VERTEX-STRENGTH OF FUZZY GRAPHS

CHANGIZ ESLAHCHI AND B. N. ONAGH

Received 5 June 2005; Revised 11 December 2005; Accepted 18 December 2005

The fuzzy coloring of a fuzzy graph was defined by the authors in Eslahchi and Onagh (2004). In this paper we define the chromatic fuzzy sum and strength of fuzzy graph. Some properties of these concepts are studied. It is shown that there exists an upper (a lower) bound for the chromatic fuzzy sum of a fuzzy graph.

1. Introduction

The chromatic sum of a graph G, $\sum(G)$, is introduced in the dissertation of Kubicka [3]. It is defined as the smallest possible total over all vertices that can occur among all colorings of G using natural numbers for the colors. It is known that computing the chromatic sum of an arbitrary graph is an NP-complete problem. The vertex-strength of the graph G, denoted by $s(G)$, is the smallest integer s such that $\sum(G)$ is attained using colors $\{1,2,\dots,s\}$.

In this article, we generalize these concepts to fuzzy graphs and wish to bound chromatic sum of a fuzzy graph with the e strong edges in G. We review some of the definitions of fuzzy graphs as in [2, 6, 7] and introduce some new notations.

Let X be a nonempty set and E the collection of all two-element subsets of X. A fuzzy set γ on X is a mapping $\gamma : X \to [0,1]$. Given $\alpha \in (0,1]$, the α-cut of γ is defined by $\gamma^\alpha = \{x \in X \mid \gamma(x) \geq \alpha\}$. The support and height of γ are defined by $\operatorname{supp}\gamma = \{x \in X \mid \gamma(x) > 0\}$ and $h(\gamma) = \max\{\gamma(x) \mid x \in X\}$, respectively. Fuzzy intersection of two fuzzy sets γ_1 and γ_2 is denoted by $\gamma_1 \wedge \gamma_2 = \min\{\gamma_1, \gamma_2\}$.

Let X be a finite nonempty set. The triple $G = (X,\sigma,\mu)$ is called a fuzzy graph on X where σ and μ are fuzzy sets on X and E, respectively, such that $\mu(\{x,y\}) \leq \min\{\sigma(x), \sigma(y)\}$ for all $x, y \in X$. Hereafter, we use $\mu(xy)$ for $\mu(\{x,y\})$. The fuzzy graph $G' = (X,\sigma',\mu')$ is called a fuzzy subgraph of G if for each two elements $x, y \in X$, we have $\sigma'(x) \leq \sigma(x)$ and $\mu'(xy) \leq \mu(xy)$. The fuzzy graph $G = (X,\sigma,\mu)$ is called *connected* if for every two elements $x, y \in X$, there exists a sequence of elements $x_0, x_1, \dots, x_m$ such that $x_0 = x$, $x_m = y$ and $\mu(x_i x_{i+1}) > 0$ $(0 \leq i \leq m-1)$.

Hindawi Publishing Corporation
International Journal of Mathematics and Mathematical Sciences
Volume 2006, Article ID 43614, Pages 1–9
DOI 10.1155/IJMMS/2006/43614

For any fuzzy graph $G = (X,\sigma,\mu)$ let $\mathcal{A} = \{\sigma(x) > 0 \mid x \in X\} \cup \{\mu(xy) > 0 \mid x \neq y,\ x,y \in X\}$ with k elements. Now assume $\mathcal{A} = \{\alpha_1,\alpha_2,\ldots,\alpha_k\}$ such that $\alpha_1 < \alpha_2 < \cdots < \alpha_k$. The sequence $\{\alpha_1,\alpha_2,\ldots,\alpha_k\}$ and the set $\mathcal{A}$ are called the fundamental sequence and the fundamental set of G, respectively. Given $\alpha \in (0,1]$, the α-cut of G is the graph $G^\alpha = (X^\alpha, U^\alpha)$, where $X^\alpha = \{x \in X \mid \sigma(x) \geq \alpha\}$ and $U^\alpha = \{xy \in E \mid \mu(xy) \geq \alpha\}$. It is obvious that a fuzzy graph will have a finite number of different α-cuts. In fact, the α-cuts do not change through the following intervals: $(0,\alpha_1],\ldots,(\alpha_{k-1},\alpha_k]$.

Let $\Gamma = \{\gamma_1,\ldots,\gamma_k\}$ be a finite family of fuzzy sets defined on X. The set of α-cuts of γ_i's is denoted by $\Gamma^\alpha = \{\gamma_1^\alpha,\ldots,\gamma_k^\alpha\}$ and the fuzzy set $\bigvee\Gamma$ on X is defined by $\bigvee\Gamma(x) = \max_i \gamma_i(x)$.

For fuzzy graph $G = (X,\sigma,\mu)$ the elements of X and E are called vertices and edges of G, respectively. Two vertices x and y in G are called *adjacent* if $(1/2)\min\{\sigma(x),\sigma(y)\} \leq \mu(xy)$. The edge xy of G is called *strong* if x and y are adjacent and it called *weak* otherwise. The *degree* of vertex x in G, denoted by $\deg_G x$, is the number of adjacent vertices to x and the maximum degree of G is defined by $\Delta(G) = \max\{\deg_G x \mid x \in X\}$. A connected fuzzy graph G is called a *cycle* if every vertex of X has degree 2. The fuzzy graph G with n vertices is called *complete* of order n if each vertex of G has degree $n-1$ and it is called *r-partite*, if a partition $\{A_1,\ldots,A_r\}$ of X exists such that for every two vertices $x, y \in A_i$, the edge xy is weak $(1 \leq i \leq r)$.

Definition 1.1. A family $\Gamma = \{\gamma_1,\ldots,\gamma_k\}$ of fuzzy sets on X is called a k-fuzzy coloring of $G = (X,\sigma,\mu)$ if

(a) $\bigvee\Gamma = \sigma$,

(b) $\gamma_i \wedge \gamma_j = 0$,

(c) for every strong edge xy of G, $\min\{\gamma_i(x),\gamma_i(y)\} = 0$ $(1 \leq i \leq k)$.

The least value of k for which G has a k-fuzzy coloring, denoted by $\chi^f(G)$, is called the *fuzzy chromatic number* of G.

When σ and μ take their values in $\{1\}$ and $\{0,1\}$, respectively, G is a crisp graph Definition 1.1(c) implies that for every i, $\operatorname{supp}\gamma_i$ is an independent set in G. Therefore, in this case, Γ induces a k-coloring of graph G.

The concept of chromatic number of fuzzy graphs was introduced by Muñoz et al. in [5]. They consider only fuzzy graphs with crisp vertices ($\sigma(i) = 1$ for all $i \in X$) and fuzzy edges. Let $G = (X,\mu)$ be such a fuzzy graph where $X = \{1,2,\ldots,n\}$ and μ is a fuzzy number on the set of all 2-subsets of X. Assume that $\mathcal{A} = \{\alpha_1 < \alpha_2 < \cdots < \alpha_k\}$ denote the fundamental set of G and $I = \mathcal{A} \cup \{0\}$. For each $\alpha \in I$, G_α denote the crisp graph $G_\alpha = (X, E_\alpha)$ where $E_\alpha = \{ij \mid 1 \leq i < j \leq n,\ \mu(ij) \geq \alpha\}$ and $\chi_\alpha = \chi(G_\alpha)$ denote the chromatic number of crisp graph G_α. By their definition the chromatic number of fuzzy graph G is the following fuzzy number:

$$\chi(G) = \{(i,\nu(i)) \mid i \in X\}, \tag{1.1}$$

where $\nu(i) = \max\{\alpha \in I \mid i \in A_\alpha\}$ and $A_\alpha = \{1,2,\ldots,\chi_\alpha\}$.

In our definition, the chromatic number of fuzzy graph G is a number, but in their definition they use a fuzzy number. It is easy to see that if we consider a fuzzy graph G with a crisp set of vertices and fuzzy set of edges, then there exist $\alpha \in I$ such that $\chi^f(G) = \chi_\alpha$.

But in a general fuzzy graph $G = (X,\sigma,\mu)$, it is possible that for each $\alpha \in I$, $\chi^f(G) \neq \chi_\alpha$. For example, consider the fuzzy graph $G = (X,\sigma,\mu)$ given by $X = \{1,2,3,4,5\}$,

$$\gamma(i) = \begin{cases} 1, & i \in \{1,2,3\}, \\ 0.4, & i \in \{4,5\}, \end{cases} \qquad \mu(xy) = \begin{cases} 1, & xy \in \{12,13\}, \\ 0.3, & xy \in \{23,24,25,34,35\}, \\ 0, & \text{otherwise}. \end{cases} \tag{1.2}$$

We have $\chi_0 = 5$, $\chi_{0.3} = 4$, $\chi_1 = 2$, but $\chi^f(G) = 3$. It is clear that for $0 \leq c \leq 1$, $\chi^f(G) = \chi_{\alpha_0}$, where $\alpha_0 = \min\{\alpha \in I \mid c \leq \alpha\}$, whenever we change the definition of strong edges to the following case: xy is strong if $\mu(xy) \geq c$. But if we define the strong edge by xy is strong if $\mu(xy) \geq c\min\{\sigma(x),\sigma(y)\}$, then there always exists a fuzzy graph G such that $\chi^f(G) \notin \{\chi_\alpha \mid \alpha \in I\}$ (in the above example replace 0.3 by $(c - \varepsilon)0.4$ for positive and enough small ε).

2. Chromatic fuzzy sum of fuzzy graphs

Let G be an undirected simple graph with n vertices. A coloring of the vertices of G is a mapping $f : V(G) \to N$ such that adjacent vertices are assigned different colors. In the minimum sum coloring problem (MSC), we are looking for a coloring in which the sum of the assigned colors of all the vertices of G is minimized. The MSC problem has a natural application in scheduling theory. One of the application is the problem of resource allocation with constraints imposed by conflicting resource requirements. In a common representation of the distributed resource allocation problem [1, 4], the constraints are given by a conflict graph G, in which the vertices represent processors, and the edges indicate competition on resources, that is, two vertices are adjacent if the corresponding processors cannot run their jobs simultaneously. The objective is to minimize the average response time, or equivalently to minimize the sum of the job completion times. Assuming some fix execution time for the jobs, this problem is MSC problem. Now consider scheduling n jobs on a single machine. At any given time the machine capable to perform any number of tasks, as long as these tasks are independent or the conflicts between the tasks are less than a number which depend on the choice of the problem. Any of the tasks consume some time of the machine. Let x and y be two tasks with some conflict. Suppose that the machine is capable to perform on x and y simultaneously. In this case, the amount of time that machine spends on x (or y) depends on the individual amount of time which previously was spent on x (or y) together with the measure of the conflict between x and y. Our goal is to minimize the average response time, or equivalently to minimize the sum of the task completion time. In order to solve this problem we define the fuzzy graph $G = (X,\sigma,\mu)$, where X is the set of all tasks, $\sigma(x)$ is the amount of the consuming time of the machine for each $x \in X$, and $\mu(xy)$ is the measure of the conflict between the tasks x and y. Finding the minimum value of the job completion times for this problem is equivalent to the chromatic fuzzy sum of G which we will study in this section.

Let $G = (X,\sigma,\mu)$ be a fuzzy graph, $U \subseteq X$ and $X' = X - U$. The fuzzy graph (X',σ',μ'), where for every two distinct vertices $x, y \in X'$, $\sigma'(x) = \sigma(x)$ and $\mu'(xy) = \mu(xy)$, is denoted by $G - U$.

Definition 2.1. For a k-fuzzy coloring $\Gamma = \{\gamma_1,\ldots,\gamma_k\}$ of the fuzzy graph G, Γ-*chromatic fuzzy sum* of G, denoted by $\sum_\Gamma(G)$, is defined as

$$\sum_\Gamma(G) = 1\sum_{x\in C_1}\theta_1(x)+\cdots+k\sum_{x\in C_k}\theta_k(x), \tag{2.1}$$

where $C_i = \operatorname{supp}\gamma_i$ and $\theta_i(x) = \max\{\sigma(x)+\mu(xy) \mid y \in C_i\}$.

Definition 2.2. The chromatic fuzzy sum of G, denoted by $\sum(G)$, is defined as follows:

$$\sum(G) = \inf\left\{\sum_\Gamma(G) \mid \Gamma \text{ is a fuzzy coloring of } G\right\}. \tag{2.2}$$

The number of fuzzy colorings of G is finite and therefore, there exists a fuzzy coloring Γ_0 which is called a *minimal fuzzy sum coloring* of G such that $\sum(G) = \sum_{\Gamma_0}(G)$.

Definition 2.3. The vertex-strength of fuzzy graph G, denoted by $s(G)$, is defined to be as follows:

$$s(G) = \min\left\{|\Gamma| \mid \sum(G) = \sum_\Gamma(G)\right\}. \tag{2.3}$$

It means that $s(G)$ is the minimum number of colors such that we can find the minimal fuzzy sum coloring of G by it. It is obvious that $s(G) \geq \chi^f(G)$. Note that this inequality can be strict. The fuzzy graph introduced in Example 2.13 has fuzzy chromatic number 2, but its vertex-strength is 3.

Definition 2.4. Let G be a fuzzy graph and x a vertex of G. The neighbor of x in G is defined to be the set $N(x) = \{y \mid xy \text{ is a strong edge of } G\}$.

THEOREM 2.5. *Let G be a fuzzy graph and $\Gamma_0 = \{\gamma_1,\ldots,\gamma_s\}$ a minimal fuzzy sum coloring of G. Then the following results are true.*

(1) $\sum_{x\in C_1}\theta_1(x) \geq \sum_{x\in C_2}\theta_2(x) \geq \cdots \geq \sum_{x\in C_s}\theta_s(x)$.

(2) *Let $x_0 \in C_i$. Then for each $j < i$ one of the following happens:*

  (a) $C_j \cap N(x_0) \neq \emptyset$,

  (b) $C_j\cap N(x_0) = \emptyset$ *and* $j[\sum_{x\in C_j'}\theta_j(x)-\sum_{x\in C_j}\theta_j(x)] \geq i[\sum_{x\in C_i}\theta_i(x)-\sum_{x\in C_i'}\theta_i(x)]$ *where* $C_i' = C_i - \{x_0\}$ *and* $C_j' = C_j \cup \{x_0\}$.

Proof. (1) Suppose that for some $i < j$ we have $\sum_{x\in C_i}\theta_i(x) < \sum_{x\in C_j}\theta_j(x)$. Consider the fuzzy coloring $\Gamma_0' = \{\gamma_1',\ldots,\gamma_s'\}$ defined by

$$\gamma_r' = \begin{cases} \gamma_r, & r \notin \{i,j\}, \\ \gamma_j, & r = i, \\ \gamma_i, & r = j. \end{cases} \tag{2.4}$$

Now we have

$$\sum_{\Gamma_0'}(G) - \sum_{\Gamma_0}(G) = (i-j)\left[\sum_{x\in C_j}\theta_j(x) - \sum_{x\in C_i}\theta_i(x)\right] < 0. \tag{2.5}$$

Therefore, $\sum_{\Gamma_0'}(G) < \sum_{\Gamma_0}(G)$ which contradicts the minimality of Γ_0.

(2) Let $j < i$ and $C_j \cap N(x_0) = \varnothing$. Consider the fuzzy coloring $\Gamma' = \{\gamma_1', \dots, \gamma_s'\}$ defined by $\gamma_k' = \gamma_k$ if $k \notin \{i, j\}$,

$$\gamma_i'(x) = \begin{cases} \gamma_i(x), & x \neq x_0, \\ 0, & x = x_0, \end{cases} \qquad \gamma_j'(x) = \begin{cases} \gamma_j(x), & x \neq x_0, \\ \sigma(x_0), & x = x_0. \end{cases} \tag{2.6}$$

Now since Γ_0 is the minimal fuzzy coloring, we have

$$0 \leq \sum_{\Gamma'}(G) - \sum_{\Gamma_0}(G) = j\left[\sum_{x\in C_j'}\theta_j(x) - \sum_{x\in C_j}\theta_j(x)\right] - i\left[\sum_{x\in C_i}\theta_i(x) - \sum_{x\in C_i'}\theta_i(x)\right] \tag{2.7}$$

and the proof is complete. □

It seems that by the hypothesis of Theorem 2.5, we have $\sum(G - C_1) = \sum_{\Gamma_0}(G) - \sum_{i=1}^{s}[\sum_{x\in C_i}\theta_i(x)]$ and $s(G - C_1) = s(G) - 1$. But, in the following example, we show that these equalities do not always hold.

Example 2.6. Let $X = \{x_1, x_2, \dots, x_{26}\}$. Construct the fuzzy graph $G = (X, \sigma, \mu)$ as follows:

$$\sigma(x_i) = 1, \quad 1 \leq i \leq 26,$$

$$\mu(x_i x_j) = \begin{cases} 1, & 1 \leq i \leq 10,\ 11 \leq j \leq 26, \\ 1, & 11 \leq i \leq 17,\ 19 \leq j \leq 26, \\ 1, & i = 18,\ 20 \leq j \leq 26, \\ 0.2, & i = 18,\ j = 19, \\ 0, & 1 \leq i < j \leq 10, \\ 0.3, & 11 \leq i < j \leq 17 \text{ or } 20 \leq i < j \leq 26, \\ 0.4, & i = 18,\ 11 \leq j \leq 17 \text{ or } i = 19,\ 20 \leq j \leq 26. \end{cases} \tag{2.8}$$

One can check that $s(G) = 4$ and $\Gamma_0 = \{\gamma_1, \dots, \gamma_4\}$ is the minimal fuzzy sum coloring of G where

$$\gamma_1(x_i) = \begin{cases} 1, & 1 \leq i \leq 10, \\ 0, & \text{otherwise}, \end{cases} \qquad \gamma_2(x_i) = \begin{cases} 1, & 11 \leq i \leq 17, \\ 0, & \text{otherwise}, \end{cases}$$
$$\gamma_3(x_i) = \begin{cases} 1, & 20 \leq i \leq 26, \\ 0, & \text{otherwise}, \end{cases} \qquad \gamma_4(x_i) = \begin{cases} 1, & 18 \leq i \leq 19, \\ 0, & \text{otherwise}, \end{cases} \tag{2.9}$$

and $\sum_{\Gamma_0}(G) = 65.1$. But $s(G - C_1) = 2$ and $\Gamma_1 = \{\xi_1, \xi_2\}$ is a minimal fuzzy sum coloring of $G - C_1$ where

$$\xi_1(x_i) = \begin{cases} 1, & 11 \le i \le 18, \\ 0, & \text{otherwise}, \end{cases} \qquad \xi_2(x_i) = \begin{cases} 1, & 19 \le i \le 26, \\ 0, & \text{otherwise}, \end{cases} \tag{2.10}$$

and $\sum_{\Gamma}(G - C_1) = 33.6$. (If Γ is a 3-fuzzy coloring of $G - C_1$, one can check that $\sum_{\Gamma}(G - C_1) > 33.6$, therefore $s(G - C_1) = 2$.) On the other hand, $\sum_{\Gamma_0}(G) - \sum_{i=1}^{4}[\sum_{x\in C_i}\theta_i(x)] = 34.5$, hence $\sum(G - C_1) < \sum_{\Gamma_0}(G) - \sum_{i=1}^{4}[\sum_{x\in C_i}\theta_i(x)]$.

Actually, we never achieved minimal fuzzy sum coloring of G by extending Γ_1 to a fuzzy coloring of G. If by extending Γ_1 we find a minimal fuzzy sum coloring of G, then by Theorem 2.5(1), this extension is as $\Gamma' = \{\xi_1', \xi_2', \xi_3'\}$ where,

$$\begin{gathered} \xi_1'(x_i) = \begin{cases} 1, & \text{if } x \in C_1, \\ 0, & \text{otherwise}, \end{cases} \\ \xi_2'(x_i) = \begin{cases} \xi_1(x_i), & \text{if } x_i \in G - C_1, \\ 0, & \text{otherwise}, \end{cases} \\ \xi_3'(x_i) = \begin{cases} \xi_2(x_i), & \text{if } x_i \in G - C_2, \\ 0, & \text{otherwise}. \end{cases} \end{gathered} \tag{2.11}$$

But $\sum_{\Gamma'}(G) = 66$ and Γ' is not a minimal fuzzy sum coloring of G.

Definition 2.7. Let G be a fuzzy graph with the fuzzy chromatic number $\chi^f(G)$. Define $\sum_{\chi^f(G)}(G)$ as

$$\sum_{\chi^f(G)}(G) = \min\left\{\sum_{\Gamma}(G) \mid |\Gamma| = \chi^f(G)\right\}. \tag{2.12}$$

THEOREM 2.8. *For fuzzy graph $G = (X, \sigma, \mu)$,*

$$\sum_{\chi^f(G)}(G) \le \frac{3(\chi^f(G)+1)}{4} h(\sigma)|X|. \tag{2.13}$$

Proof. Let Γ_1 be a $\chi^f(G)$-fuzzy coloring of G such that $\sum_{\chi^f(G)}(G) = \sum_{\Gamma_1}(G)$. Similar to Theorem 2.5(1), we have $\sum_{x\in C_1}\theta_1(x) \ge \sum_{x\in C_2}\theta_2(x) \ge \cdots \ge \sum_{x\in C_{\chi^f(G)}}\theta_{\chi^f(G)}(x)$. Hence, for each i, $1 \le i \le \chi^f(G)$, we have

$$\begin{aligned} & i\sum_{x\in C_i}\theta_i(x) + (\chi^f(G) - i + 1)\sum_{x\in C_{\chi^f(G)}}\theta_{\chi^f(G)}(x) \\ & \quad \le \frac{\chi^f(G)+1}{2}\left[\sum_{x\in C_i}\theta_i(x) + \sum_{x\in C_{\chi^f(G)}}\theta_{\chi^f(G)}(x)\right]. \end{aligned} \tag{2.14}$$

So,

$$\begin{aligned}\sum_{i=1}^{\chi^f(G)}\left[i\sum_{x\imath nC_i}\theta_i(x)+\left(\chi^f(G)-i+1\right)\sum_{x\in C_{\chi^f(G)}}\theta_{\chi^f(G)}(x)\right]\\ \leq\sum_{i=1}^{\chi^f(G)}\frac{\chi^f(G)+1}{2}\left[\sum_{x\in C_i}\theta_i(x)+\sum_{x\in C_{\chi^f(G)}}\theta_{\chi^f(G)}(x)\right].\end{aligned}\tag{2.15}$$

Then,

$$\sum_{i=1}^{\chi^f(G)}i\sum_{x\in C_i}\theta_i(x)\leq\frac{\chi^f(G)+1}{2}\sum_{i=1}^{\chi^f(G)}\sum_{x\in C_i}\theta_i(x).\tag{2.16}$$

But since $\theta_i(x)\leq 3h(\sigma)/2$, we have $\sum_{i=1}^{\chi^f(G)}\sum_{x\in C_i}\theta_i(x)\leq(3h(\sigma)/2)|X|$ and the proof is complete. □

Now by Theorem 2.8, we find an upper bound for $\sum(G)$.

Corollary 2.9. *For the fuzzy graph G,*

$$\sum(G)\leq\frac{3\left(\chi^f(G)+1\right)}{4}h(\sigma)|X|.\tag{2.17}$$

Definition 2.10. For the fuzzy graph $G=(X,\sigma,\mu)$, define

$$w=\min\{\sigma(x)+\mu(xy)>0\mid x\in X,\ xy\text{ is a weak edge of }G\}.\tag{2.18}$$

Theorem 2.11. *Let $G=(X,\sigma,\mu)$ be a connected fuzzy graph with e strong edges. Then,*

$$w\sqrt{8e}\leq\sum(G).\tag{2.19}$$

Proof. Let $\mathscr{A}$ be the fundamental set of G. Among all connected fuzzy graphs with e strong edges, fundamental set $\mathscr{A}'\subset\mathscr{A}$, and minimum chromatic fuzzy sum value, select $G_0=(X_0,\sigma_0,\mu_0)$ and its minimal fuzzy coloring $\Gamma=\{\gamma_1,\dots,\gamma_s\}$ to have the largest $\sum_{x\in\operatorname{supp}\gamma_1}\theta_1(x)$. Suppose $s(G_0)\geq 3$, $C_3=\operatorname{supp}\gamma_3=\{x_1,\dots,x_k\}$, and $y_1,\dots,y_k$ are new vertices. Consider the fuzzy graph $G_1=(X_1,\sigma_1,\mu_1)$ with $X_1=X_0\cup\{y_1,\dots,y_k\}$ as follows:

$$\begin{gathered}\sigma_1(a)=\begin{cases}\sigma_0(a), & a\in X_0,\\ \sigma_0(x_i), & a=y_i,\end{cases}\\ \mu_1(xy)=\begin{cases}\mu_0(xy), & x,y\notin C_3\cup\{y_1,\dots,y_k\},\\ \min\{\sigma_0(x),\sigma_0(y)\}, & x=x_i,\ y\in C_1,\\ \min\{\sigma_1(y_i),\sigma_0(y)\}, & x=y_i,\ y\in C_2,\\ \mu_0(xy), & x,y\in C_3,\\ \mu_0(x_ix_j), & x=y_i,\ y=y_j,\\ 0, & \text{otherwise}.\end{cases}\end{gathered}\tag{2.20}$$

Now $\Gamma_1 = \{\gamma'_1,\dots,\gamma'_{s-1}\}$ defined by

$$\gamma'_1(x) = \begin{cases} \gamma_1(x), & x \in X_0, \\ \sigma_1(y_i), & x = y_i, \end{cases}$$
$$\gamma'_2(x) = \begin{cases} \gamma_2(x), & x \in X_0 - C_3, \\ \sigma_0(x), & x \in C_3, \\ 0, & \text{otherwise}, \end{cases} \tag{2.21}$$
$$\gamma'_i = \gamma_{i-1}, \quad 4 \le i \le s.$$

Γ_1 is a fuzzy coloring of G_1 with $\sum_{\Gamma_1}(G_1) = \sum_{\Gamma}(G_0)$. But, G_1 has at least e strong edges. By deleting some strong edges from G_1, we find a connected fuzzy graph G_2 with exactly e strong edges and fundamental set $\mathcal{A}' \subset \mathcal{A}$ such that $\sum(G_2) \le \sum_{\Gamma_1}(G_1) = \sum(G_0)$. But, by the minimality of $\sum(G_0)$, we have $\sum(G_2) = \sum(G_0)$. On the other hand, $\sum_{x\in \operatorname{supp}\gamma'_1} \theta_1(x)$ is greater than $\sum_{x\in\operatorname{supp}\gamma_1} \theta_1(x)$ which is a contradiction. Therefore, $s(G_0) \le 2$, $\Gamma = \{\gamma_1,\gamma_2\}$ and G_0 is bipartite. Hence,

$$\begin{aligned} \sum(G_0) &= \sum_{x\in\operatorname{supp}\gamma_1} \theta_1(x) + 2\sum_{x\in\operatorname{supp}\gamma_2} \theta_2(x) \\ &\ge w|\operatorname{supp}\gamma_1| + 2w|\operatorname{supp}\gamma_2| \\ &= w(|\operatorname{supp}\gamma_1| + 2|\operatorname{supp}\gamma_2|). \end{aligned} \tag{2.22}$$

Now since the number of strong edges in a bipartite fuzzy graph with fuzzy coloring $\{\gamma_1,\gamma_2\}$ is at most $|\operatorname{supp}\gamma_1|\cdot|\operatorname{supp}\gamma_2|$, then $\sqrt{8e} \le |\operatorname{supp}\gamma_1| + 2|\operatorname{supp}\gamma_2|$. Hence $\sum(G_0) \ge w\sqrt{8e}$ and the proof is complete. □

We will end this section by giving a counter example to show that a well-known theorem in graph theory is no longer valid in the case of fuzzy graphs. Also we will give a conjecture for upper bond of $s(G)$.

We have known the following theorem for graphs [2].

Theorem 2.12. *Let G be a connected graph. Then, $s(G) \le \Delta(G) + 1$ and equality hold if and only if G is a complete graph or an odd cycle.*

In the following example, we show that the above theorem is not hold for fuzzy version.

Example 2.13. Let $G = (\sigma,\mu)$ be a fuzzy graph with vertex set $X = \{1,2,3,4,5,6\}$, $\sigma(i) = 1$ for each $i \in X$ and

$$\mu(xy) = \begin{cases} 0.49, & xy \in \{13,35,24,46\}, \\ 1, & \text{otherwise}. \end{cases} \tag{2.23}$$

We have $\Delta(G) = 2$. If $s(G) = 2$ we have a fuzzy coloring Γ of size 2 such that $\sum_{\Gamma}(G) = \sum(G)$. But every 2-fuzzy coloring of G is as $\Gamma = \{\gamma_1,\gamma_2\}$,

$$\gamma_1 = \begin{cases} 1, & x \in A, \\ 0, & x \in B, \end{cases} \qquad \gamma_2 = \begin{cases} 0, & x \in A, \\ 1, & x \in B, \end{cases} \tag{2.24}$$

(or $\gamma_1 |_B = 1$ and $\gamma_2 |_A = 1$) where $A = \{1,3,5\}$ and $B = \{2,4,6\}$. In this case, $\sum_\Gamma(G) = 13.41$. Now, consider the fuzzy coloring $\Gamma' = \{\gamma_1, \gamma_2, \gamma_3\}$ as

$$\gamma_1(x) = \begin{cases} 1, & x \in \{1,4\}, \\ 0, & \text{otherwise}, \end{cases} \qquad \gamma_2(x) = \begin{cases} 1, & x \in \{2,5\}, \\ 0, & \text{otherwise}, \end{cases} \qquad \gamma_3(x) = \begin{cases} 1, & x \in \{3,6\}, \\ 0, & \text{otherwise}. \end{cases} \tag{2.25}$$

In this case, $\sum_{\Gamma'}(G) = 12$ which implies $s(G) > 2$. It is easy to show that $s(G) = 3$. But, G is neither an odd cycle nor a complete fuzzy graph.

Conjecture 2.14. *Let G be a fuzzy graph. Then,*

(1) $s(G) \leq 2\Delta(G) + 1$,

(2) *for every integer $k \geq 2$, there exists a fuzzy graph G_k such that $k - 1 \leq \Delta(G_k)$ and $s(G_k) = \Delta(G_k) + k$.*

References

[1] K. Chandy and J. Misra, *The drinking philosophers problem*, ACM Transactions on Programming Languages and Systems **6** (1984), no. 4, 632–646.

[2] Ch. Eslahchi and B. N. Onagh, *Fuzzy coloring of fuzzy graphs*, 2004, submitted.

[3] E. Kubicka, *The chromatic sum of a graph*, Ph.D. dissertation, Western Michigan University, Michigan, 1989.

[4] N. A. Lynch, *Upper bounds for static resource allocation in a distributed system*, Journal of Computer and System Sciences **23** (1981), no. 2, 254–278.

[5] S. Muñoz, T. Ortuño, J. Ramirez, and J. Yáñez, *Coloring Fuzzy graphs*, Omega **32** (2005), 211–221.

[6] A. Rosenfeld, *Fuzzy graphs*, Fuzzy Sets and Their Applications to Cognitive and Decision Processes (Proceeding of U.S.-Japan Sem., University of California, Berkeley, Calif, 1974) (L. A. Zadeh, K. S. Fu, and M. Shimura, eds.), Academic Press, New York, 1975, pp. 77–95.

[7] L. A. Zadeh, *Fuzzy sets*, Information and Computation **8** (1965), 338–353.

Changiz Eslahchi: Department of Mathematical Sciences, Shahid Beheshti University, Tehran 1983963113, Iran
E-mail address: ch-eslahchi@sbu.ac.ir

B. N. Onagh: Department of Mathematical Sciences, Shahid Beheshti University, Tehran 1983963113, Iran
E-mail address: b_onagh@yahoo.com