

THREE-DIMENSIONAL FLUCTUATING COUETTE FLOW THROUGH THE POROUS PLATES WITH HEAT TRANSFER

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Unsteady Couette flow of a viscous incompressible fluid between two horizontal porous flat plates is considered. The stationary plate is subjected to a periodic suction and the plate in uniform motion is subjected to uniform injection. Approximate solutions have been obtained for the velocity and the temperature fields, skin friction by using perturbation technique. The heat transfer characteristic has also been studied on taking viscous dissipation into account. It is found that the main flow velocity decreases with increase in frequency parameter. On the other hand, the magnitude of the cross-flow velocity increases with increase in frequency parameter. It is seen that the amplitude of the shear stress due to main flow decreases while that due to cross-flow increases with increase in frequency parameter. It is also seen that the tangent of phase shifts both due to the main and cross-flows decrease with increase in frequency parameter. It is observed that the temperature increases with increase in frequency parameter.

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1. Introduction

Couette flow is important in numerous mechanisms involving the relative motion of two surfaces. The problem of Couette flow is considered important in transpiration cooling. In this process several engines can be protected from the influence of hot gases. This process is used in turbojet and rocket engines, like combustion chamber walls, exhaust nozzles, and gas turbine blades. The solution is well known when both surfaces are flat and moving in their own planes. An exact solution of Navier-Stokes equations between two parallel plates without suction is well known in Schlichting [3]. Two-dimensional plane Couette flow with transpiration cooling applying uniform injection and suction at the porous plates is discussed by Eckert [1]. Gersten and Gross [2] studied the three-dimensional flow and heat transfer along a flat plate by applying periodic suction. Singh et al. [7] analyzed the three-dimensional flow and heat transfer past a vertical wall. Singh et al. [6] and Singh [4] studied the three-dimensional flow and heat transfer past a porous

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plate. Recently Singh [5] discussed the Couette flow between two parallel plates with transverse sinusoidal injection of the fluid at the stationary plate and the constant suction at the plate in motion.

However, the application of the transverse sinusoidal injection or suction velocity in the unsteady problem of transpiration cooling has not yet received much attention. In the present paper we have studied the unsteady Couette flow and heat transfer between two horizontal parallel porous flat plates with periodic suction at the stationary plate and constant injection at the plate in motion. We assume that the periodic suction velocity is time-dependent and perpendicular to the flow direction. Due to the periodic suction the flow becomes three dimensional. The main flow velocity profile, cross-flow velocity profile, and shear stress have been calculated and plotted. The heat transfer characteristic has also been studied on taking viscous dissipation into account. It is found that the main flow velocity increases with the increase in either Reynolds number or suction parameter, while it decreases with the increase in frequency parameter. On the other hand, the magnitude of the cross-flow velocity increases with the increase in either suction parameter or frequency parameter, while it increases near the stationary plate and decreases near the moving plate with the increase in Reynolds number. It is seen that the amplitude of the shear stress due to main flow increases with the increase in Reynolds number but decreases with the increase in frequency parameter. The phase shift decreases with the increase in frequency parameter. But for very small values of Reynolds number, it increases and decreases for large Reynolds number. Also it is seen that the amplitude of the shear stress due to cross-flow increases with the increase in either Reynolds number or frequency parameter. The phase shift decreases with the increase in frequency parameter while it increases with the increase in Reynolds number. It is observed that the temperature increases with the increase in frequency parameter, on the other hand, it increases near the stationary plate and decreases near the moving plate with the increase in Reynolds number.

2. Formulation of the problem

Consider the unsteady flow of a viscous incompressible fluid between two horizontal flat porous plates separated by a distance d . The upper plate moves with a uniform velocity U in the direction of the flow. We choose a cartesian coordinate system with its origin on the lower stationary plate, x^* -axis is in the direction of the flow, y^* -axis is perpendicular to the plate, and z^* -axis normal to the $x^* y^*$ -plane.

The upper plate is subjected to a constant injection $-V_0$ and the lower plate to a transverse sinusoidal suction velocity distribution of the form

$$v^* = -V_0 \left[1 + \epsilon \cos \left(\frac{\pi z^*}{d} - ct^* \right) \right], \quad (2.1)$$

where ϵ ($\ll 1$) is the amplitude of the suction velocity. Denoting velocity components u^* , v^* , w^* in the directions x^* -, y^* -, and z^* -axes, respectively, the flow is governed by

the following equations:

$$\begin{aligned} \frac{\partial v^*}{\partial y^*} + \frac{\partial w^*}{\partial z^*} &= 0, \\ \frac{\partial u^*}{\partial t^*} + v^* \frac{\partial u^*}{\partial y^*} + w^* \frac{\partial u^*}{\partial z^*} &= \nu \left(\frac{\partial^2 u^*}{\partial y^{*2}} + \frac{\partial^2 u^*}{\partial z^{*2}} \right), \\ \frac{\partial v^*}{\partial t^*} + v^* \frac{\partial v^*}{\partial y^*} + w^* \frac{\partial v^*}{\partial z^*} &= -\frac{1}{\rho} \frac{\partial p^*}{\partial y^*} + \nu \left(\frac{\partial^2 v^*}{\partial y^{*2}} + \frac{\partial^2 v^*}{\partial z^{*2}} \right), \\ \frac{\partial w^*}{\partial t^*} + v^* \frac{\partial w^*}{\partial y^*} + w^* \frac{\partial w^*}{\partial z^*} &= -\frac{1}{\rho} \frac{\partial p^*}{\partial z^*} + \nu \left(\frac{\partial^2 w^*}{\partial y^{*2}} + \frac{\partial^2 w^*}{\partial z^{*2}} \right), \end{aligned} \quad (2.2)$$

where ν is the kinematic viscosity, ρ is the density, p^* is the fluid pressure.

The boundary conditions of the problem are

$$\begin{aligned} u^* = 0, \quad v^* = -V_0 \left[1 + \cos \left(\frac{\pi}{d} z^* - ct^* \right) \right], \quad w^* = 0 \quad \text{at } y^* = 0, \\ u^* = U, \quad v^* = -V_0, \quad w^* = 0 \quad \text{at } y^* = d. \end{aligned} \quad (2.3)$$

Introducing the nondimensional variables

$$y = \frac{y^*}{d}, \quad z = \frac{z^*}{d}, \quad t = ct^*, \quad p = \frac{p^*}{\rho U^2}, \quad u = \frac{u^*}{U}, \quad v = \frac{v^*}{U}, \quad w = \frac{w^*}{U}, \quad (2.4)$$

equation (2.2) becomes

$$\begin{aligned} \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} &= 0, \\ \omega \frac{\partial u}{\partial t} + \text{Re} \left(v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} \right) &= \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}, \\ \omega \frac{\partial v}{\partial t} + \text{Re} \left(v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} \right) &= -\text{Re} \frac{\partial p}{\partial y} + \left(\frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} \right), \\ \omega \frac{\partial w}{\partial t} + \text{Re} \left(v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z} \right) &= -\text{Re} \frac{\partial p}{\partial z} + \left(\frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} \right), \end{aligned} \quad (2.5)$$

where $\text{Re} = Ud/\nu$, the Reynolds number; $S = V_0/U$, the suction parameter; and $\omega = cd^2/\nu$, the frequency parameter. Using (2.4) the boundary conditions (2.3) become

$$\begin{aligned} u = 0, \quad v = -S[1 + \epsilon \cos(\pi z - t)], \quad w = 0 \quad \text{at } y = 0, \\ u = 1, \quad v = -S, \quad w = 0 \quad \text{at } y = 1. \end{aligned} \quad (2.6)$$

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3. Solution of the problem

In order to solve the differential equation (2.5), we assume the solution of the following form:

$$\begin{aligned}
 u &= u_0(y) + \epsilon u_1(y, z, t) + \epsilon^2 u_2(y, z, t) + \dots, \\
 v &= v_0(y) + \epsilon v_1(y, z, t) + \epsilon^2 v_2(y, z, t) + \dots, \\
 w &= w_0(y) + \epsilon w_1(y, z, t) + \epsilon^2 w_2(y, z, t) + \dots, \\
 p &= p_0(y) + \epsilon p_1(y, z, t) + \epsilon^2 p_2(y, z, t) + \dots.
 \end{aligned}
 \tag{3.1}$$

When $\epsilon = 0$ the flow becomes two-dimensional with constant suction and injection at both the plates. In this case (2.5) is reduced to

$$\begin{aligned}
 v_0' &= 0, \\
 u_0'' + SRe u_0' &= 0,
 \end{aligned}
 \tag{3.2}$$

where the primes denote differentiation with respect to y , and the corresponding boundary conditions are

$$\begin{aligned}
 u_0 &= 0, \quad v_0 = -S \quad \text{at } y = 0, \\
 u_0 &= 1, \quad v_0 = -S \quad \text{at } y = 1.
 \end{aligned}
 \tag{3.3}$$

The solutions of (3.2) using (3.3) become

$$v_0(y) = -S, \quad u_0(y) = \frac{1 - e^{-SRey}}{1 - e^{-SRe}}.
 \tag{3.4}$$

When $\epsilon \neq 0$, for small values of ϵ , we take only upto $O(\epsilon)$. Substituting (3.1) in (2.5) and comparing the coefficients of ϵ from both sides, we get

$$\frac{\partial v_1}{\partial y} + \frac{\partial w_1}{\partial z} = 0,
 \tag{3.5}$$

$$\omega \frac{\partial u_1}{\partial t} + Re \left(-S \frac{\partial u_1}{\partial y} + v_1 \frac{\partial u_0}{\partial y} \right) = \frac{\partial^2 u_1}{\partial y^2} + \frac{\partial^2 u_1}{\partial z^2},
 \tag{3.6}$$

$$\omega \frac{\partial v_1}{\partial t} - SRe \frac{\partial v_1}{\partial y} = -Re \frac{\partial p_1}{\partial y} + \frac{\partial^2 v_1}{\partial y^2} + \frac{\partial^2 v_1}{\partial z^2},
 \tag{3.7}$$

$$\omega \frac{\partial w_1}{\partial t} - SRe \frac{\partial w_1}{\partial y} = -Re \frac{\partial p_1}{\partial z} + \frac{\partial^2 w_1}{\partial y^2} + \frac{\partial^2 w_1}{\partial z^2}.
 \tag{3.8}$$

The corresponding boundary conditions are reduced to

$$\begin{aligned} u_1 = 0, \quad v_1 = -S \cos(\pi z - t), \quad w_1 = 0 \quad \text{at } y = 0, \\ u_1 = 0, \quad v_1 = 0, \quad w_1 = 0 \quad \text{at } y = 1. \end{aligned} \quad (3.9)$$

These are the linear partial differential equations describing the three-dimensional flow. To solve (3.6), (3.7), and (3.8) we assume u_1 , v_1 , w_1 , and p_1 of the following form:

$$\begin{aligned} u_1(y, z, t) &= u_{11}(y)e^{i(\pi z - t)}, \\ v_1(y, z, t) &= v_{11}(y)e^{i(\pi z - t)}, \\ w_1(y, z, t) &= \frac{i}{\pi} v'_{11}(y)e^{i(\pi z - t)}, \\ p_1(y, z, t) &= p_{11}(y)e^{i(\pi z - t)}. \end{aligned} \quad (3.10)$$

The corresponding boundary conditions are

$$\begin{aligned} u_{11} = 0, \quad v_{11} = -S, \quad v'_{11} = 0 \quad \text{at } y = 0, \\ u_{11} = 0, \quad v_{11} = 0, \quad v'_{11} = 0 \quad \text{at } y = 1. \end{aligned} \quad (3.11)$$

Substituting (3.10) in (3.6), (3.7), and (3.8), we get the following differential equations:

$$\begin{aligned} u''_{11} + S \operatorname{Re} u'_{11} - (\pi^2 - i\omega)u_{11} &= \operatorname{Re} v_{11} u'_0, \\ v''_{11} + S \operatorname{Re} v'_{11} - (\pi^2 - i\omega)v_{11} &= \operatorname{Re} p'_{11}, \\ v'''_{11} + S \operatorname{Re} v''_{11} - (\pi^2 - i\omega)v'_{11} &= \operatorname{Re} \pi^2 p_{11}. \end{aligned} \quad (3.12)$$

Solving (3.12) under the boundary conditions (3.11), we get

$$\begin{aligned} u_1(y, z, t) &= \left[A_3 e^{-r_1 y} + A_4 e^{-r_2 y} + C_1 e^{-(r_1 + S \operatorname{Re})y} + C_2 e^{-(r_2 + S \operatorname{Re})y} \right. \\ &\quad \left. + C_3 e^{(\pi - S \operatorname{Re})y} + C_4 e^{-(\pi + S \operatorname{Re})y} \right] e^{i(\pi z - t)}, \end{aligned} \quad (3.13)$$

$$v_1(y, z, t) = \left[A e^{-r_1 y} + B e^{-r_2 y} + C e^{\pi y} + D e^{-\pi y} \right] e^{i(\pi z - t)}, \quad (3.14)$$

$$w_1(y, z, t) = -\frac{i}{\pi} \left[A r_1 e^{-r_1 y} + B r_2 e^{-r_2 y} - C \pi e^{\pi y} + D \pi e^{-\pi y} \right] e^{i(\pi z - t)}, \quad (3.15)$$

$$p_1(y, z, t) = \left[A_1 e^{\pi y} + A_2 e^{-\pi y} \right] e^{i(\pi z - t)}, \quad (3.16)$$

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where

$$\begin{aligned}
 r_1 &= \frac{S\text{Re} + \sqrt{S^2\text{Re}^2 + 4(\pi^2 - i\omega)}}{2}, & r_2 &= \frac{S\text{Re} - \sqrt{S^2\text{Re}^2 + 4(\pi^2 - i\omega)}}{2}, \\
 A &= \frac{Se^\pi(\pi r_4 + r_6) + Se^{-\pi}(r_6 - \pi r_4)}{2(r_3 r_6 - r_4 r_5)}, & B &= \frac{-Se^\pi(r_5 + \pi r_3) - Se^{-\pi}(r_5 - \pi r_3)}{2(r_3 r_6 - r_4 r_5)}, \\
 C &= -\frac{1}{2\pi}[A(\pi - r_1) + B(\pi - r_2) + S\pi], & D &= -\frac{1}{2\pi}[A(\pi + r_1) + B(\pi + r_2) + S\pi], \\
 r_3 &= e^{-r_1} - \frac{(\pi - r_1)}{2\pi}e^\pi - \frac{(\pi + r_1)}{2\pi}e^{-\pi}, & r_4 &= e^{-r_2} - \frac{(\pi - r_2)}{2\pi}e^\pi - \frac{(\pi + r_2)}{2\pi}e^{-\pi}, \\
 r_5 &= r_1 e^{-r_1} + \frac{1}{2}e^\pi(\pi - r_1) - \frac{1}{2}e^{-\pi}(\pi + r_1), & r_6 &= r_2 e^{-r_2} + \frac{1}{2}e^\pi(\pi - r_2) - \frac{1}{2}e^{-\pi}(\pi + r_2), \\
 C_1 &= \frac{S\text{Re}^2}{(1 - e^{-S\text{Re}})} \frac{A}{(r_1^2 + S\text{Re}r_1 - \pi^2 + i\omega)}, & C_2 &= \frac{S\text{Re}^2}{(1 - e^{-S\text{Re}})} \frac{B}{(r_2^2 + S\text{Re}r_2 - \pi^2 + i\omega)}, \\
 C_3 &= \frac{S\text{Re}^2}{(1 - e^{-S\text{Re}})} \frac{C}{(i\omega - \pi S\text{Re})}, & C_4 &= \frac{S\text{Re}^2}{(1 - e^{-S\text{Re}})} \frac{D}{(i\omega + \pi S\text{Re})}, \\
 A_1 &= \frac{S\text{Re}\pi + i\omega}{\text{Re}\pi} C, & A_2 &= \frac{S\text{Re}\pi - i\omega}{\text{Re}\pi} D, \\
 A_3 &= -(A_4 + C_1 + C_2 + C_3 + C_4), \\
 A_4 &= \frac{1}{(e^{-r_1} - e^{-r_2})} \left[C_1(e^{-r_1 - S\text{Re}} - e^{-r_1}) + C_2(e^{-r_2 - S\text{Re}} - e^{-r_2}) \right. \\
 &\quad \left. + C_3(e^{\pi - S\text{Re}} - e^{-r_1}) + C_4(e^{-\pi - S\text{Re}} - e^{-r_1}) \right].
 \end{aligned} \tag{3.17}$$

4. Result and discussion

We have presented the nondimensional main flow velocity u against y for different values of Reynolds number Re , suction parameter S , and frequency parameter ω for $z = 0.0$, $t = 0.2$, $\epsilon = 0.2$ in Figures 4.1, 4.2, and 4.3. From the figure it is seen that the main flow velocity u increases with increase in either Re or S , while it decreases with the increase in ω . The cross-flow velocity profile is shown in Figures 4.4 and 4.5 against y for $z = 0.5$, $t = 0.2$, $\epsilon = 0.2$. It is observed that the magnitude of the cross-flow velocity w increases with the increase in either S or ω but it increases near the stationary plate and decreases near the moving plate with the increase in Re . This is due to the fact that suction at the stationary plate and injection at the moving plate are two exactly opposite processes. Also

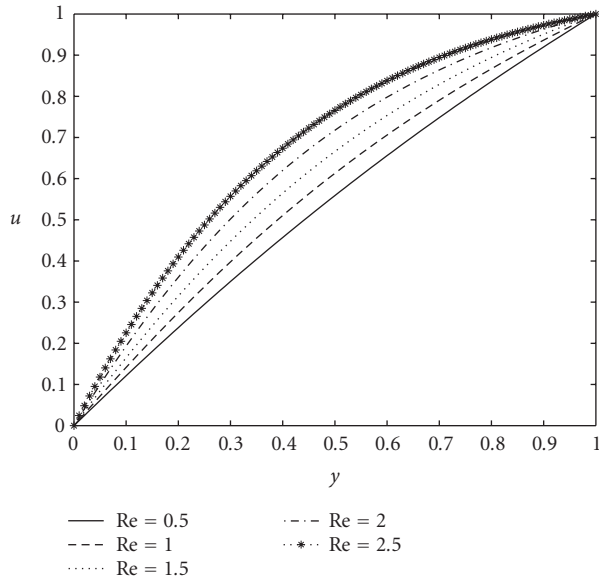


Figure 4.1. Main velocity u for $\omega = 6.0, S = 1.0, t = 0.2, z = 0.0, \epsilon = 0.2$.

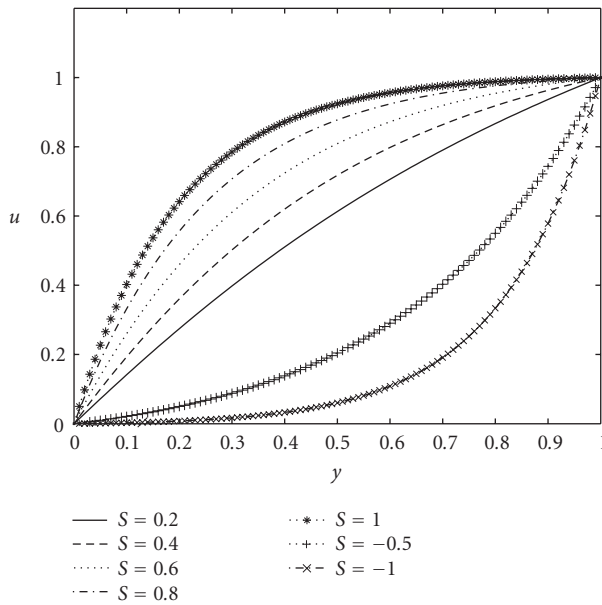


Figure 4.2. Main velocity u for $\omega = 6.0, Re = 5.0, Z = 0.0, t = 0.2, \epsilon = 0.2$.

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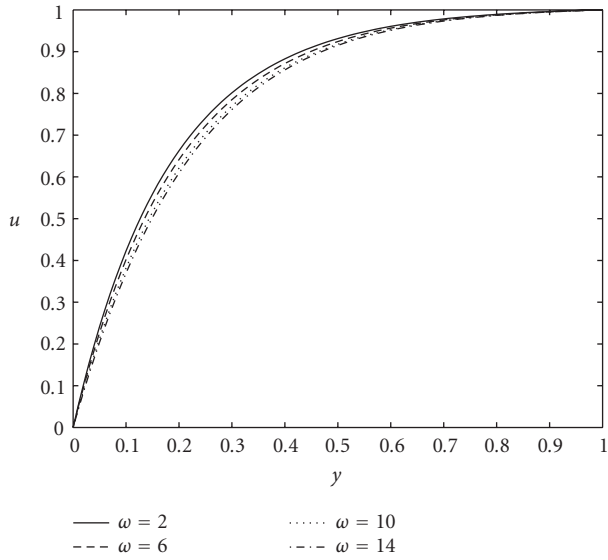


Figure 4.3. Main velocity u for $S = 1.0$, $Re = 5.0$, $t = 0.2$, $z = 0.0$, $\epsilon = 0.2$.

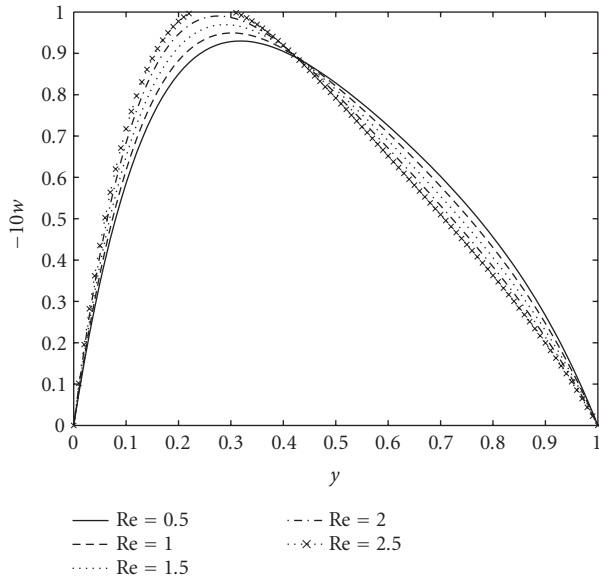


Figure 4.4. Cross-velocity $-10w$ for $\omega = 6.0$, $S = 1.0$, $z = 0.5$, $t = 0.2$, $\epsilon = 0.2$.

the variations of cross-velocity w for different values of ω are shown in Table 4.1. From the table it is observed that the magnitude of w increases with increase in ω .

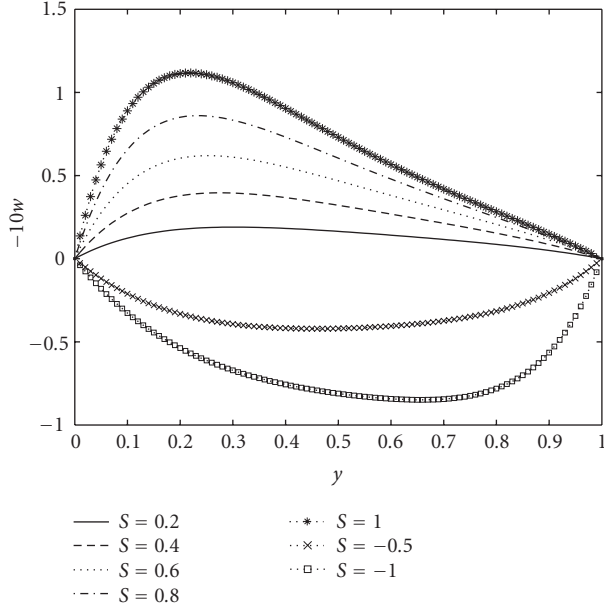


Figure 4.5. Cross-velocity $-10w$ for $\omega = 6.0$, $\text{Re} = 5.0$, $z = 0.5$, $t = 0.2$, $\epsilon = 0.2$.

The nondimensional shear stress components due to the main flow and cross-flow at the plate $y = 0$ are, respectively,

$$\begin{aligned}
 \tau_x &= \frac{d\tau_x^*}{\mu U} = \left(\frac{du_0}{dy} \right)_{y=0} + \epsilon \left(\frac{du_1}{dy} \right)_{y=0} \\
 &= \left(\frac{du_0}{dy} \right)_{y=0} + \epsilon \left(\frac{du_{11}}{dy} \right)_{y=0} e^{i(\pi z - t)} \\
 &= \tau_{u_0} + \epsilon \text{Re}_1 \cos(\pi z - t + \phi_1), \\
 \tau_z &= \frac{d\tau_z^*}{\mu U} = \left(\frac{dw_0}{dy} \right)_{y=0} + \epsilon \left(\frac{dw_1}{dy} \right)_{y=0} \\
 &= \epsilon \frac{i}{\pi} \left(\frac{dv'_{11}}{dy} \right)_{y=0} e^{i(\pi z - t)} \\
 &= \epsilon \text{Re}_2 \cos(\pi z - t + \phi_2),
 \end{aligned} \tag{4.1}$$

where

$$\begin{aligned}
 \text{Re}_1 &= \{X_1^2 + Y_1^2\}^{1/2}, & \tan \phi_1 &= \frac{Y_1}{X_1}, \\
 \text{Re}_2 &= \{X_2^2 + Y_2^2\}^{1/2}, & \tan \phi_2 &= \frac{Y_2}{X_2}.
 \end{aligned} \tag{4.2}$$

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Table 4.1. Cross-flow velocity for $S = 1.0$, $Re = 5.0$, $z = 0.5$, $t = 0.2$, $\epsilon = 0.2$.

y	-10w			
	$\omega = 2.0$	$\omega = 6.0$	$\omega = 10.0$	$\omega = 14.0$
0.00	0.000000	0.000000	0.000000	0.000000
0.20	1.109254	1.113393	1.118991	1.125880
0.40	0.907241	0.903539	0.898763	0.893008
0.60	0.568972	0.565611	0.561192	0.555843
0.80	0.282000	0.281842	0.281629	0.281395
1.00	0.000011	0.000000	-0.00500302	0.000001

Table 4.2. Shear stress due to main flow.

Re	Re ₁			-tan ϕ_1		
	$\omega = 5.0$	$\omega = 10.0$	$\omega = 15.0$	$\omega = 5.0$	$\omega = 10.0$	$\omega = 15.0$
0.4	0.3133	0.1787	0.1344	1.0171	1.2728	1.2516
0.6	0.4892	0.2891	0.2182	0.6059	0.9704	1.0421
0.8	0.6606	0.4119	0.3132	0.3236	0.7402	0.8681
1.0	0.8185	0.5452	0.4191	0.0970	0.5541	0.7193

Table 4.3. Shear stress due to cross-flow.

Re	Re ₂			-tan ϕ_2		
	$\omega = 5.0$	$\omega = 10.0$	$\omega = 15.0$	$\omega = 5.0$	$\omega = 10.0$	$\omega = 15.0$
0.4	3.9497	4.1925	4.5407	6.2946	3.3011	2.3627
0.6	4.0614	4.2989	4.6407	6.5018	3.4055	2.4331
0.8	4.1760	4.4080	4.7430	6.7209	3.5155	2.5070
1.0	4.2937	4.5198	4.8478	6.9523	3.6316	2.5847

The shear stresses due to main flow and cross-flow are shown in Tables 4.2 and 4.3 for $S = 1.0$. From the table it is seen that the amplitude Re_1 of the shear stress due to main flow increases with the increase in Re but decreases with the increase in ω . The magnitude of the tangent of phase shift $\tan \phi_1$ decreases with the increase in ω while it increases for small values of Re and decreases for large values of Re . It is also seen from Table 4.2 that there is always a phase lag. Table 4.3 shows the shear stress due to cross-flow for $S = 1.0$. It is observed that the amplitude Re_2 of shear stress due to cross-flow increases with the increase in either Re or ω . It is also seen that there is always a phase lag. The magnitude of the tangent of phase angle increases with the increase in Re but decreases with the increase in ω .

5. Heat transfer

To find the temperature distribution we consider the energy equation

$$\frac{\partial T^*}{\partial t^*} + v^* \frac{\partial T^*}{\partial y^*} + w^* \frac{\partial T^*}{\partial z^*} = \alpha \left(\frac{\partial^2 T^*}{\partial y^2} + \frac{\partial^2 T^*}{\partial z^2} \right) + \frac{\mu}{\rho C_p} \Phi^*, \quad (5.1)$$

where Φ^* is the viscous dissipation function given by

$$\Phi^* = 2 \left[\left(\frac{\partial v^*}{\partial y^*} \right)^2 + \left(\frac{\partial w^*}{\partial z^*} \right)^2 \right] + \left(\frac{\partial u^*}{\partial y^*} \right)^2 + \left(\frac{\partial w^*}{\partial y^*} + \frac{\partial v^*}{\partial z^*} \right)^2 + \left(\frac{\partial u^*}{\partial z^*} \right)^2, \quad (5.2)$$

where C_p is the specific heat at constant pressure, and μ is the viscosity, α is the thermal diffusivity of the fluid. We assume the temperature at the lower and upper plates are T_0 and T_1 ($T_1 > T_0$), respectively.

The temperature boundary conditions are

$$\begin{aligned} T^* &= T_0 \quad \text{at } y^* = 0, \\ T^* &= T_1 \quad \text{at } y^* = d. \end{aligned} \quad (5.3)$$

Introducing the nondimensional variables

$$\theta = \frac{T^* - T_0}{T_1 - T_0}, \quad \text{Pr} = \frac{\mu}{\alpha}, \quad E_c = \frac{U^2}{C_p(T_1 - T_0)} \quad (5.4)$$

and on using (2.4), (5.1), and (5.2) become

$$\text{Pr} \omega \frac{\partial \theta}{\partial t} + \text{Re} \text{Pr} \left(v \frac{\partial \theta}{\partial y} + w \frac{\partial \theta}{\partial z} \right) = \frac{\partial^2 \theta}{\partial y^2} + \frac{\partial^2 \theta}{\partial z^2} + \text{Pr} E_c \Phi, \quad (5.5)$$

where

$$\Phi = 2 \left[\left(\frac{\partial v}{\partial y} \right)^2 + \left(\frac{\partial w}{\partial z} \right)^2 \right] + \left(\frac{\partial u}{\partial y} \right)^2 + \left(\frac{\partial u}{\partial z} \right)^2 + \left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} \right)^2. \quad (5.6)$$

The corresponding boundary condition is reduced to

$$\theta(0) = 0 \quad \theta(1) = 1. \quad (5.7)$$

We assume the solution of the temperature equation in the form

$$\theta = \theta_0(y) + \epsilon \theta_1(y, z, t) + \epsilon^2 \theta_2(y, z, t) + \dots \quad (5.8)$$

Substitute (5.8), (3.1) in (5.13) and equate the term independent of ϵ and the coefficients of ϵ and neglect the higher order of ϵ as $\epsilon \ll 1$. Equating the terms independent of ϵ ,

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we get

$$\theta_0'' + S\text{RePr}\theta_0' = -\text{PrEc}u_0'^2. \quad (5.9)$$

The corresponding temperature boundary conditions are

$$\theta_0(0) = 0, \quad \theta_0(1) = 1. \quad (5.10)$$

Solving (5.9) under the boundary conditions (5.10), we get

$$\begin{aligned} \theta_0(y) &= h_1(1 - e^{-S\text{RePr}y}) + h_2(e^{-2S\text{Re}y} - 1) \quad \text{for } \text{Pr} \neq 2.0, \\ \theta_0(y) &= D_1(1 - e^{-2S\text{Re}y}) + D_3ye^{-2S\text{Re}y} \quad \text{for } \text{Pr} = 2.0, \end{aligned} \quad (5.11)$$

where

$$\begin{aligned} h_1 &= \frac{1 + h_2(1 - e^{-2S\text{Re}})}{1 - e^{-S\text{RePr}}}, & h_2 &= \frac{-\text{PrEc}}{2(1 - e^{-S\text{Re}})^2(2 - \text{Pr})} \quad \text{for } \text{Pr} \neq 2.0, \\ D_1 &= \frac{1 - D_3e^{-2S\text{Re}}}{1 - e^{-2S\text{Re}}}, & D_3 &= \frac{E_c S\text{Re}}{(1 - e^{-S\text{Re}})^2} \quad \text{for } \text{Pr} = 2.0. \end{aligned} \quad (5.12)$$

Equating the coefficient of ϵ from both sides, we have

$$\text{Pr}\omega \frac{\partial \theta_1}{\partial t} + \text{RePr} \left[v_1 \frac{\partial \theta_0}{\partial y} + v_0 \frac{\partial \theta_1}{\partial y} \right] = \frac{\partial^2 \theta_1}{\partial y^2} + \frac{\partial^2 \theta_1}{\partial z^2} + 2\text{PrEc}u_0'u_1'. \quad (5.13)$$

The corresponding conditions for θ_1 are

$$\theta_1(0) = 0, \quad \theta_1(1) = 0. \quad (5.14)$$

We assume the solution of the above differential equation (5.13) of the form

$$\theta_1(y, z, t) = \theta_{11}(y)e^{i(\pi z - t)}. \quad (5.15)$$

Substituting (5.15) in (5.13) and on using (3.10), we get

$$\theta_{11}'' + S\text{RePr}\theta_{11}' - (\pi^2 - i\text{Pr}\omega)\theta_{11} = \text{RePr}\theta_0'v_{11} - 2\text{PrEc}u_0'u_{11}'. \quad (5.16)$$

The corresponding boundary conditions are reduced to

$$\theta_{11}(0) = 0, \quad \theta_{11}(1) = 0. \quad (5.17)$$

The solution of (5.16) subject to the boundary conditions (5.17) and on using (3.4), (3.13), (3.14), and (5.11) are

$$\begin{aligned}
\theta_{11} = & L\{e^{-m_1 y} - e^{-m_2 y}\} \\
& + S^2 \text{Re}^2 \text{Pr}^2 h_1 \left[\frac{A}{\lambda_1} \{e^{-(r_1+S\text{Re})y} - e^{-m_2 y}\} + \frac{B}{\lambda_2} \{e^{-(r_2+S\text{Re})y} - e^{-m_2 y}\} \right. \\
& \quad \left. + \frac{C}{\lambda_3} \{e^{(\pi-S\text{Re})y} - e^{-m_2 y}\} + \frac{D}{\lambda_4} \{e^{-(\pi+S\text{Re})y} - e^{-m_2 y}\} \right] \\
& - 2S\text{RePr} \left[\left\{ Ah_2 \text{Re} - \frac{\text{Ec} C_1 (r_1 + S\text{Re})}{(1 - e^{-S\text{Re}})} \right\} \frac{1}{\lambda_5} \{e^{-(r_1+2S\text{Re})y} - e^{-m_2 y}\} \right. \\
& \quad + \left\{ Bh_2 \text{Re} - \frac{\text{Ec} C_2 (r_2 + S\text{Re})}{(1 - e^{-S\text{Re}})} \right\} \frac{1}{\lambda_6} \{e^{-(r_2+2S\text{Re})y} - e^{-m_2 y}\} \\
& \quad + \left\{ Ch_2 \text{Re} + \frac{\text{Ec} C_3 (\pi - S\text{Re})}{(1 - e^{-S\text{Re}})} \right\} \frac{1}{\lambda_7} \{e^{(\pi-2S\text{Re})y} - e^{-m_2 y}\} \\
& \quad \left. + \left\{ Dh_2 \text{Re} - \frac{\text{Ec} C_4 (\pi + S\text{Re})}{(1 - e^{-S\text{Re}})} \right\} \frac{1}{\lambda_8} \{e^{-(\pi+2S\text{Re})y} - e^{-m_2 y}\} \right] \\
& + \frac{2\text{PrEcSRe}}{(1 - e^{-S\text{Re}})} \left[\frac{A_3 r_1}{\lambda_9} \{e^{-(r_1+S\text{Re})y} - e^{-m_2 y}\} \right. \\
& \quad \left. + \frac{A_4 r_2}{\lambda_{10}} \{e^{-(r_2+S\text{Re})y} - e^{-m_2 y}\} \right] \quad \text{for Pr} \neq 2.0,
\end{aligned} \tag{5.18}$$

and for $\text{Pr} = 2.0$,

$$\begin{aligned}
\theta_{11}(y) = & L_1 e^{-m_3 y} + M_1 e^{-m_4 y} + \left[2\text{Re}AD_4 + \frac{C_1 (r_1 + S\text{Re})}{\beta} \right] \frac{1}{\mu_1} e^{-(r_1+2S\text{Re})y} \\
& + \left[2\text{Re}BD_4 + \frac{C_2 (r_2 + S\text{Re})}{\beta} \right] \frac{1}{\mu_2} e^{-(r_2+2S\text{Re})y} + \left[2\text{Re}CD_4 - \frac{C_3 (\pi - S\text{Re})}{\beta} \right] \frac{1}{\mu_3} \\
& \times e^{(\pi-2S\text{Re})y} + \left[2\text{Re}DD_4 + \frac{C_4 (\pi + S\text{Re})}{\beta} \right] \frac{1}{\mu_4} e^{-(\pi+2S\text{Re})y} \\
& + \frac{1}{\beta} \left[\frac{A_3 r_1}{\mu_5} e^{-(r_1+S\text{Re})y} + \frac{A_4 r_2}{\mu_6} e^{-(r_2+S\text{Re})y} \right] \\
& - 4S\text{Re}^2 D_3 \left[\frac{A}{\mu_1} \left\{ y + \frac{2(r_1 + S\text{Re})}{\mu_1} \right\} \right. \\
& \quad \times e^{-(r_1+2S\text{Re})y} + \frac{B}{\mu_2} \left\{ y + \frac{2(r_2 + S\text{Re})}{\mu_2} \right\} e^{-(r_2+2S\text{Re})y} + \frac{C}{\mu_3} \\
& \quad \times \left\{ y - \frac{2(\pi - S\text{Re})}{\mu_3} \right\} e^{(\pi-2S\text{Re})y} + \frac{D}{\mu_4} \left\{ y + \frac{2(\pi + S\text{Re})}{\mu_4} \right\} e^{-(\pi+2S\text{Re})y} \left. \right],
\end{aligned} \tag{5.19}$$

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where

$$\begin{aligned}
 m_{1,2} &= \frac{SRePr}{2} \pm \frac{1}{2} \{S^2 Re^2 Pr^2 + 4(\pi^2 - iPr\omega)\}^{1/2}, \\
 \lambda_{1,2} &= SRe r_{1,2}(Pr + 1.0) + i\omega(Pr - 1.0), \\
 \lambda_{3,4} &= Pr(i\omega \mp S\pi Re), \\
 \lambda_{5,6} &= SRe r_{1,2}(5 - Pr) + 2S^2 Re^2(2 - Pr) + i\omega(Pr - 1.0), \\
 \lambda_{7,8} &= \pm \pi SRe(Pr - 4.0) + 2S^2 Re^2(2 - Pr) + iPr\omega, \\
 \lambda_{9,10} &= SRe r_{1,2}(2 - Pr) + (1 - Pr)(S^2 Pr^2 - i\omega), \\
 L &= \frac{1}{(e^{-m_2} - e^{-m_1})} \\
 &\times \left\{ S^2 Re^2 Pr^2 h_1 \left[\frac{A}{\lambda_1} \{e^{-(r_1 + SRePr)} - e^{-m_2}\} \right. \right. \\
 &\quad + \frac{B}{\lambda_2} \{e^{-(r_2 + SRePr)} - e^{-m_2}\} + \frac{C}{\lambda_3} \{e^{(\pi - SRePr)} - e^{-m_2}\} \\
 &\quad \left. \left. + \frac{D}{\lambda_4} \{e^{-(\pi + SRePr)} - e^{-m_2}\} \right] \right. \\
 &\quad - 2SRePr \left[\left\{ Ah_2 Re - \frac{Ec C_1 (r_1 + SRe)}{(1 - e^{-SRe})} \right\} \frac{1}{\lambda_5} \{e^{-(r_1 + 2SRe)} - e^{-m_2}\} \right. \\
 &\quad + \left\{ Bh_2 Re - \frac{Ec C_2 (r_2 + SRe)}{(1 - e^{-SRe})} \right\} \frac{1}{\lambda_6} \{e^{-(r_2 + 2SRe)} - e^{-m_2}\} \\
 &\quad + \left\{ Ch_2 Re + \frac{Ec C_3 (\pi - SRe)}{(1 - e^{-SRe})} \right\} \frac{1}{\lambda_7} \{e^{(\pi - 2SRe)} - e^{-m_2}\} \\
 &\quad \left. \left. + \left\{ Dh_2 Re - \frac{Ec C_4 (\pi + SRe)}{(1 - e^{-SRe})} \right\} \frac{1}{\lambda_8} \{e^{-(\pi + 2SRe)} - e^{-m_2}\} \right] \right\} \\
 &\quad + \frac{2Pr Ec SRe}{(1 - e^{-SRe})} \left[\frac{A_3 r_1}{\lambda_9} \{e^{-(r_1 + SRe)} - e^{-m_2}\} + \frac{A_4 r_2}{\lambda_{10}} \{e^{-(r_2 + SRe)} - e^{-m_2}\} \right] \Bigg\} \quad \text{for } Pr \neq 2.0,
 \end{aligned} \tag{5.20}$$

and for $Pr = 2.0$,

$$\begin{aligned}
 m_{3,4} &= SRe \pm \{S^2 Re^2 + (\pi^2 - 2i\omega)\}^{1/2}, & \mu_{1,2} &= 3SRe r_{1,2} + i\omega, \\
 \mu_{3,4} &= 2i\omega \mp 2\pi SRe, & \mu_{5,6} &= SRe (r_{1,2} - SRe) + i\omega, & \beta &= \frac{(1 - e^{-SRe})}{4EcSRe},
 \end{aligned} \tag{5.21}$$

L_1 and M_1 are not given here to save space.

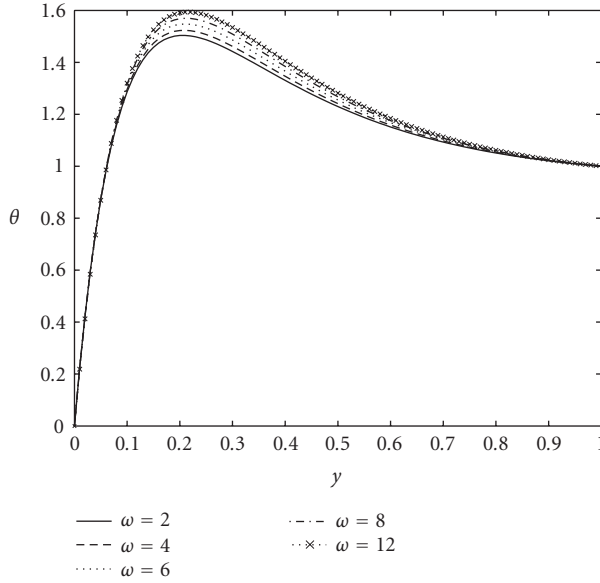


Figure 5.1. Temperature profile for $Re = 5.0$, $Pr = 0.72$, $S = 1.0$, $Ec = 10.0$, $z = 0.5$, $t = 0.2$, $\epsilon = 0.2$.

The temperature profile has been plotted against y for different values of frequency parameter ω and Reynolds number Re for $Pr = 0.72$, $Ec = 10.0$, $S = 1.0$, $z = 0.5$, $t = 0.2$ in Figures 5.1 and 5.2. It is observed that the temperature increases with the increase in frequency parameter ω . On the other hand, it increases near the stationary plate and decreases away from the stationary plate with the increase in Reynolds number Re .

One of the most important characteristics of the problem is the rate of heat transfer at the plate. The rate of heat transfer from the plate $y = 0$ to the fluid may be calculated using the formula $q_w = -(\partial T/\partial y)_{y=0}$ and can be put in nondimensional form in terms of Nusselt number as

$$\begin{aligned}
 Nu &= -\frac{q_w d}{k(T_1 - T_0)} = -\left(\frac{d\theta}{dy}\right)_{y=0} = -\left(\frac{d\theta_0}{dy}\right)_{y=0} - \epsilon \left(\frac{d\theta_1}{dy}\right)_{y=0} \\
 &= -\left(\frac{d\theta_0}{dy}\right)_{y=0} - \epsilon \left(\frac{d\theta_{11}}{dy}\right)_{y=0} e^{i(\pi z - t)} \\
 &= -\theta'_0 - \epsilon Re_3 \cos(\pi z - t + \phi_3) \quad \text{for } Pr \neq 2.0,
 \end{aligned} \tag{5.22}$$

where

$$Re_3 = \{X_3^2 + Y_3^2\}^{1/2}, \quad \tan \phi_3 = \frac{Y_3}{X_3}, \tag{5.23}$$

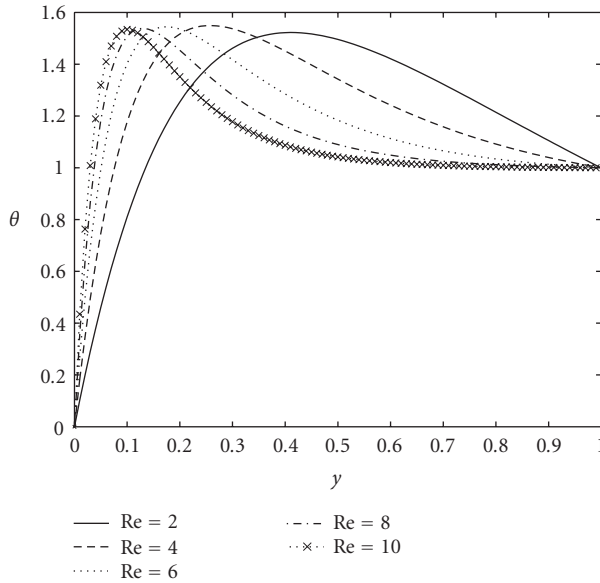


Figure 5.2. Temperature profile for $Pr = 0.72, \omega = 6.0, Ec = 10.0, z = 0.5, t = 0.2, \epsilon = 0.2$.

and for $Pr = 2.0$,

$$\begin{aligned}
 Nu &= -\frac{q_w d}{k(T_1 - T_0)} = -\left(\frac{d\theta}{dy}\right)_{y=0} = -\left(\frac{d\theta_0}{dy}\right)_{y=0} - \epsilon \left(\frac{d\theta_1}{dy}\right)_{y=0} \\
 &= -\left(\frac{d\theta_0}{dy}\right)_{y=0} - \epsilon \left(\frac{d\theta_{11}}{dy}\right)_{y=0} e^{i(\pi z - t)} \\
 &= -\theta'_0 - \epsilon Re_4 \cos(\pi z - t + \phi_4),
 \end{aligned} \tag{5.24}$$

where

$$Re_4 = \{X_4^2 + Y_4^2\}^{1/2}, \quad \tan \phi_4 = \frac{Y_4}{X_4}. \tag{5.25}$$

The rate of heat transfer due to the unsteady part of the temperature is shown in terms of Nusselt number. The amplitude Re_3 and the phase shift $\tan \phi_3$ in Nusselt number are shown in Figures 5.3 and 5.4 against Re for $Pr = 0.72, S = 1.0$, and $Ec = 10.0$ for different values of ω . It is found that Re_3 decreases with the increase in ω . This means that for higher frequencies, rate of heat transfer decreases. Figure 5.1 shows the variation of the phase shift of Nusselt number, $\tan \phi_3$ with Re for different values of ω . It is found that for

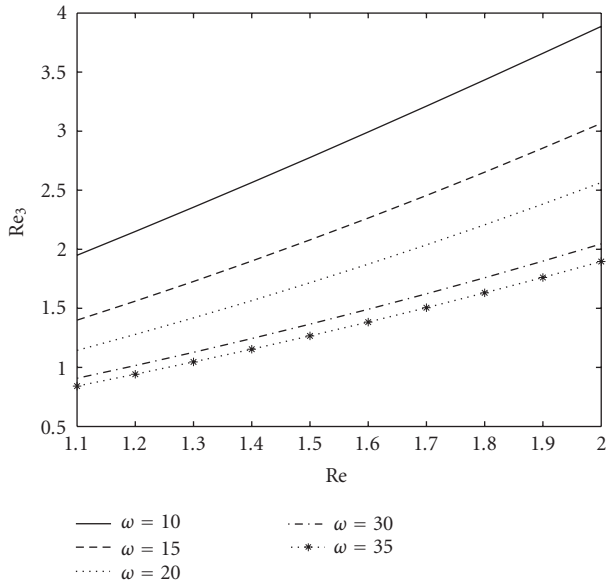


Figure 5.3. The amplitude Re_3 of Nusselt number for $Pr = 0.72, S = 1.0, Ec = 10$.

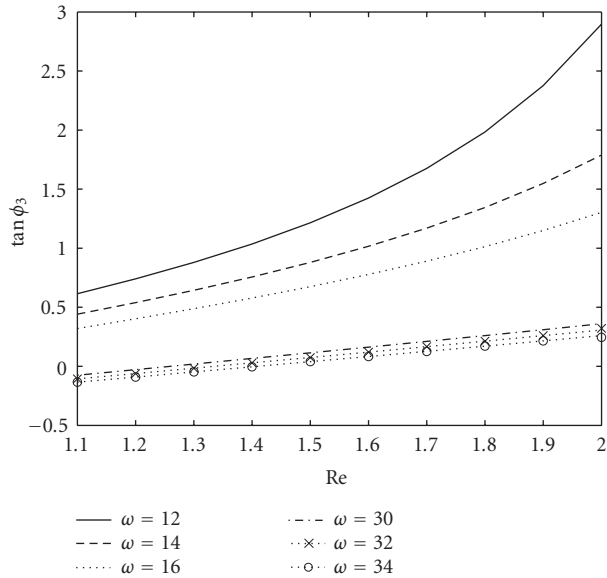


Figure 5.4. Tangent of phase shift of Nusselt number $\tan \phi_3$ for $S = 1.0, Pr = 0.72, Ec = 10$.

small frequencies there is a phase lead but for large frequencies there is a phase lag. The tangent of the phase shift $\tan \phi_3$ increases with the increase in Re .

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