ON THE MEAN VALUE PROPERTY OF SUPERHARMONIC AND SUBHARMONIC FUNCTIONS

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We prove a converse of the mean value property for superharmonic and subharmonic functions. The case of harmonic functions was treated by Epstein and Schiffer.

Recall that a function u is harmonic (superharmonic, subharmonic) in an open set $U \subset \mathbb{R}^n$ $(n \geq 1)$ if $u \in C^2(U)$ and $\Delta u = 0$ $(\Delta u \leq 0, \Delta u \geq 0)$ on U. Denote by $H(U)$ the space of harmonic functions in U and $SH(U)$ $(sH(U))$ the subset of $C^2(U)$ consisting of superharmonic (subharmonic) functions in U. If $A \subset \mathbb{R}^n$ is Lebesgue measurable, $L^1(A)$ denotes the space of Lebesgue integrable functions on A and $|A|$ denotes the Lebesgue measure of A when A is bounded.

We also recall the mean value property of harmonic, superharmonic, and subharmonic functions in U ([2]): if $x \in U$ and $B(x,r) = \{y \in \mathbb{R}^n; \|y - x\| < r\}$, $r > 0$, is such that $\overline{B}(x,r) \subset U$, then for all $u \in H(U)$ $(SH(U), sH(U))$,

$$u(x) = (\geq, \leq) \frac{1}{|B(x,r)|} \int_{B(x,r)} u(y)\,dy. \tag{1}$$

Using the Lebesgue-dominated convergence theorem we see that the conclusion above holds whenever $B(x,r) \subset U$ if $u \in H(U) \cap L^1(B(x,r))$ $(SH(U) \cap L^1(B(x,r)), sH(U) \cap L^1(B(x,r)))$. Epstein and Schiffer [1] proved the following converse.

Theorem 1. *Let $\Omega \subset \mathbb{R}^n$ $(n \geq 1)$ be a bounded open set. Suppose that there exists $x_0 \in \Omega$ such that*

$$u(x_0) = \frac{1}{|\Omega|} \int_{\Omega} u(x)\,dx \tag{2}$$

for every $u \in H(\Omega) \cap L^1(\Omega)$. Then Ω is a ball with center x_0.

A more general result was obtained by Kuran [3]. In this note we give a proof of the following converse.

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Theorem 2. *Let $\Omega \subset \mathbb{R}^n$ $(n \geq 1)$ be a bounded open set. Suppose that there exists $x_0 \in \Omega$ such that*

$$u(x_0) \geq (\leq) \frac{1}{|\Omega|} \int_\Omega u(x)dx \tag{3}$$

for every $u \in SH(\Omega) \cap L^1(\Omega) \backslash H(\Omega)$ $(sH(\Omega) \cap L^1(\Omega) \backslash H(\Omega))$. Then Ω is a ball with center x_0.

Proof. Clearly it is enough to consider the case of superharmonic functions. Since Ω is bounded, there exists a largest open ball B centered at x_0 of radius r_1 which lies in Ω. The compactness of $\partial\Omega$ implies that there is some $x_1 \in \partial\Omega$ such that $\|x_1 - x_0\| = r_1$. We will show that $\Omega = B$. Define

$$h(x) = r_1^{n-2}\left(\|x - x_0\|^2 - r_1^2\right)\|x - x_1\|^{-n} \tag{4}$$

for $x \in \mathbb{R}^n \backslash \{x_1\}$. Then $h \in H(\mathbb{R}^n \backslash \{x_1\})$ and $h(x_0) = -1$. Now let $R > r_1$ be such that $\Omega \subset B(x_0, R)$. For $k \in \mathbb{N}^*$ we set

$$u_k(x) = 1 + h(x) + \frac{1}{2nk}\left(R^2 - \|x - x_0\|^2\right), \quad x \in \Omega. \tag{5}$$

Obviously $u_k \in C^2(\Omega)$ and $\Delta u_k = -1/k$ in Ω, hence $u_k \in SH(\Omega) \backslash H(\Omega)$. Moreover $u_k \in L^1(\Omega)$ and $u_k(x) \geq 1$ for $x \in \Omega \backslash B$. Since $1 + h \in H(\Omega) \cap L^1(\Omega)$, we have

$$0 = 1 + h(x_0) = \int_B (1 + h(x))dx. \tag{6}$$

Now using (6) we can write

$$\begin{aligned}
\frac{R^2}{2nk} = u_k(x_0) &\geq \frac{1}{|\Omega|}\int_\Omega u_k(x)dx = \frac{1}{|\Omega|}\int_{\Omega\backslash B} u_k(x)dx + \frac{1}{|\Omega|}\int_B u_k(x)dx \\
&= \frac{1}{|\Omega|}\int_{\Omega\backslash B} u_k(x)dx + \frac{1}{2nk|\Omega|}\int_B \left(R^2 - \|x - x_0\|^2\right)dx \\
&\geq \frac{|\Omega\backslash B|}{|\Omega|} + \frac{\omega_n r_1^n}{2nk|\Omega|}\left(\frac{R^2}{n} - \frac{r_1^2}{n+2}\right) \\
&\geq \frac{|\Omega\backslash \overline{B}|}{|\Omega|} + \frac{\omega_n r_1^n}{2nk|\Omega|}\left(\frac{R^2}{n} - \frac{r_1^2}{n+2}\right)
\end{aligned} \tag{7}$$

for all $k \in \mathbb{N}^*$, where ω_n denotes the measure of the unit sphere in $\mathbb{R}^n$. This implies that $|\Omega\backslash\overline{B}| = 0$. Then the open set $\Omega\backslash\overline{B}$ must be empty, hence $\Omega \subset \overline{B}$. Since Ω is open and $B \subset \Omega \subset \overline{B}$, we deduce that $\Omega = B$. □

References

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