ON THE BASIS NUMBER OF THE CORONA OF GRAPHS

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Received 5 February 2005; Revised 19 June 2006; Accepted 22 June 2006

The basis number $b(G)$ of a graph G is defined to be the least integer k such that G has a k-fold basis for its cycle space. In this note, we determine the basis number of the corona of graphs, in fact we prove that $b(v \circ T) = 2$ for any tree and any vertex v not in T, $b(v \circ H) \leq b(H) + 2$, where H is any graph and v is not a vertex of H, also we prove that if $G = G_1 \circ G_2$ is the corona of two graphs G_1 and G_2, then $b(G_1) \leq b(G) \leq \max\{b(G_1), b(G_2) + 2\}$, moreover we prove that if G is a Hamiltonian graph, then $b(v \circ G) \leq b(G) + 1$, where v is any vertex not in G, and finally we give a sequence of remarks which gives the basis number of the corona of some of special graphs.

1. Introduction

In this note, we consider only finite, undirected, simple graphs. Our terminology and notation will be standard except as indicated. For undefined terms, see [7]. Let G be a (p,q) graph (i.e., G has p vertices and q edges), and let $e_1, e_2, \ldots, e_q$ be an ordering of its edges. Then any subset E of edges in G corresponds to $(0,1)$-vector $(v_1, \ldots, v_q)$ with $v_i = 1$ if $e_i \in E$ and $v_i = 0$ if $e_i \notin E$. The vectors form a q-dimensional vector space over the field of two elements Z_2 and is denoted by $(Z_2)^q$. The vectors in $(Z_2)^q$ which correspond to the cycles in G generate a subspace called the cycle space of G and is denoted by $C(G)$, we will say that the cycles themselves, instead of saying the vectors corresponding to the cycles, generate $C(G)$. It is well known (see [7, page 39]) that if G is a (p,q) graph with k components, then $\dim C(G) = \gamma(G) = q - p + k$, where $\gamma(G)$ is the cyclomatic number of G. A basis for $C(G)$ is called a k-fold basis if each edge of G occurs in at most k of the cycles in the basis. The basis number of G denoted by $b(G)$ is the smallest integer k such that $C(G)$ has a k-fold basis. The corona (see [7, page 167]) of two graphs G_1 and G_2, denoted by $G_1 \circ G_2$, is defined to be the graph G obtained by taking one copy of G_1 (which has p_1 vertices) and p_1 copies of G_2, and then joining the vertex of G_1 to every vertex in the ith copy of G_2. If G_1 is a (p_1,q_1) graph and G_2 is a (p_2,q_2) graph, then it follows from the definition of the corona that $G_1 \circ G_2$ has $p_1(1+p_2)$ vertices and $q_1 + p_1q_2 + p_1p_2$ edges

Hindawi Publishing Corporation
International Journal of Mathematics and Mathematical Sciences
Volume 2006, Article ID 53712, Pages 1–3
DOI 10.1155/IJMMS/2006/53712

(see [7, page 168]). It is clear that if G_1 is connected, then $G_1 \circ G_2$ is connected, and in general $G_1 \circ G_2$ is not isomorphic to $G_2 \circ G_1$.

In the rest of this note, P_n, C_n, S_n, and W_n stand for the path, the cycle, the star, and the wheel of n vertices. A theta graph θ_n is defined to be a cycle C_n with n vertices, respectively, to which we add a new edge that joins two nonadjacent vertices of C_n.

MacLane [8] proved that a graph G is planar if and only if $b(G) \le 2$. Schmeichel [9] proved that for $n \ge 5$, $b(K_n) = 3$ and for $m,n \ge 5$, $b(K_{m,n}) = 4$ except for $K_{6,10}$, $K_{5,n}$, and $K_{6,n}$, where $n = 5, 6, 7$, and 8. Banks and Schmeichel [6] proved that for $n \ge 7$, $b(Q_n) = 7$, where Q_n is the n-cube. Ali [1] proved that $b(K_{n,n,\dots,n}) \le 9$, $b(K_{n,n,n}) = 3$ for all $n \ge 3$, and $b(K_{l,m,n}) \le 4$. Moreover, Ali [2] proved that $b(C_m \wedge P_n) \le 2$, and $b(C_m \wedge C_n) = 3$. Al-Rhayyel [4] proved that $b(P_2 \times \theta_n) = 2$ and $b(\theta_n \times \theta_m) = 3$ for all $n,m \ge 4$. Al-Rhayyel [5] proved that $b(P_2 \wedge W_n) = 2$ and $b(P_m \wedge W_n) = 3$ for all $m \ge 3$, $n \ge 4$, and n is even where $\wedge$ and $\times$ are the direct and the cartesian products of graphs, respectively. Next we restate [3, Theorem 2.3].

Theorem 1.1. *Let G' be a graph obtained from G by deleting an edge e of at most 2-fold in a basis B for $C(G)$. Then $b(G) - 1 \le b(G') \le b(G)$.*

The purpose of this note is to investigate the basis number of the corona of graphs, in fact we prove that for any two graphs G_1 and G_2, if $G = G_1 \circ G_2$, then $b(G_1) \le b(G) \le \max\{b(G_1), b(G_2)+2\}$ and we give the exact basis number of the corona of some special graphs.

2. Main results

This section is devoted for proving the main results of this note, and this is done by writing a sequence of theorems and remarks.

Remark 2.1. We note that if v is not a vertex of G, then $b(v \circ G) = 2$, where G is any one of the following graphs: P_n, C_n or S_n and $b(v \circ G) = 3$ if G is either W_n or K_n $(n \ge 4)$.

Lemma 2.2. *Let T be a tree with p vertices $(p \ge 3)$ if v is any point which is not a vertex of T, and if $G = v \circ T$, then $b(G) = 2$, and hence G is planar.*

Proof. Assume that G is not planar. Then, by Kuratowski's theorem, G contains a subdivision of K_5 or $K_{3,3}$. Then $G - x$ cannot be acyclic graph for any $x \in V(G)$, while $G - v$ is a tree. This is a contradiction, and hence G is planar. Therefore, $b(G) \le 2$. If $b(G) = 1$, then G has a 1-fold basis, which implies that $\dim C(G) \le |E(G)|/3$ since each cycle contains at least three edges. Since $|E(G)| = 2p - 1$ and $\dim C(G) = p - 1$, we have $p - 1 \le (2p - 1)/3$, which implies that $p \le 2$. This is a contradiction. Therefore, $b(G) = 2$. □

Lemma 2.3. *Let H be any connected (p,q) graph and let v be any vertex which is not a vertex of H. If $G = v \circ H$, then $b(G) \le b(H)+2$.*

Proof. Let $u_1, u_2, \dots, u_p$ be the vertices of H. Since $\dim C(G) = q$ and $\dim C(H) = q - p + 1$, $\dim C(G) - \dim C(H) = p - 1$. Let T be a spanning tree of H. Then $b(v \circ T) = 2$, $\dim C(v \circ T) = p - 1$, and each cycle in $v \circ T$ must contain an edge of the form vu_i for some $i \in \{1,2,\dots,p\}$. Thus the cycles in $v \circ T$ are independent from the cycles in H. Let B_1 be a $b(H)$-fold basis for $C(H)$, and let B_2 be a 2-fold basis for $v \circ T$. Then clearly

$B = B_1 \cup B_2$ is an independent set of cycles with $|B| = \dim C(G)$, hence B is a basis for $C(G)$. Note that if e is an edge of G, then either e is an edge of H or $e = vu_i$ for some $i \in \{1,2,\ldots,p\}$. If $e = vu_i$, then $f_B(e) \le f_{B_2}(e) \le 2$, and if e is an edge of H, then clearly $f_B(e) \le b(H)+2$. Thus, $b(G) \le b(H)+2$. □

Theorem 2.4. *Let G_1 and G_2 be two connected graphs. If $G = G_1 \circ G_2$, then $b(G_1) \le b(G) \le \max\{b(G_1), b(G_2)+2\}$.*

Proof. Clearly $b(G_1) \le b(G)$. Let $v_1,\ldots,v_n$ be the vertices of G_1 and let $H_k = v_k \circ G_2$ and let B_k be the basis of H_k, for each $k = 1,\ldots,n$. Clearly $E(H_i) \cap E(H_j) = \phi$, for all $i \neq ji$, $j \in \{1,\ldots,n\}$. Therefore, $\bigcup_{k=1}^{n} B_k$ is linearly independent. Let $B = (\bigcup_{k=1}^{n} B_k) \cup B(G_1)$, where $B(G_1)$ is a $b(G_1)$-fold basis for G_1. Since $E(G_1) \cap E((\bigcup_{k=1}^{n} H_k)) = \phi$, as a result B is linearly independent. Since $|E(B)| = \dim C(G)$, B is a basis of $C(G)$. By Lemma 2.3, $b(H_k) \le b(G_2)+2$ for each $k \in \{1,\ldots,n\}$. Therefore, $b(G_1) \le b(G) \le \max\{b(G_1), b(G_2)+2\}$. □

Lemma 2.5. *Let H be a Hamiltonian graph, and let v be any point which is not a vertex of H. If $G = v \circ H$, then $b(G) \le b(H)+1$.*

Proof. Let $C = u_1u_2,\ldots,u_nu_1$ be a spanning cycle of H, then G is obtained from H by joining every vertex v_i of H to the vertex v. Let $B = \{vu_iu_{i+1} : i = 1,2,\ldots,n-1\} \cup \mathrm{B}(H)$, where $B(H)$ is a $b(H)$-fold basis of $C(H)$. Then, clearly that B is a basis of $C(G)$ and given any edge of H, then it occurs in at most one of these cycles, hence $b(G) \le b(H)+1$. □

Corollary 2.6. *If G_1 and G_2 are two graphs such that $b(G_1) \ge b(G_2)+2$, then $b(G_1 \circ G_2) = b(G_1)$. Moreover, if G_2 is Hamiltonian, and $b(G_1) \ge b(G_2)+1$, then $b(G_1 \circ G_2) = b(G_1)$.*

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