \omega_2 \in \Omega$, the closed line segment $[\omega_1,\omega_2] = \{(1-t)\omega_1 + t\omega_2 : 0 \le t \le 1\}$ lies in Ω. Fixing $\omega_1 = 0$ brings the definition of starlike region. If $\mathcal{A}$ denotes the class of functions $f(z)$ that are analytic in the unit disk $\mathcal{U} = \{z : |z| < 1\}$ and normalized by $f(0) = f'(0) - 1 = 0$, then a function $f \in \mathcal{A}$ is called *convex* or *starlike* if it maps $\mathcal{U}$ into a convex or starlike region, respectively. Corresponding classes are denoted by K and S^*. It is well known that $K \subset S^*$, and it is well known that both are subclasses of the class of univalent functions and have the following analytical representations:

$$
\begin{aligned}
f \in K &\Longleftrightarrow \operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0, \quad z \in \mathcal{U}, \\
f \in S^* &\Longleftrightarrow \operatorname{Re}\frac{zf'(z)}{f(z)} > 0, \quad z \in \mathcal{U}.
\end{aligned}
\tag{1.1}
$$

More about these classes may be found in [2].

Further, let $f, g \in \mathcal{A}$. Then we say that $f(z)$ is *subordinate* to $g(z)$, and we write $f(z) \prec g(z)$, ifthere exists a function $\omega(z)$, analytic in the unit disk $\mathcal{U}$, such that $\omega(0) = 0$,

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$|\omega(z)| < 1$, and $f(z) = g(\omega(z))$ for all $z \in \mathcal{U}$. Specially, if $g(z)$ is univalent in $\mathcal{U}$, then $f(z) \prec g(z)$ if and only if $f(0) = g(0)$ and $f(\mathcal{U}) \subseteq g(\mathcal{U})$.

In terms of subordination, we have

$$S^* = \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec \frac{1+z}{1-z} \right\}, \qquad K = \left\{ f \in \mathcal{A} : 1 + \frac{zf''(z)}{f'(z)} \prec \frac{1+z}{1-z} \right\}. \tag{1.2}$$

If $-1 \le B < A \le 1$, then a generalization of class S^* is

$$S^*[A,B] = \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec \frac{1+Az}{1+Bz} \right\}. \tag{1.3}$$

Geometrically, this means that the image of $\mathcal{U}$ by $zf'(z)/f(z)$ is inside the open disk centered on the real axis with diameter endpoints $(1-A)/(1-B)$ and $(1+A)/(1+B)$. Special selection of A and B leads us to the following classes: $S^*[1,-1] \equiv S^*$, $S^*[1-2\alpha,-1] \equiv S^*(\alpha)$-class of starlike functions of order α, $0 \le \alpha < 1$, and $K(\alpha)$ is the class of convex functions of order α, $0 \le \alpha < 1$, defined by $f(z) \in K(\alpha)$ if and only if $zf'(z) \in S^*(\alpha)$, that is,

$$\operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha, \quad z \in \mathcal{U}. \tag{1.4}$$

These classes are widely studied during the past decades, mainly in two different directions: for developing criteria for starlikeness or convexity and for obtaining properties of the Maclaurin coefficients of a starlike or convex function. In this paper sufficient conditions (some of them sharp) that embed the class

$$G_{\lambda,\alpha} = \left\{ f \in \mathcal{A} : \left| \frac{1-\alpha+\alpha zf''(z)/f'(z)}{zf'(z)/f(z)} - (1-\alpha) \right| < \lambda, z \in \mathcal{U} \right\}, \tag{1.5}$$

$0 < \alpha \le 1$, $\lambda > 0$, into the classes $S^*[A,B]$ and $K(\delta)$, $0 \le \delta < 1$, will be given, together with sharp upper bound of the Fekete-Szegö functional $|a_3 - \mu a_2^2|$, $\mu \in \mathbb{R}$. Sufficient motivation for studying the class $G_{\lambda,\alpha}$ is the fact that it makes close connection between classes,

$$\begin{gathered} G_{\lambda,1/2} = \left\{ f \in \mathcal{A} : \left| \frac{1+zf''(z)/f'(z)}{zf'(z)/f(z)} - 1 \right| < 2\lambda, z \in \mathcal{U} \right\}, \\ G_{\lambda,1} = \left\{ f \in \mathcal{A} : \left| \frac{f(z)f''(z)}{f'^2(z)} \right| < \lambda, z \in \mathcal{U} \right\}, \\ G_{\lambda,1/(2-\gamma)} = \left\{ f \in \mathcal{A} : \left| \frac{1-\gamma+zf''(z)/f'(z)}{zf'(z)/f(z)} - (1-\gamma) \right| < \lambda(2-\gamma), z \in \mathcal{U} \right\}, \end{gathered} \tag{1.6}$$

studied in [1, 6–9, 11–13] and other references.

2. Conditions for starlikeness and convexity

For obtaining the result for convexity and starlikeness of the class $G_{\lambda,\alpha}$, we will use the method of differential subordinations. Valuable reference on this topic is [5]. The general theory of differential subordinations, as well as the theory of first-order differential subordinations, was introduced by Miller and Mocanu in [3, 4]. Namely, if $\phi : \mathbb{C}^2 \to \mathbb{C}$ is analytic in a domain D, if $h(z)$ is univalent in $\mathcal{U}$, and if $p(z)$ is analytic in $\mathcal{U}$ with $(p(z), zp'(z)) \in D$ when $z \in \mathcal{U}$, then $p(z)$ is said to satisfy a first-order differential subordination if

$$\phi(p(z), zp'(z)) \prec h(z). \tag{2.1}$$

The univalent function $q(z)$ is said to be a *dominant* of the differential subordination (2.1) if $p(z) \prec q(z)$ for all $p(z)$ satisfying (2.1). If $\tilde{q}(z)$ is a dominant of (2.1) and $\tilde{q}(z) \prec q(z)$ for all dominants of (2.1), then we say that $\tilde{q}(z)$ is the *best dominant* of the differential subordination (2.1).

From the theory of first-order differential subordinations, we will make use of the following lemma.

Lemma 2.1 (see [4]). *Let $q(z)$ be univalent in the unit disk $\mathcal{U}$, and let $\theta(\omega)$ and $\phi(\omega)$ be analytic in a domain D containing $q(\mathcal{U})$, with $\phi(\omega) \neq 0$ when $\omega \in q(\mathcal{U})$. Set $Q(z) = zq'(z)\phi(q(z))$, $h(z) = \theta(q(z)) + Q(z)$, and suppose that*

(i) $Q(z) \in S^*$;

(ii) $\operatorname{Re}(zh'(z)/Q(z)) = \operatorname{Re}\{\theta'(q(z))/\phi(q(z)) + zQ'(z)/Q(z)\} > 0$, $z \in \mathcal{U}$.

If $p(z)$ is analytic in $\mathcal{U}$, with $p(0) = q(0)$, $p(\mathcal{U}) \subseteq D$, and

$$\theta(p(z)) + zp'(z)\phi(p(z)) \prec \theta(q(z)) + zq'(z)\phi(q(z)) = h(z), \tag{2.2}$$

then $p(z) \prec q(z)$, and $q(z)$ is the best dominant of (2.2).

In the beginning, using Lemma 2.1 we will prove the following result.

Theorem 2.2. *Let $f \in \mathcal{A}$, $-1 \le B < A \le 1$, and $(1+|A|)/(3+|A|) \le \alpha \le 1$. If*

$$\frac{1-\alpha+\alpha zf''(z)/f'(z)}{zf'(z)/f(z)} \prec \alpha + (1-2\alpha)\frac{1+Bz}{1+Az} + \frac{\alpha z(A-B)}{(1+Az)^2} \equiv h(z), \tag{2.3}$$

then $f \in S^*[A,B]$. This result is sharp.

Proof. We choose $p(z) = f(z)/zf'(z)$, $q(z) = (1+Bz)/(1+Az)$, $\theta(\omega) = (1-2\alpha)\omega + \alpha$, and $\phi(\omega) = -\alpha$. Then $q(z)$ is convex, thus univalent, because $1 + zq''(z)/q'(z) = (1 - Az)/(1+Az)$; $\theta(\omega)$ and $\phi(\omega)$ are analytic in the domain $D = \mathbb{C}$ which contains $q(\mathcal{U})$ and $\phi(\omega)$ when $\omega \in q(\mathcal{U})$. Further,

$$Q(z) = zq'(z)\phi(q(z)) = \frac{\alpha(A-B)z}{(1+Az)^2} \tag{2.4}$$

is starlike because $zQ'(z)/Q(z) = (1-Az)/(1+Az)$. Further,

$$\begin{gathered} h(z) = \theta(q(z)) + Q(z) = \alpha + (1-2\alpha)\frac{1+Bz}{1+Az} + \frac{\alpha z(A-B)}{(1+Az)^2}, \\ \operatorname{Re}\frac{zh'(z)}{Q(z)} = \operatorname{Re}\left(1 - \frac{1}{\alpha} + \frac{2}{1+Az}\right) > 1 - \frac{1}{\alpha} + \frac{2}{1+|A|}, \end{gathered} \tag{2.5}$$

$z \in \mathcal{U}$, which is greater or equal to zero if and only if $\alpha \geq (1+|A|)/(3+|A|)$. Therefore from Lemma 2.1, it follows that $p(z) \prec q(z)$, that is, $f \in S^*[A,B]$.

The result is sharp as the functions ze^{Az} and $z(1+Bz)^{A/B}$ show in the cases $B = 0$ and $B \neq 0$, respectively. □

Remark 2.3. According to the definition of subordination, the sharpness of the result of Theorem 2.2 means that $h(\mathcal{U})$ is the greatest region in the complex plane with the property that if

$$\frac{1-\alpha+\alpha z f''(z)/f'(z)}{zf'(z)/f(z)} \in h(\mathcal{U}) \tag{2.6}$$

for all $z \in \mathcal{U}$, then $f(z) \in S^*[A,B]$.

The following corollary gives sharp sufficient conditions that embed $G_{\lambda,\alpha}$ into $S^*[A,B]$.

Corollary 2.4. *Let* $-1 \leq B < A \leq 1$ *and* $(1+|A|)/(3+|A|) \leq \alpha \leq 1$. *Then*

$$\lambda = (A-B)\cdot\frac{(1-2\alpha)|A|-(1-3\alpha)}{(1+|A|)^2} \tag{2.7}$$

is the greatest number such that $G_{\lambda,\alpha} \subseteq S^*[A,B]$.

Proof. In order to prove this corollary, due to Theorem 2.2 it is enough to show that

$$\lambda = \min\{|h(z)-(1-\alpha)| : |z| = 1\} \equiv \hat{\lambda}, \tag{2.8}$$

where $h(z)$ is defined as in the statement of the theorem and

$$h(z)-(1-\alpha) = -z(A-B)\cdot\frac{A(1-2\alpha)z+1-3\alpha}{(1+Az)^2}. \tag{2.9}$$

Further, let

$$\begin{aligned}\psi(t) &\equiv \left|h(e^{i\gamma\pi/2})-(1-\alpha)\right|^2\\ &= (A-B)^2\cdot\frac{\left[(1-2\alpha)^2A^2+2(1-3\alpha)(1-2\alpha)At+(1-3\alpha)^2\right]}{(1+2At+A^2)^2},\end{aligned} \tag{2.10}$$

$t = \cos(\gamma\pi/2) \in [-1,1]$. Thus $\hat{\lambda} = \min\{\sqrt{\psi(t)} : -1 \leq t \leq 1\}$.

If $\alpha \leq 1/2$, then $1-2\alpha \geq 0$ and having in mind that $1-3\alpha \leq -2|A|/(3+|A|) \leq 0$, we receive that $\psi(t)$ is a monotone function and

$$\hat{\lambda} = \min\left\{\sqrt{\psi(-1)},\sqrt{\psi(1)}\right\} = \min\{|h(-1)-(1-\alpha)|,|h(1)-(1-\alpha)|\} = \lambda. \tag{2.11}$$

The last equality holds because $1-3\alpha \pm A(1-2\alpha) \geq 0$ is equivalent to $\alpha \geq (1+|A|)/(3+|A|) \geq (1-|A|)/(3-2|A|)$.

If $\alpha > 1/2$, we have the following analysis. Equation $\psi'_t(t) = 0$ has unique solution

$$t_* = -\frac{A^2(1-\alpha)(1-2\alpha)+(1-3\alpha)(1-4\alpha)}{2A(1-2\alpha)(1-3\alpha)}. \tag{2.12}$$

It can be verified that $|t_*| > 1$ is equivalent to

$$\varphi(A,\alpha) \equiv A^2(1-\alpha)(1-2\alpha) - 2|A|(1-2\alpha)(1-3\alpha) + (1-3\alpha)(1-4\alpha) > 0. \tag{2.13}$$

Now, $\varphi(A,\alpha)$ is a decreasing function of $|A| \in [0,1]$ which implies that $\varphi(A,\alpha) \geq \varphi(1,\alpha) = 2\alpha^2 > 0$. Thus, $|t_*| > 1$, which implies that $\psi(t)$ is a monotone function on $[-1,1]$ leading to $\hat{\lambda} = \min\{\sqrt{\psi(t)} : -1 \leq t \leq 1\} = \min\{\sqrt{\psi(-1)}, \sqrt{\psi(1)}\} = \min\{|h(-1)-(1-\alpha)|, |h(1)-(1-\alpha)|\}$. At the end, the function

$$\eta(A,\alpha) \equiv \left|h(1)-(1-\alpha)\right| - \left|h(-1)-(1-\alpha)\right| = 2A \cdot \frac{1-A^2-2\alpha(2-A^2)}{(1+A)^2(1-A)^2} \tag{2.14}$$

has the opposite sign of the sign of coefficient A. Therefore,

$$\hat{\lambda} = \begin{Bmatrix} \left|h(1)-(1-\alpha)\right|, & A \geq 0 \\ \left|h(-1)-(1-\alpha)\right|, & A < 0 \end{Bmatrix} = \lambda. \tag{2.15}$$

Sharpness of the result follows from the sharpness of Theorem 2.2 (see Remark 2.3) and the fact that the obtained λ is the greatest, which embeds the disk $|\omega - (1-\alpha)| < \lambda$ in $h(\mathcal{U})$. □

The following example exhibits some concrete conclusions that can be obtained from the results of the previous section by specifying the values α, A, B.

Example 2.5. Let $-1 \leq B < A \leq 1$.

(i) $G_{\lambda,1/2} \subseteq S^*[A,B]$ when $\lambda = (A-B)/2(1+|A|)^2$.

(ii) $G_{\lambda,1} \subseteq S^*[A,B]$ when $\lambda = (A-B)\cdot(2-|A|)/2(1+|A|)^2$.

(iii) $G_{\lambda,1/(2-\gamma)} \subseteq S^*[A,B]$ when $\gamma \geq -(1-|A|)/(1+|A|)$ and $\lambda = (A-B)\cdot(1+\gamma-\gamma|A|)/2(1+|A|)^2$.

(iv) $G_{\lambda,\alpha} \subseteq S^*$ when $1/2 \leq \alpha \leq 1$ and $\lambda = \alpha/2$.

(v) $G_{\lambda,\alpha} \subseteq S^*[0,B] \subset S^*(1/(1-B))$ when $1/3 \leq \alpha \leq 1$, $-1 \leq B < 0$ and $\lambda = B(1-3\alpha)$.

The value of λ in each of the above cases is the greatest that makes the corresponding inclusion true.

Remark 2.6. The result from Example 2.5(i) is the same as in [13, Corollary 2.6]. Also, for $\alpha = 1/2$ in Example 2.5(v), we receive the same result as in [6, Theorem 1]. Finally, for $\alpha = 1$ and $B = -1$ in Example 2.5(v), we receive the same result as in [11, Corollary 2].

Next theorem studies connection between $G_{\lambda,\alpha}$ and the class of convex functions of some order.

THEOREM 2.7. $G_{\lambda,\alpha} \subseteq K(2-1/\alpha)$ *when* $1/2 \leq \alpha < 1$ *and* $\lambda = (1-\alpha)(3\alpha-1)/\sqrt{2(5\alpha^2-4\alpha+1)}$.

Proof. Let $f \in G_{\lambda,\alpha}$ and $B = \lambda/(1-3\alpha)$. Then, by Example 2.5(v) we have $f \in S^*[0,B]$, that is, $|f(z)/zf'(z) - 1| < B$, $z \in \mathcal{U}$. Further,

$$1 + \frac{zf''(z)}{f'(z)} - \left(2 - \frac{1}{\alpha}\right) = \frac{zf'(z)}{\alpha f(z)} \cdot \frac{1-\alpha+\alpha zf''(z)/f'(z)}{zf'(z)/f(z)}, \tag{2.16}$$

and for all $z \in \mathcal{U}$, we obtain

$$\begin{aligned} \left|\arg\left(1+\frac{zf''(z)}{f'(z)}-2+\frac{1}{\alpha}\right)\right| &\leq \left|\arg\frac{zf'(z)}{f(z)}\right| + \left|\arg\frac{1-\alpha+\alpha zf''(z)/f'(z)}{zf'(z)/f(z)}\right| \\ &\leq \arcsin|B| + \arcsin\frac{\lambda}{1-\alpha} \\ &= \arcsin\left(\frac{\lambda}{1-\alpha}\sqrt{1-B^2}+|B|\sqrt{1-\frac{\lambda^2}{(1-\alpha)^2}}\right) \\ &= \arcsin 1 = \frac{\pi}{2}, \end{aligned} \tag{2.17}$$

that is, $f \in K(2-1/\alpha)$. □

Example 2.8. For $\alpha = 1/2$ and $\alpha = 1/(2-\gamma)$ in the previous theorem, we get
(i) $G_{\lambda,1/2} \subseteq K$ when $\lambda = \sqrt{2}/4$;
(ii) $G_{\lambda,1/(2-\gamma)} \subseteq K(\gamma)$ when $0 \leq \gamma < 1$ and $\lambda = (1-\gamma^2)/[(2-\gamma)\sqrt{2(1+\gamma^2)}]$.

Remark 2.9. By putting $\alpha = 1/(2-\gamma)$, $0 \leq \gamma < 1$, we get the result from [10, Theorem 2].

3. Sharp estimate of the Fekete-Szegö functional

In this section we give sharp estimates of $|a_2|$ and of the Fekete-Szegö functional $|a_3 - \mu a_2^2|$ for a function $f \in G_{\lambda,\alpha}$. We will use following lemmas.

LEMMA 3.1 [2, page 41]. *Let $p \in \mathcal{P}$, that is, let p be analytic in $\mathcal{U}$, be given by $p(z) = 1+\sum_{n=1}^{\infty} p_n z^n$ and $\operatorname{Re} p(z) > 0$ for $z \in \mathcal{U}$. Then $|p_n| \leq 2$ and for all $n \in \mathbb{N}$, $|p_2 - p_1^2/2| \leq 2 - |p_1|^2/2$.*

LEMMA 3.2. *Let $\omega(z) = \sum_{n=1}^{\infty} c_n z^n$ be an analytic function in the unit disk $\mathcal{U}$ and $|\omega(z)| < 1$, $z \in \mathcal{U}$. Then $|c_1| \leq 1$ and $|c_2| \leq 1 - |c_1|^2$.*

Proof. Define a function $p(z) = 1+\sum_{n=1}^{\infty} p_n z^n \in \mathcal{P}$ by $p(z) = (1-\omega(z))/(1+\omega(z))$. Then $c_1 = -p_1/2$, $c_2 = (p_1^2/2 - p_2)/2$ and the rest follows from Lemma 3.1. □

THEOREM 3.3. *Let $f(z) = z+\sum_{n=2}^{\infty} a_n z^n \in G_{\lambda,\alpha}$ for some $\lambda > 0$ and $0 \leq \alpha \leq 1$. Then $|a_2| \leq \lambda/|1-3\alpha|$ and for any complex μ, the following bound is sharp:*

$$|a_3 - \mu a_2^2| \leq \max\left\{\frac{\lambda}{2|4\alpha-1|}, \frac{\lambda^2|1-\mu|}{(1-3\alpha)^2}\right\}. \tag{3.1}$$

Proof. If $f \in G_{\lambda,\alpha}$, then

$$(1-\alpha)f(z)f''(z)+\alpha zf(z)f''(z) = zf'^2(z)[1-\alpha+\lambda\omega(z)], \tag{3.2}$$

where $\omega(z) = \sum_{n=1}^{\infty} c_n z^n$ is such that $|\omega(z)| < 1$, $z \in \mathcal{U}$. After equating the coefficients, we get $a_2 = \lambda c_1/(3\alpha-1)$ and

$$a_3 = \frac{\lambda c_2}{2(4\alpha-1)} + \frac{\lambda^2 c_1^2}{(1-3\alpha)^2}. \tag{3.3}$$

From Lemma 3.2, we get $|a_2| \le \lambda/|1-3\alpha|$. Further,

$$a_3 - \mu a_2^2 = \frac{\lambda c_2}{2(4\alpha-1)} + \frac{\lambda^2 c_1^2}{(1-3\alpha)^2}(1-\mu). \tag{3.4}$$

So, for $x = |c_1| \le 1$,

$$|a_3 - \mu a_2^2| \le Ax^2 + \frac{\lambda}{2|1-4\alpha|} \equiv H(x), \tag{3.5}$$

where $A = \lambda^2|1-\mu|/(1-3\alpha)^2 - \lambda/2|1-4\alpha|$ and

$$|a_3 - \mu a_2^2| \le \begin{cases} H(1) = \dfrac{\lambda^2|1-\mu|}{(1-3\alpha)^2}, & A \ge 0, \\ H(0) = \dfrac{\lambda}{2|1-4\alpha|}, & A < 0. \end{cases} \tag{3.6}$$

The upper bound is sharp due to the functions $f_1(z) = z(1-3\alpha)/(1-3\alpha+\lambda z)$ and $f_2(z) = z \cdot \sqrt{(1-4\alpha)/(1-4\alpha+\lambda z^2)}$. □

Remark 3.4. By putting $\alpha = 1/(2-\gamma)$, $0 \le \gamma < 1$, we get the result from [10, Theorem 3].

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