

MAPPINGS WHICH PRESERVE REGULAR DODECAHEDRONS

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We prove that if a one-to-one mapping $f : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ preserves regular dodecahedrons, then f is a linear isometry up to translation.

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1. Introduction

Let us begin with the properties of an isometry. For normed spaces X and Y , a mapping $f : X \rightarrow Y$ is called an isometry if f satisfies the equality

$$\|f(x) - f(y)\| = \|x - y\| \quad (1.1)$$

for all $x, y \in X$. A distance $r > 0$ is said to be preserved by a mapping $f : X \rightarrow Y$ if $\|f(x) - f(y)\| = r$ for all $x, y \in X$ whenever $\|x - y\| = r$.

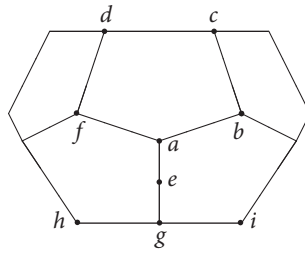
If f is an isometry, then every distance $r > 0$ is preserved by f , and conversely. We can now raise a question whether each mapping that preserves certain distances is an isometry. Indeed, Aleksandrov [1] had raised a question whether a mapping $f : X \rightarrow X$ preserving a distance $r > 0$ is an isometry, which is now known to us as the Aleksandrov problem.

Beckman and Quarles [2] solved the Aleksandrov problem for finite-dimensional real Euclidean spaces $X = \mathbb{R}^n$ (see also [3–6, 10, 12–17]).

THEOREM 1.1 (Beckman and Quarles). *If a mapping $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ ($2 \leq n < \infty$) preserves a distance $r > 0$, then f is a linear isometry up to translation.*

It is an interesting question whether the “distance $r > 0$ ” in the above theorem can be replaced by some properties characterized by “geometrical figures” without loss of its validity.

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Dodecahedron A

Figure 2.1

In [7–9], the authors proved that if a one-to-one mapping $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ maps every unit circle (or unit sphere, tetrahedron) onto a unit circle (or a unit sphere, tetrahedron, resp.), then f is a linear isometry up to translation.

In this connection, we will extend these results to the more general three-dimensional objects, that is, we prove in this note that if a one-to-one mapping $f : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ maps every regular dodecahedron onto a regular dodecahedron, then f is a linear isometry up to translation.

2. Main theorem

In the following, by a regular dodecahedron we mean a regular dodecahedron with its side length one. We first make our terms precise as follows. In Figure 2.1, we will call the point a a “vertex” and the line \overline{ab} an “edge” and the plane bounded by the five edges \overline{ab} , \overline{bc} , \overline{cd} , \overline{df} , \overline{fa} “face $abcdf$ ” or simply a “face.” Further by a dodecahedron we will mean the 12 faces only and not the three-dimensional open set bounded by those 12 faces. Let us denote the three-dimensional open set bounded by the dodecahedron A as “Inside of A ” or simply as $\text{Ins}A$.

Now we begin with the following lemma.

LEMMA 2.1. *Suppose that a one-to-one mapping $f : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ maps every dodecahedron onto a dodecahedron. Then, for any dodecahedrons A and B , if $\text{Ins}A \cap \text{Ins}B = \phi$, $\text{Ins}f(A) \cap \text{Ins}f(B) = \phi$.*

Proof. First, we show that if $q \notin \text{Ins}A$, then $f(q) \notin \text{Ins}f(A)$. In other words, we show that if $f(q) \in \text{Ins}f(A)$, then $q \in \text{Ins}A$. Suppose that $q \in A$. Then $f(q) \in f(A)$ and so $f(q) \notin \text{Ins}f(A)$. Suppose that $q \notin \text{Ins}A$ and $q \notin A$. Then choose another dodecahedron B such that $q \in B$ and $B \cap A = \phi$. Then $f(B) \cap f(A) = \phi$ and therefore $f(q) \notin \text{Ins}f(A)$.

Suppose now that $\text{Ins}f(A) \cap \text{Ins}f(B) \neq \phi$. Then $\text{Ins}f(A) \cap f(B) \neq \phi$, which means that for some $b \in B$, $f(b) \in \text{Ins}f(A)$. Therefore $b \in \text{Ins}A$ and $(\text{Ins}A) \cap B \neq \phi$ by which we conclude that $\text{Ins}A \cap \text{Ins}B \neq \phi$. \square

We show now that if any one-to-one mapping preserves regular dodecahedrons, then it is actually an isometry. More precisely, we have the following.

THEOREM 2.2. *If a one-to-one mapping $f : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ maps every regular dodecahedron onto a regular dodecahedron, then f is a linear isometry up to translation.*

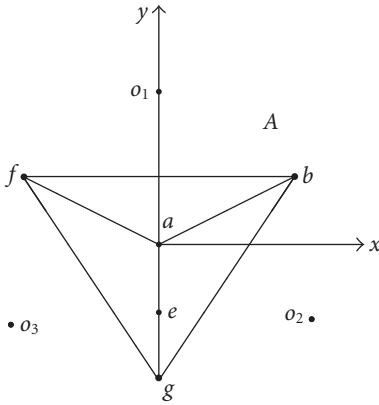


Figure 2.2

$$\begin{aligned}
 a &= (0,0,0) \\
 b &= (\sin 54^\circ, \frac{1}{\sqrt{3}} \sin 54^\circ, -z_0) \\
 f &= (-\sin 54^\circ, \frac{1}{\sqrt{3}} \sin 54^\circ, -z_0) \\
 g &= (0, -\frac{2}{\sqrt{3}} \sin 54^\circ, -z_0) \\
 e &= (0, -\frac{1}{\sqrt{3}} \sin 54^\circ, -\frac{1}{2}z_0) \\
 z_0 &= \sqrt{1 - \frac{4}{3} \sin^2 54^\circ}
 \end{aligned}$$

Proof. We show f preserves the distance 1. Then the theorem of Beckman and Quarles implies that f is an isometry.

We will use solid angle arguments. Let A be a regular dodecahedron. For any $p \in A$, let us denote the solid angle that $\text{Ins}A$ subtends with respect to $p \in A$ as $\Omega(A, p)$. We first find the solid angles at a vertex and at an edge point. To do so, let us choose suitable coordinate axes so that the vertex a of Figure 2.1 is located at the origin (see Figure 2.2). To get the coordinate of b , first we note that the $\angle fab$ is a part of a regular pentagon $abcdf$ with side length one and therefore it is 108° . Therefore the length of \overline{bf} is $2 \sin 54^\circ$ and the triangle $bf g$ is a regular triangle with side length $2 \sin 54^\circ$. Then the x coordinate of b is $\sin 54^\circ$, the y coordinate is $(1/3) \times \sqrt{3} \sin 54^\circ$, and we can find the z coordinate $-z_0$ with the condition that the length of \overline{ab} is one. Therefore $z_0 = \sqrt{1 - (4/3) \sin^2 54^\circ}$. Let $e = (1/2)g$ which belongs to an edge of A . We find $\Omega(A, e)$ first.

Numerically these points are $b = (0.809, 0.467, -0.357)$, $f = (-0.809, 0.467, -0.357)$, $g = (0, -0.934, -0.357)$, $e = (0, -0.467, -0.1784)$, $z_0 = 0.357$.

If we call C_1 the pentagon $abcdf$, then the radius r of the circle which circumscribes C_1 satisfies $2r \cos 54^\circ = 1$ and therefore $r = 1/2 \cos 54^\circ = 0.851$. Now the center o_1 of C_1 is the distance $r = 0.851$ away from the origin a and located along the direction of $(\vec{ab} + \vec{af})/|\vec{ab} + \vec{af}| = (0, (1/\sqrt{3}) \tan 54^\circ, -z_0/\cos 54^\circ)$. Therefore $o_1 = r((\vec{ab} + \vec{af})/|\vec{ab} + \vec{af}|) = (0, 0.676, -0.517)$. Similarly the center o_2 of the pentagon C_2 which contains the vertices g, a, b is located at $o_2 = r((\vec{ab} + \vec{ag})/|\vec{ab} + \vec{ag}|) = (0.586, -0.338, -0.517)$ and similarly $o_3 = r((\vec{ag} + \vec{af})/|\vec{ag} + \vec{af}|) = (-0.586, -0.338, -0.517)$. The two pentagons C_2 and C_3 meet at the edge \overline{ag} and the vectors $e\vec{o}_2$ and $e\vec{o}_3$ are both perpendicular to \overline{ag} . Therefore, to find the (ordinary) angle between the two planes C_2 and C_3 , we have only to find the angle between the two vectors $e\vec{o}_2$ and $e\vec{o}_3$. Since $e\vec{o}_2 = a\vec{o}_2 - a\vec{e} = a\vec{o}_2 - (1/2)a\vec{g}$ and $e\vec{o}_3 = a\vec{o}_3 - a\vec{e} = a\vec{o}_3 - (1/2)a\vec{g}$, the angle θ between these two vectors is given by

$$\cos \theta = \frac{e\vec{o}_2 \cdot e\vec{o}_3}{|e\vec{o}_2| |e\vec{o}_3|} = -0.4468 \quad (2.1)$$

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and we get $\theta = 116.5^\circ$ or 2.034 rad. This angle is often called the dihedral angle of the dodecahedron. Now we are ready to find the solid angle $\Omega(A, e)$. We use the unit of the solid angle such that the solid angle of the whole sphere with respect to its center is 4π . Then the solid angle that $\text{Ins}A$ subtends with respect to $e \in A$ is

$$\Omega(A, e) = 4\pi \frac{\theta}{2\pi} = 2\theta = 4.068 \equiv \Omega_e. \quad (2.2)$$

Having found the solid angle Ω_e at the edge point of the regular dodecahedron, let us now find the solid angle that $\text{Ins}A$ subtends at the vertex a . For that we use the result of [11], where it is shown that given any three vectors $\mathbf{T}_1, \mathbf{T}_2, \mathbf{T}_3$, all starting from the origin, the solid angle Ω that these three vectors subtend with respect to the origin (i.e., the solid angle that the (not necessarily regular) tetrahedron whose vertices are the end tips of these three vectors and the origin subtends with respect to the origin) is

$$\tan\left(\frac{1}{2}\Omega\right) = \frac{[\mathbf{T}_1\mathbf{T}_2\mathbf{T}_3]}{T_1T_2T_3 + (\mathbf{T}_1 \cdot \mathbf{T}_2)T_3 + (\mathbf{T}_2 \cdot \mathbf{T}_3)T_1 + (\mathbf{T}_3 \cdot \mathbf{T}_1)T_2}. \quad (2.3)$$

Here $[\mathbf{T}_1\mathbf{T}_2\mathbf{T}_3] \equiv |\mathbf{T}_1 \cdot \mathbf{T}_2 \times \mathbf{T}_3|$ and $T_i = |\mathbf{T}_i|$ ($i = 1, 2, 3$). Note that this expression does not depend on the lengths of the three vectors. For example, when $\mathbf{T}_1, \mathbf{T}_2, \mathbf{T}_3$ are mutually orthogonal, we get $\Omega = \pi/2$.

Now we are ready to find the solid angle that $\text{Ins}A$ subtends at the vertex a , $\Omega(A, a)$. To find Ω , let $\mathbf{T}_1 = \vec{a}\vec{g}$, $\mathbf{T}_2 = \vec{a}\vec{f}$, and $\mathbf{T}_3 = \vec{a}\vec{b}$. Then using (2.3) we can compute the solid angle we want. We find $[\mathbf{T}_1\mathbf{T}_2\mathbf{T}_3] = 0.809$, $\mathbf{T}_1 \cdot \mathbf{T}_2 = \mathbf{T}_2 \cdot \mathbf{T}_3 = \mathbf{T}_3 \cdot \mathbf{T}_1 = -0.309$, and $T_1 = T_2 = T_3 = 1$. Therefore we get

$$\Omega = 2 \tan^{-1} \left(\frac{0.809}{1 - 3 \times 0.309} \right) = 2 \tan^{-1} 11.082, \quad (2.4)$$

$$\Omega(A, a) = 2 \tan^{-1} 11.082 = 2.962 \equiv \Omega_v.$$

Let us summarize our results above. Suppose that $p \in A$ where p is a point and A is a regular dodecahedron. If p is a vertex, say $p = a$, then the solid angle that $\text{Ins}A$ subtends with respect to p is $\Omega(A, p) = \Omega_v = 2.962$. If p is a point which belongs to an edge and is not a vertex, then $\Omega(A, p) = \Omega_e = 4.068$. If $p \in A$ is neither a vertex nor an edge point, then $\Omega(A, p) = 2\pi$.

Suppose now that a is a vertex of a regular dodecahedron $A = A_1$. We show then that $f(a)$ is a vertex of $f(A)$ too. Or we have only to show that $\Omega(f(A), f(a)) = \Omega_v$. Construct three more regular dodecahedrons A_i ($i = 2, 3, 4$) such that a is the common vertex of all four regular dodecahedrons A_i ($i = 1, \dots, 4$) and $\text{Ins}A_i \cap \text{Ins}A_j = \emptyset$ for any $i \neq j$ (see the appendix). Then $f(a)$ belongs to $f(A_i)$ for $i = 1, \dots, 4$ and by the above lemma $\text{Ins}f(A_i) \cap \text{Ins}f(A_j) = \emptyset$ for $i \neq j$. Now the solid angle that $\text{Ins}f(A_i)$ subtends with respect to $f(a)$

can be Ω_v , Ω_e , or 2π . Further

$$\sum_{i=1}^4 \Omega(f(A_i), f(a)) \leq 4\pi. \quad (2.5)$$

Since

$$3\Omega_v + \Omega_e = 3 \times (2.962) + 4.068 = 12.954 > 4\pi = 12.566, \quad (2.6)$$

we conclude that $\Omega(f(A_i), f(a)) = \Omega_v$ ($i = 1, \dots, 4$) and that $f(a)$ is a vertex of $f(A) = f(A_1)$.

Now we prove the statement of the theorem. Given any two points a and b which are distanced by the unit distance from each other, form two regular dodecahedrons A and B such that the following three conditions are met. (1) Both a and b are common vertices of A and B . (2) $\text{Ins}A \cap \text{Ins}B = \phi$. (3) No other vertices are common vertices of A and B . This means that the two dodecahedrons A and B share exactly one edge which has end points a and b . Now it is obvious that the two dodecahedrons $f(A)$ and $f(B)$ also satisfy the above three conditions with the common vertices $f(a)$ and $f(b)$. Therefore the distance between $f(a)$ and $f(b)$ is again one too. Since f preserves the unit distance, by the theorem of Beckman and Quarles, we conclude that f is a linear isometry up to translation. \square

Appendix

We show that we can form four regular dodecahedrons A_i ($i = 1, \dots, 4$) such that $g \in A_i$ is a common vertex of A_i ($i = 1, \dots, 4$) and $\text{Ins}A_i \cap \text{Ins}A_j = \phi$ for any $i \neq j$ ($i, j = 1, \dots, 4$). Let $A = A_1$ be a regular dodecahedron as shown in Figure 2.1. Choose a coordinate such that g is at the origin and the pentagon containing the vertices h, g, i is contained completely in the xz plane. Then $i = (\sin 54^\circ, 0, -\cos 54^\circ)$ and $h = (-\sin 54^\circ, 0, -\cos 54^\circ)$. Further $a = (0, \sin 54^\circ \sqrt{3 - \tan^2 54^\circ}, -\cos 108^\circ / \cos 54^\circ)$. The coordinate of a is uniquely determined by the condition that a is in the yz plane with $|\vec{g}a| = 1$ and $|\vec{h}a| = 2 \sin 54^\circ$. Numerically $g = (0, 0, 0) \equiv g_1$, $i = (0.809, 0, -0.588) \equiv i_1$, $h = (-0.809, 0, -0.588) \equiv h_1$, and $a = (0, 0.8507, 0.5257) \equiv a_1$.

Let L_1 be a reflection through the xz plane, that is, $L_1(x, y, z) = (x, -y, z)$. Call $L_1(A_1) = A_2$. Let L_2 be a 90° rotation around z axis followed by the reflection through the xy plane, that is, $L_2(x, y, z) = (-y, x, -z)$. Call $L_2(A_1) = A_3$ and $L_2(A_2) = A_4$ (see Figure A.1). Note that $A_4 = L_2L_1(A_1)$ and $L_2L_1(x, y, z) = (y, x, -z)$. Therefore $a_4 = L_2L_1(a_1) = (0.8507, 0, -0.5257)$ and $i_4 = L_2L_1(i_1) = (0, 0.809, 0.588)$. We claim that $\text{Ins}A_i \cap \text{Ins}A_j = \phi$ for any $i \neq j$ ($i, j = 1, \dots, 4$).

By symmetry we have only to check the part where $x \geq 0$ and $y \geq 0$. We think of z coordinate as a height function. In Figure A.1, $g_{1,2} = g_{3,4} = (0, 0, 0) \equiv g$. The "slope" of $\vec{g}a_1$ is $a_{1z}/a_{1y} = 0.5257/0.8507 = 0.6180$, which is smaller than the slope of $\vec{g}i_4$ that is $i_{4z}/i_{4y} = 0.588/0.809 = 0.7268$. Similarly, The slope of $\vec{g}i_1$ is $i_{1z}/i_{1x} = -0.588/0.809 = -0.7268$, which is smaller than the slope of $\vec{g}a_4$ that is $a_{4z}/a_{4x} = -0.5257/0.8507 = -0.6180$. This

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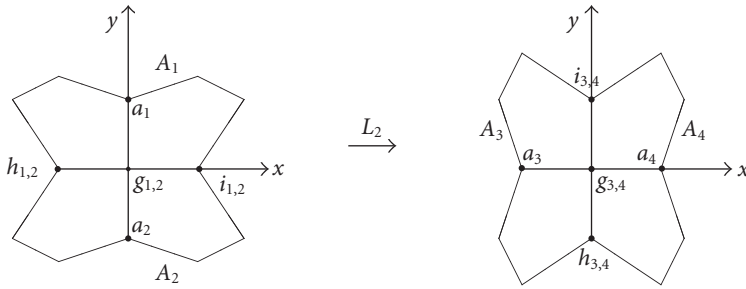


Figure A.1

means that A_4 is “higher” than A_1 except at the origin g . Therefore we conclude that $\text{Ins}A_i \cap \text{Ins}A_j = \emptyset$ for any $i \neq j$ ($i, j = 1, \dots, 4$).

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