SUBCLASSES OF α-SPIRALLIKE FUNCTIONS ASSOCIATED WITH RUSCHEWEYH DERIVATIVES

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Making use of the Ruscheweyh derivatives, we introduce the subclasses $T(n,\alpha,\lambda)$ ($n \in \{0,1,2,3,\dots\}$, $-\pi/2 < \alpha < \pi/2$, and $0 \le \lambda \le \cos^2\alpha$) of functions $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$ which are analytic in $|z| < 1$. Subordination and inclusion relations are obtained. The radius of α-spirallikeness of order ρ is calculated. A convolution property and a special member of $T(n,\alpha,\lambda)$ are also given.

1. Introduction

Let A denote the class of functions f of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \tag{1.1}$$

which are analytic in the unit disk $U = \{z : |z| < 1\}$. Let $S \subset A$ consist of univalent functions in U. For $0 \le \rho < 1$, a function $f \in S$ is said to be starlike of order ρ if

$$\operatorname{Re}\frac{zf'(z)}{f(z)} > \rho \quad (z \in U). \tag{1.2}$$

The class of such functions we denote by $S^*(\rho)$ $(0 \le \rho < 1)$. A function $f \in S$ is said to be convex in U if

$$\operatorname{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > 0 \quad (z \in U). \tag{1.3}$$

We denote by K the class of all convex functions in U. For $-\pi/2 < \alpha < \pi/2$ and $0 \le \rho < 1$, a function $f \in S$ is said to be α-spirallike of order ρ in U if

$$\operatorname{Re}\left\{e^{i\alpha}\frac{zf'(z)}{f(z)}\right\} > \rho\cos\alpha \quad (z \in U). \tag{1.4}$$

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Further let $UCV \subset K$ be the class of functions introduced by Goodman [2] called uniformly convex in U. It was shown in [4, 7] that $f \in A$ is in UCV if and only if

$$\operatorname{Re}\left\{1+\frac{zf''(z)}{f'(z)}\right\} > \left|\frac{zf''(z)}{f'(z)}\right| \quad (z \in U). \tag{1.5}$$

In [7], Ronning investigated the class S_p defined by

$$S_p = \{f \in S^*(0) : f(z) = zg'(z),\ g \in UCV\}. \tag{1.6}$$

The uniformly convex and related functions have been studied by several authors (see, e.g., [1–4, 7, 6, 8, 12]).

If $f(z) = z+\sum_{k=2}^{\infty} a_k z^k \in A$ and $g(z) = z+\sum_{k=2}^{\infty} b_k z^k \in A$, then the Hadamard product or convolution of f and g is defined by $(f * g)(z) = z+\sum_{k=2}^{\infty} a_k b_k z^k$. Let

$$D^n f(z) = \frac{z}{(1-z)^{n+1}} * f(z), \tag{1.7}$$

for $f \in A$ and $n \in N_0 = \{0,1,2,3,\ldots\}$. Then

$$D^n f(z) = \frac{z\left(z^{n-1} f(z)\right)^{(n)}}{n!}. \tag{1.8}$$

This symbol $D^n f$ is called the Ruscheweyh derivative of order n of f. It was introduced by Ruscheweyh [9].

In this paper we introduce and investigate the subclasses $T(n,\alpha,\lambda)$ of A as follows.

Definition 1.1. A function $f \in A$ is said to be in $T(n,\alpha,\lambda)$ if

$$\left(\operatorname{Re}\left\{e^{i\alpha}\frac{z(D^n f(z))'}{D^n f(z)}\right\}\right)^2 + \lambda > \left|\frac{z(D^n f(z))'}{D^n f(z)} - 1\right|^2 \quad (z \in U), \tag{1.9}$$

where $n \in N_0, -\pi/2 < \alpha < \pi/2$, and $0 \le \lambda \le \cos^2\alpha$.

Note that, for $\lambda = 0$,

$$T(n,\alpha,0) = \left\{f \in A : \operatorname{Re}\left\{e^{i\alpha}\frac{z(D^n f(z))'}{D^n f(z)}\right\} > \left|\frac{z(D^n f(z))'}{D^n f(z)} - 1\right| \ (z \in U)\right\}. \tag{1.10}$$

In particular, $T(0,0,0) = S_p$ and $T(1,0,0) = UCV$.

2. Properties of $T(n,\alpha,\lambda)$

Let f and g be analytic in U. Then we say that f is subordinate to g in U, written $f \prec g$, if there exists an analytic function w in U such that $|w(z)| \le |z|$ and $f(z) = g(w(z))$ for $z \in U$. If g is univalent in U, then $f \prec g$ is equivalent to $f(0) = g(0)$ and $f(U) \subset g(U)$.

THEOREM 2.1. *Let* $n \in N_0, \alpha \in (-\pi/2, \pi/2)$, *and* $\lambda \in [0, \cos^2\alpha]$. *A function* $f \in A$ *belongs to* $T(n,\alpha,\lambda)$ *if and only if*

$$e^{i\alpha}\frac{z(D^n f(z))'}{D^n f(z)} \prec h(z)\cos\alpha + i\sin\alpha \quad (z \in U), \tag{2.1}$$

where

$$h(z) = 1 - \frac{\lambda}{2\cos^2\alpha} + \frac{2}{\pi^2}\left(\log\frac{1+\sqrt{(z+\beta)/(1+\beta z)}}{1-\sqrt{(z+\beta)/(1+\beta z)}}\right)^2, \tag{2.2}$$

with

$$\beta = \left(\frac{e^\mu - 1}{e^\mu + 1}\right)^2, \qquad \mu = \frac{\sqrt{\lambda}\pi}{2\cos\alpha}. \tag{2.3}$$

Proof. Let us define $w = u + iv$ by

$$e^{i\alpha}\frac{z(D^n f(z))'}{D^n f(z)} = w(z)\cos\alpha + i\sin\alpha \quad (z \in U). \tag{2.4}$$

Then $w(0) = 1$ and the inequality (1.9) can be rewritten as

$$u > \frac{1}{2}\left(v^2 + 1 - \frac{\lambda}{\cos^2\alpha}\right). \tag{2.5}$$

Thus

$$w(U) \subset G = \{w = u + iv : u \text{ and } v \text{ satisfy (2.5)}\}. \tag{2.6}$$

It follows from (2.2) that

$$h(0) = 1 - \frac{\lambda}{2\cos^2\alpha} + \frac{2}{\pi^2}\left(\log\frac{1+\sqrt{\beta}}{1-\sqrt{\beta}}\right)^2 = 1. \tag{2.7}$$

In order to prove the theorem, it suffices to show that the function $w = h(z)$ defined by (2.2) maps U conformally onto the parabolic region G.

Note that

$$0 \le \frac{1}{2}\left(1 - \frac{\lambda}{\cos^2\alpha}\right) < 1 - \frac{\lambda}{2\cos^2\alpha} \le 1, \tag{2.8}$$

for $0 \le \lambda \le \cos^2\alpha$. Consider the transformations

$$w_1 = \sqrt{w - \left(1 - \frac{\lambda}{2\cos^2\alpha}\right)}, \qquad w_2 = e^{\sqrt{2}\pi w_1}, \qquad t = \frac{1}{2}\left(w_2 + \frac{1}{w_2}\right). \tag{2.9}$$

It is easy to verify that the composite function

$$t=\varphi(w)=\operatorname{ch}\left(\pi\sqrt{2w-\left(2-\frac{\lambda}{\cos^2\alpha}\right)}\right) \tag{2.10}$$

maps $G^+=G\bigcap\{w=u+iv:v>0\}$ conformally onto the upper half plane $\operatorname{Im}(t)>0$ so that $w=(1/2)(1-\lambda/\cos^2\alpha)$ corresponds to $t=-1$ and $w=1-\lambda/2\cos^2\alpha$ to $t=1$. Applying the symmetry principle, the function $t=\varphi(w)$ maps G conformally onto $\Omega=\{t: |\arg(t+1)|<\pi\}$. Since $t=2((1+\zeta)/(1-\zeta))^2-1$ maps the unit disk $|\zeta|<1$ onto Ω, we see that

$$\begin{aligned} w=\varphi^{-1}(t)&=1-\frac{\lambda}{2\cos^2\alpha}+\frac{1}{2\pi^2}\left(\log\left(t+\sqrt{t^2-1}\right)\right)^2 \\ &=1-\frac{\lambda}{2\cos^2\alpha}+\frac{2}{\pi^2}\left(\log\frac{1+\sqrt{\zeta}}{1-\sqrt{\zeta}}\right)^2=g(\zeta) \end{aligned} \tag{2.11}$$

maps $|\zeta|<1$ conformally onto G so that $\zeta=\beta$ $(0\le\beta<1)$ corresponds to $w=1$. Therefore the function

$$w=h(z)=g\left(\frac{z+\beta}{1+\beta z}\right) \quad (z\in U) \tag{2.12}$$

maps U conformally onto G and the proof of the theorem is complete. □

Corollary 2.2. *Let $f\in T(n,\alpha,\lambda)$, $n\in N_0$, $\alpha\in(-\pi/2,\pi/2)$, $\lambda\in[0,\cos^2\alpha]$, and h be given by (2.2). Then*

$$\frac{D^n f(z)}{z}\prec\exp\left(e^{-i\alpha}\cos\alpha\int_0^z\frac{h(t)-1}{t}dt\right), \tag{2.13}$$

$$\exp\left(\int_0^1\frac{h(-\rho|z|)-1}{\rho}d\rho\right)\le\left|\left(\frac{D^n f(z)}{z}\right)^{e^{i\alpha}\sec\alpha}\right|\le\exp\left(\int_0^1\frac{h(\rho|z|)-1}{\rho}d\rho\right), \tag{2.14}$$

for $z\in U$. The bounds in (2.14) are sharp with the extremal function $f_0\in A$ defined by

$$D^n f_0(z)=z\exp\left(e^{-i\alpha}\cos\alpha\int_0^z\frac{h(t)-1}{t}dt\right). \tag{2.15}$$

Proof. From Theorem 2.1 we have

$$\frac{e^{i\alpha}}{\cos\alpha}\left(\frac{z\left(D^n f(z)\right)'}{D^n f(z)}-1\right)\prec h(z)-1, \tag{2.16}$$

for $f \in T(n,\alpha,\lambda)$. Since the function $h-1$ is univalent and starlike (with respect to the origin) in U, using (2.16) and the result of Suffridge [11, Theorem 3], we obtain

$$\frac{e^{i\alpha}}{\cos\alpha}\log\frac{D^n f(z)}{z} = \frac{e^{i\alpha}}{\cos\alpha}\int_0^z \left(\frac{(D^n f(t))'}{D^n f(t)} - \frac{1}{t}\right)dt \prec \int_0^z \frac{h(t)-1}{t}dt. \tag{2.17}$$

This implies (2.13).

Noting that the univalent function h maps the disk $|z| < \rho$ $(0 < \rho \le 1)$ onto a region which is convex and symmetric with respect to the real axis, we get

$$h(-\rho|z|) \le \operatorname{Re} h(\rho z) \le (\rho|z|) \quad (z \in U). \tag{2.18}$$

Now, (2.17) and (2.18) lead to

$$\int_0^1 \frac{h(-\rho|z|)-1}{\rho}d\rho \le \log\left|\left(\frac{D^n f(z)}{z}\right)^{e^{i\alpha}\sec\alpha}\right| \le \int_0^1 \frac{h(\rho|z|)-1}{\rho}d\rho, \tag{2.19}$$

for $z \in U$, which yields (2.14).

The bounds in (2.14) are best possible since the equalities are attained for the function f_0 in $T(n,\alpha,\lambda)$ defined by (2.15). □

Theorem 2.3. *Let $f \in T(n,\alpha,\lambda)$, $n \in N_0$, $\alpha \in (-\pi/2,\pi/2)$, $\lambda \in [0,\cos^2\alpha]$. Then $D^n f$ is α-spirallike of order ρ in $|z| < r$, where*

$$\begin{aligned} r = r(\rho,\alpha,\lambda) &= \frac{\beta + \left(\tan\left((\pi/4)\sqrt{2(1-\rho)-\lambda/\cos^2\alpha}\right)\right)^2}{1+\beta\left(\tan\left((\pi/4)\sqrt{2(1-\rho)-\lambda/\cos^2\alpha}\right)\right)^2} \\ &\quad \times \left(\frac{1}{2}\left(1-\frac{\lambda}{\cos^2\alpha}\right) \le \rho < 1-\frac{\lambda}{2\cos^2\alpha}\right) \end{aligned} \tag{2.20}$$

and β is given by (2.2).The result is sharp.

Proof. It follows from (2.20) and (2.2) that

$$0 < 2(1-\rho) - \frac{\lambda}{\cos^2\alpha} \le 1, \quad 0 \le \beta < r \le 1. \tag{2.21}$$

Let h be given by (2.2). Then

$$\begin{aligned} h(-r) &= 1 - \frac{\lambda}{2\cos^2\alpha} + \frac{2}{\pi^2}\left(\log\frac{1+i\sqrt{(r-\beta)/(1-\beta r)}}{1-i\sqrt{(r-\beta)/(1-\beta r)}}\right)^2 \\ &= 1 - \frac{\lambda}{2\cos^2\alpha} - \frac{8}{\pi^2}\left(\arctan\sqrt{\frac{r-\beta}{1-\beta r}}\right)^2 \end{aligned} \tag{2.22}$$

and hence

$$\inf_{|z|<r} \operatorname{Re} h(z) = h(-r) = \rho. \tag{2.23}$$

If $f \in T(n,\alpha,\lambda)$, then from Theorem 2.1 and (2.23) we have

$$\operatorname{Re}\left\{ e^{i\alpha} \frac{z(D^n f(z))'}{D^n f(z)} \right\} > \rho \cos\alpha \quad (|z| < r), \tag{2.24}$$

that is, $D^n f$ is α-spirallike of order ρ in $|z| < r$. Further, the result is sharp with the extremal function f_0 defined by (2.15).

Taking $\rho = (1/2)(1 - \lambda/\cos^2\alpha)$, Theorem 2.3 yields. □

Corollary 2.4. *Let $f \in T(n,\alpha,\lambda)$, $n \in N_0$, $\alpha \in (-\pi/2,\pi/2)$, $\lambda \in [0,\cos^2\alpha]$. Then $D^n f$ is α-spirallike of order $(1/2)(1-\lambda/\cos^2\alpha)$ in U and the result is sharp.*

Theorem 2.5. *Let $f \in T(n,\alpha,\lambda)$, $n \in N_0$, $\alpha \in (-\pi/2,\pi/2)$, $\lambda \in [0,\cos^2\alpha]$. Then $D^n f \in S^*((1-\lambda)/2)$ and the order $(1-\lambda)/2$ is sharp.*

Proof. Let h be given by (2.2). Then it follows from the proof of Theorem 2.1 that

$$\partial h(U) = \left\{ w = u + iv : u = \frac{1}{2}\left(v^2 + 1 - \frac{\lambda}{\cos^2\alpha}\right)\right\}. \tag{2.25}$$

Hence

$$\min_{|z|=1\,(z\neq 1)} \operatorname{Re}\{e^{-i\alpha}(h(z)\cos\alpha + i\sin\alpha)\} = \min_{u \geq (1/2)(1-\lambda/\cos^2\alpha)} g(u)\cos\alpha + \sin^2\alpha, \tag{2.26}$$

where

$$g(u) = u\cos\alpha - |\sin\alpha|\sqrt{2u - 1 + \frac{\lambda}{\cos^2\alpha}} \quad \left(u \geq \frac{1}{2}\left(1 - \frac{\lambda}{\cos^2\alpha}\right)\right). \tag{2.27}$$

Since

$$g'(u) = \cos\alpha - \frac{|\sin\alpha|}{\sqrt{2u - 1 + \lambda/\cos^2\alpha}} \quad \left(u > \frac{1}{2}\left(1 - \frac{\lambda}{\cos^2\alpha}\right)\right), \tag{2.28}$$

the function g attains its minimum value at $u = (1-\lambda)/2\cos^2\alpha$. Thus

$$\min_{|z|=1\,(z\neq 1)} \operatorname{Re}\{e^{-i\alpha}(h(z)\cos\alpha + i\sin\alpha)\} = g\left(\frac{1-\lambda}{2\cos^2\alpha}\right)\cos\alpha + \sin^2\alpha = \frac{1-\lambda}{2}. \tag{2.29}$$

Let $f \in T(n,\alpha,\lambda)$. Then, by Theorem 2.1 and (2.29), we conclude that $D^n f$ is starlike of order $(1-\lambda)/2$ in U, and the function f_0 defined by (2.15) shows that the order $(1-\lambda)/2$ is sharp. □

Theorem 2.6. *$T(n+1,\alpha,\lambda) \subset T(n,\alpha,\lambda)$, where $n \in N_0$, $\alpha \in (-\pi/2,\pi/2)$, $\lambda \in [0,\cos^2\alpha]$.*

Proof. It follows from (1.7) that

$$z(D^n f(z))' = (n+1)D^{n+1} f(z) - nD^n f(z) \quad (z \in U), \tag{2.30}$$

for $f \in A$. Set

$$p(z) = e^{i\alpha}\frac{z(D^n f(z))'}{D^n f(z)} \quad (z \in U). \tag{2.31}$$

Then (2.30) and (2.31) lead to

$$\frac{D^{n+1} f(z)}{D^n f(z)} = \frac{e^{-i\alpha}p(z)+n}{n+1} \quad (z \in U). \tag{2.32}$$

Differentiating both sides of (2.32) logarithmically and using (2.31), we get

$$e^{i\alpha}\frac{z(D^{n+1} f(z))'}{D^{n+1} f(z)} = p(z) + \frac{zp'(z)}{e^{-i\alpha}p(z)+n} \quad (z \in U). \tag{2.33}$$

If $f \in T(n+1,\alpha,\lambda)$, then by Theorem 2.1 and (2.33) we have

$$p(z) + \frac{zp'(z)}{e^{-i\alpha}p(z)+n} \prec h(z)\cos\alpha + i\sin\alpha \quad (z \in U), \tag{2.34}$$

where h is given by (2.2). The function $Q(z) = e^{-i\alpha}(h(z)\cos\alpha + i\sin\alpha) + n$ is univalent and convex in U and

$$\operatorname{Re} Q(z) > \frac{1-\lambda}{2} + n \geq 0 \quad (z \in U) \tag{2.35}$$

because of (2.29). Hence an application of the result of Miller and Mocanu [5, Corollary 1.1] yields

$$p(z) = e^{i\alpha}\frac{z(D^n f(z))'}{D^n f(z)} \prec h(z)\cos\alpha + i\sin\alpha \quad (z \in U). \tag{2.36}$$

Now, by Theorem 2.1, we know that $f \in T(n,\alpha,\lambda)$ and the theorem is proved. □

Remark 2.7. Combining Theorem 2.6 with Corollary 2.4, we see that each function in $T(n,\alpha,\lambda)$ is α-spirallike of order $(1/2)(1-\lambda/\cos^2\alpha)$ in U. In view of Theorems 2.5 and 2.6 we have $T(n,\alpha,\lambda) \subset S^*((1-\lambda)/2)$.

Theorem 2.8. *A function $f \in A$ is in $T(n,\alpha,\lambda)$ if and only if*

$$F(z) = \frac{n+1}{z^n}\int_0^z t^{n-1} f(t)dt \tag{2.37}$$

is in $T(n+1,\alpha,\lambda)$, where $n \in N_0$, $\alpha \in (-\pi/2,\pi/2)$, $\lambda \in [0,\cos^2\alpha]$.

Proof. It follows from (2.37) that $F \in A$ and

$$(n+1)f(z) = nF(z) + zF'(z) \quad (z \in U), \tag{2.38}$$

for $f \in A$. By using (2.30) and (2.38), we obtain

$$D^n f(z) = \frac{nD^n F(z) + z\left(D^n F(z)\right)'}{n+1} = D^{n+1}F(z) \quad (z \in U), \tag{2.39}$$

which proves the assertions of the theorem. □

Let $R(\rho)$ be the class of prestarlike functions of order ρ in U consisting of functions $f \in A$ satisfying

$$\frac{z}{(1-z)^{2-2\rho}} * f(z) \in S^*(\rho), \tag{2.40}$$

for some ρ $(0 \le \rho < 1)$. The following lemma is due to Ruscheweyh [10].

Lemma 2.9. *If $f \in S^*(\rho)$ and $g \in R(\rho)$ $(0 \le \rho < 1)$, then for any analytic function F in U,*

$$\frac{g * (Ff)}{g * f}(U) \subset \overline{\text{co}}(F(U)), \tag{2.41}$$

where $\overline{\text{co}}(F(U))$ stands for the convex hull of $F(U)$.

Applying the lemma, we derive the following.

Theorem 2.10. *Let $f \in T(n,\alpha,\lambda)$ and $g \in R((1-\lambda)/2)$. Then*

$$f * g \in T(n,\alpha,\lambda), \tag{2.42}$$

where $n \in N_0$, $\alpha \in (-\pi/2,\pi/2)$, $\lambda \in [0,\cos^2\alpha]$.

Proof. Let $f \in T(n,\alpha,\lambda)$. Making use of Theorems 2.1 and 2.5, we have

$$F(z) = \frac{z(D^n f(z))'}{D^n f(z)} \prec e^{-i\alpha}(h(z)\cos\alpha + i\sin\alpha), \quad D^n f \in S^*\left(\frac{1-\lambda}{2}\right). \tag{2.43}$$

If we put $\varphi = f * g$, then for $z \in U$,

$$\begin{aligned} \frac{z(D^n\varphi(z))'}{D^n\varphi(z)} &= \frac{z(g(z) * D^n f(z))'}{g(z) * D^n f(z)} = \frac{g(z) * (z(D^n f(z))')}{g(z) * D^n f(z)} \\ &= \frac{g(z) * (F(z)D^n f(z))}{g(z) * D^n f(z)}. \end{aligned} \tag{2.44}$$

Since the univalent function $e^{-i\alpha}(h(z)\cos\alpha + i\sin\alpha)$ is convex in U and $g \in R((1-\lambda)/2)$, from (2.43), (2.44), and the lemma we deduce that

$$\frac{z(D^n\varphi(z))'}{D^n\varphi(z)} \prec e^{-i\alpha}(h(z)\cos\alpha + i\sin\alpha). \tag{2.45}$$

Therefore, by using Theorem 2.1, $\varphi \in T(n,\alpha,\lambda)$ and the proof is complete. □

Note that $R(1/2) = S^*(1/2)$. Since $R(\rho_1) \subset R(\rho_2)$ for $0 \le \rho_1 < \rho_2 < 1$ (see [10]), we have $K = R(0) \subset R((1-\lambda)/2)$. Thus Theorem 2.10 yields the following.

Corollary 2.11. (i) *If $f \in T(n,\alpha,0)$, $n \in N_0$, $\alpha \in (-\pi/2,\pi/2)$, and $g \in S^*(1/2)$, then $f * g \in T(n,\alpha,0)$.*

(ii) *If $f \in T(n,\alpha,\lambda)$, $n \in N_0$, $\alpha \in (-\pi/2,\pi/2)$, $\lambda \in [0,\cos^2\alpha]$, and $g \in K$, then $f * g \in T(n,\alpha,\lambda)$.*

Theorem 2.12. *Let $n \in N_0$, $\alpha \in (-\pi/2,\pi/2)$, $\lambda \in [0,\cos^2\alpha]$. The function $f \in A$ defined by*

$$D^n f(z) = \frac{z}{(1-bz)^{2e^{-i\alpha}\cos\alpha}} \quad (z \in U) \tag{2.46}$$

is in $T(n,\alpha,\lambda)$, where b is complex and

$$|b| = \begin{cases} \dfrac{\cos^2\alpha + \lambda}{3\cos^2\alpha - \lambda} & (0 \le \lambda \le (3-2\sqrt{2})\cos^2\alpha), \\ \sqrt{\dfrac{\sqrt{\lambda}}{2\cos\alpha + \sqrt{\lambda}}} & ((3-2\sqrt{2})\cos^2\alpha \le \lambda \le \cos^2\alpha). \end{cases} \tag{2.47}$$

The result is sharp, that is, $|b|$ cannot be increased.

Proof. Let $f \in A$ be given by (2.46). Then

$$e^{i\alpha}\frac{z(D^n f(z))'}{D^n f(z)} = \frac{1+bz}{1-bz}\cos\alpha + i\sin\alpha. \tag{2.48}$$

Hence, by Theorem 2.1, $f \in T(n,\alpha,\lambda)$ if and only if

$$\frac{1+bz}{1-bz} \prec h(z), \tag{2.49}$$

where h is given by (2.2), or, equivalently, when

$$\left\{w : \left| w - \frac{1+|b|^2}{1-|b|^2} \right| < \frac{2|b|}{1-|b|^2}\right\} \subset h(U), \tag{2.50}$$

for $0 < |b| < 1$.

Let δ denote the minimum distance from the point $(1+|b|^2)/(1-|b|^2)$ to the points on the parabola $\partial h(U)$ given by (2.25). Then

$$\delta = \min_{u \geq (1/2)(1-\lambda/\cos^2\alpha)} \sqrt{g(u)}, \qquad g(u) = \left(u - \frac{1+|b|^2}{1-|b|^2}\right)^2 + 2u - 1 + \frac{\lambda}{\cos^2\alpha}. \tag{2.51}$$

Note that

$$\frac{1+|b|^2}{1-|b|^2} > \frac{1}{2}\left(1 - \frac{\lambda}{\cos^2\alpha}\right), \qquad g'(u) = 2\left(u - \frac{2|b|^2}{1-|b|^2}\right). \tag{2.52}$$

(i) If

$$0 \leq \lambda \leq (3-2\sqrt{2})\cos^2\alpha, \qquad |b| = \frac{\cos^2\alpha + \lambda}{3\cos^2\alpha - \lambda}, \tag{2.53}$$

then $\lambda^2 - 6\lambda\cos^2\alpha + \cos^4\alpha \geq 0$. Thus

$$|b|^2 = \left(\frac{\cos^2\alpha + \lambda}{3\cos^2\alpha - \lambda}\right)^2 \leq \frac{\cos^2\alpha - \lambda}{5\cos^2\alpha - \lambda}, \qquad \frac{2|b|^2}{1-|b|^2} \leq \frac{1}{2}\left(1 - \frac{\lambda}{\cos^2\alpha}\right). \tag{2.54}$$

From (2.51), (2.52) and (2.54), we have $g'(u) \geq 0$ and hence

$$\delta = \sqrt{g\left(\frac{1}{2}\left(1 - \frac{\lambda}{\cos^2\alpha}\right)\right)} = \frac{1+|b|^2}{1-|b|^2} - \frac{1}{2}\left(1 - \frac{\lambda}{\cos^2\alpha}\right) = \frac{2|b|}{1-|b|^2}. \tag{2.55}$$

(ii) If $0 \leq \lambda < (3-2\sqrt{2})\cos^2\alpha$ and

$$\frac{\cos^2\alpha + \lambda}{3\cos^2\alpha - \lambda} < |b| < \sqrt{\frac{\cos^2\alpha - \lambda}{5\cos^2\alpha - \lambda}}, \tag{2.56}$$

then $g'(u) > 0$ and

$$\delta = \frac{1+|b|^2}{1-|b|^2} - \frac{1}{2}\left(1 - \frac{\lambda}{\cos^2\alpha}\right) < \frac{2|b|}{1-|b|^2}. \tag{2.57}$$

(iii) If

$$(3-2\sqrt{2})\cos^2\alpha \leq \lambda \leq \cos^2\alpha, \qquad |b| = \sqrt{\frac{\sqrt{\lambda}}{2\cos\alpha + \sqrt{\lambda}}}, \tag{2.58}$$

then $\lambda^2 - 6\lambda\cos^2\alpha + \cos^4\alpha \le 0$ and so

$$|b|^2 = \frac{\sqrt{\lambda}}{2\cos\alpha + \sqrt{\lambda}} \ge \frac{\cos^2\alpha - \lambda}{5\cos^2\alpha - \lambda}, \qquad \frac{2|b|^2}{1-|b|^2} \ge \frac{1}{2}\left(1 - \frac{\lambda}{\cos^2\alpha}\right). \tag{2.59}$$

Thus we have

$$\delta = \sqrt{g\left(\frac{2|b|^2}{1-|b|^2}\right)} = \sqrt{\frac{4|b|^2}{1-|b|^2} + \frac{\lambda}{\cos^2\alpha}} = \frac{2|b|}{1-|b|^2}. \tag{2.60}$$

(iv) If $(3-2\sqrt{2})\cos^2\alpha \le \lambda \le \cos^2\alpha$ and $\sqrt{\sqrt{\lambda}/(2\cos\alpha + \sqrt{\lambda})} < |b| < 1$, then

$$\delta = \sqrt{\frac{4|b|^2}{1-|b|^2} + \frac{\lambda}{\cos^2\alpha}} < \frac{2|b|}{1-|b|^2}. \tag{2.61}$$

By virtue of (2.49), (2.50), (2.55), (2.57), (2.60), and (2.61), the proof of the theorem is now complete. □

Letting $n = \alpha = 0$ in Theorem 2.12, we have the following.

Corollary 2.13. *The function $f(z) = z/(1-bz)^2$ is in $T(0,0,\lambda)$, where $\lambda \in [0,1], b$ is complex and*

$$|b| = \begin{cases} \dfrac{1+\lambda}{3-\lambda} & (0 \le \lambda \le 3-2\sqrt{2}), \\ \sqrt{\dfrac{\sqrt{\lambda}}{2+\sqrt{\lambda}}} & (3-2\sqrt{2} \le \lambda \le 1). \end{cases} \tag{2.62}$$

The result is sharp, that is, $|b|$ cannot be increased.

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