

# COMPARISON OF WAVELET APPROXIMATION ORDER IN DIFFERENT SMOOTHNESS SPACES

M. R. ISLAM, S. F. AHMED, AND S. M. A. RAHMAN

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In linear approximation by wavelet, we approximate a given function by a finite term from the wavelet series. The approximation order is improved if the order of smoothness of the given function is improved, discussed by Cohen (2003), DeVore (1998), and Siddiqi (2004). But in the case of nonlinear approximation, the approximation order is improved quicker than that in linear case. In this study we proved this assumption only for the Haar wavelet. Haar function is an example of wavelet and this fundamental example gives major feature of the general wavelet. A nonlinear space comes from arbitrary selection of wavelet coefficients, which represent the target function almost equally. In this case our computational work will be reduced tremendously in the sense that approximation error decays more quickly than that in linear case.

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## 1. Introduction

Approximation by wavelet is a new tool in mathematics, physics, and engineering. In [5, 6] Morlet et al. first introduced the idea of wavelets as a family of functions constructed from translation and dilations of a signal function called mother wavelet. Readers interested in the history of this subject may go through Debnath [2], Meyer [4], Morlet et al. [5, 6].

Wavelet analysis was originally introduced in order to improve seismic signal processing by switching from short-time Fourier analysis to new better algorithms to detect and analyze abrupt changes in signals. It may be remarked that a systematic study of approximation theory was initiated by Natanson [7] in the 1950s. Results concerning approximation by trigonometric polynomials to functions belonging to different classes of functions can be found in Zygmund [9]. By the 1970s the subject became very popular in view of its wide applications. The finite element method developed by engineers in the early 1950s found close connection with the approximation theory. French mathematician Céa observed in the early 1960s that error estimation of finite element is nothing but an approximation problem in Sobolev spaces. Approximation by Spline function attracted the

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attention of several eminent mathematicians during the 1970s and 1980s. They are not only convenient and suitable for computer calculations, but also provide optimal theoretical solution to the estimation of functions from limited data.

From the viewpoint of approximation theory and harmonic analysis, the wavelet theory is important on several counts. It gives simple and elegant unconditional wavelet bases for function spaces (Lebesgue, Sobolev, Besov, etc.).

A recent development of approximation theory is approximation of an arbitrary function by wavelet polynomials. There are different types of wavelet such as Haar wavelet, Mexican-Hat wavelet, Shannon wavelet, Daubechies wavelet, Meyer's wavelet, and so forth. In this paper we mainly focus on approximation by Haar wavelet. Haar function is an example of wavelet and this fundamental example gives major feature of the general wavelet.

Infinite series is a mathematical tool for exact representation of certain functions. When we work with the series representations in practice, we are only able to deal with finite sums. For example, if a function  $f$  has an exact representation through Fourier series, we need to have finite partial sum  $(S_N)_{N \in \mathbb{N}}$  for computer work. We need to choose  $N$  such that the partial sum  $S_N$  approximates  $f$  sufficiently well. For a good approximation  $N$  becomes very large. If we can replace the partial sum  $S_N$  by another finite sum, which approximates  $f$  equally well by using fewer coefficients. This is the idea behind nonlinear approximation.

In wavelet theory, if we approximate the target function by selecting terms of the wavelet series, for which the target function  $f$  is kept controlled only over the number of terms to be used, it is called  $N$ -term approximation. Our aim is to approximate a function via Haar wavelet. In Section 2 we give a brief discussion on Haar wavelet and its properties. In Section 3 we approximate a function by Haar wavelet in different smoothness spaces. Finally in Section 4 we use only few Haar coefficients for which it is nonlinear. In that case we get a significant improvement of approximation order in comparison to any other wavelet methods.

### 2. Haar wavelet systems

*Definition 2.1* (Haar function). A function defined on the real line  $\mathbb{R}$  as

$$\psi(t) = \begin{cases} 1 & \text{for } t \in \left[0, \frac{1}{2}\right), \\ -1 & \text{for } t \in \left[\frac{1}{2}, 1\right), \\ 0 & \text{otherwise,} \end{cases} \quad (2.1)$$

is known as Haar function.

The Haar function  $\Psi(t)$  is the simplest example of Haar wavelet. The Haar function  $\Psi(t)$  is a wavelet because it satisfies all the conditions of wavelet. Haar wavelet is discontinuous at  $t = 0, 1/2, 1$  and it is very well localized in the time domain.

*Definition 2.2* (dyadic interval). For each pair of  $j, k \in \mathbb{Z}$ , define the interval  $I_{j,k}$  by  $I_{j,k} = [2^{-j}k, 2^{-j}(k+1)]$  which is known as dyadic interval. The collection of all such intervals is called dyadic subintervals of  $\mathbb{R}$ .

*Definition 2.3* (Haar scaling function). The family of functions  $\{\varphi_{i,k}(t)\}_{i,k \in \mathbb{Z}} = 2^{j/2}\varphi(2^j t - k)$  is called the system of Haar scaling functions. For each  $j, k \in \mathbb{Z}$ , the collection of  $\{\varphi_{i,k}(t)\}_{i,k \in \mathbb{Z}}$  is called the Haar scaling function at scale  $j$ .

Haar scaling function can be defined as

$$\varphi(t) = \chi_{[0,1)}(t) = \begin{cases} 1 & \text{if } 0 \leq t < 1, \\ 0 & \text{otherwise.} \end{cases} \quad (2.2)$$

For each  $j, k \in \mathbb{Z}$ ,  $\{\varphi_{i,k}(t)\}_{i,k \in \mathbb{Z}} = 2^{j/2}\varphi(2^j t - k) = D_{2^j} T_k \varphi(t)$ , where dilation operator  $D_a f(x) = a^{1/2} f(ax)$  and the translation of operator  $T_k f(x) = f(x - k)$ .

*Definition 2.4* (Haar wavelet system). For each  $j, k \in \mathbb{Z}$ , define  $\{\psi_{i,k}(t)\}_{i,k \in \mathbb{Z}} = 2^{j/2}\psi(2^j t - k)$ . The family of functions  $\{\psi_{i,k}(t)\}_{i,k \in \mathbb{Z}}$  is called the Haar wavelet system.

Consider  $f(t)$  is defined on  $L_2[0,1]$ , has an expansion in terms of Haar functions as follows. For any integer  $J \geq 0$ ,

$$\begin{aligned} f(t) &= \sum_{k=0}^{2^J-1} \langle f, \varphi_{J,k} \rangle \varphi_{J,k}(t) + \sum_{j=J}^{\infty} \sum_{k=0}^{2^j-1} \langle f, \psi_{j,k} \rangle \psi_{j,k}(t) \\ &= \sum_{k=0}^{2^J-1} c_{J,k} \varphi_{J,k}(t) + \sum_{j=J}^{\infty} \sum_{k=0}^{2^j-1} d_{j,k} \psi_{j,k}(t) \end{aligned} \quad (2.3)$$

which is known as Haar series; and  $d_{j,k}$  and  $c_{j,k}$  are the Haar coefficients for wavelet and Haar scaling coefficients, respectively.

### 3. Approximation by Haar wavelet in different spaces

**3.1. Approximation space.** Let  $(X, \|\cdot\|_X)$  be a normed space in which the approximation takes place. Let  $(S_N)_{N \geq 0}$  be a family of subspaces of a normed space  $X$ . Our approximation comes from the space  $(S_N)_{N \geq 0} \subset X$ .

For a function  $f \in X$ , the approximation error is  $E_N(f)_X = \text{dist} \cdot (f, S_N)_X = \inf_{g \in S_N} \|f - g\|_X$ , where  $g$  is the approximating function in  $(S_N)_{N \geq 0}$ .

*For linear approximation.*  $N$  represents the number of parameters, which are needed to describe an element in  $S_N$ . That is,  $N$  is dimension of  $S_N$ . In most cases of interest  $E_N(f)$  goes to zero as  $N$  tends to infinity.

*For nonlinear approximation.*  $N$  is related to the number of free parameters. For example,  $N$  might be the number of knots in piecewise constant approximation with free knots. The  $S_N$  can be quite general spaces; in particular, they do not have to be linear.

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**3.2. Approximation in  $L_2(\mathbb{R})$ .** Let  $f$  be continuous on  $L_2(\mathbb{R})$  and the Haar wavelet series of  $f$  is  $f \approx \sum_{j=0}^{\infty} \sum_{k=0}^{2^j-1} \langle f, \psi_{j,k} \rangle \psi_{j,k}(t)$ . If  $\psi_{j,k}(t)$  is support on the interval  $I_{j,k} = [2^{-j}k, 2^{-j}(k+1)]$ , then

$$\langle f, \psi_{j,k} \rangle = \int_{\mathbb{R}} f(t) \psi_{j,k}(t) dt = 2^{-j/2} \int_{k2^{-j}}^{(k+1)2^{-j}} f(t) \psi(2^j t - k) dt. \quad (3.1)$$

For computing finite sum, let  $N = 2^J$  be coefficients for some  $J \in \mathbb{N}$ .

That is, we consider  $j = 0, 1, 2, 3, \dots, J-1$ , then  $\sum_{j=0}^{J-1} \sum_{k=0}^{2^j-1} 1 = 1 + 2 + 2^2 + \dots + 2^{J-1} = N - 1$  coefficients. For Haar wavelet we can see that for each  $j$  only one of the coefficients is nonzero and its size is  $2^{-j/2}$ . For details one can see Christensen and Christensen [1] and Walnut [8].

Then the error of the approximation in  $L_2(\mathbb{R})$  is

$$\begin{aligned} \left\| f - \sum_{j=0}^{J-1} \sum_{k=0}^{2^j-1} \langle f, \psi_{j,k} \rangle \psi_{j,k}(t) \right\|_{L_2}^2 &= \left\| \sum_{j=J}^{\infty} \sum_{k=0}^{2^j-1} \langle f, \psi_{j,k} \rangle \psi_{j,k}(t) \right\|_{L_2}^2 = \sum_{j=J}^{\infty} \sum_{k=0}^{2^j-1} |\langle f, \psi_{j,k} \rangle|^2 \\ &\approx \sum_{j=J}^{\infty} 2^{-j} \approx 2^{-J} \approx \frac{1}{N} = O(2^{-J}). \end{aligned} \quad (3.2)$$

#### 3.3. Approximation in $L_p(\mathbb{R})$ .

**THEOREM 3.1.** *If  $f \in L_p(\mathbb{R})$  and the partial sum of the Haar wavelet series of  $f$  is  $g = \sum_{j=0}^{J-1} \sum_{k=0}^{2^j-1} \langle f, \psi_{j,k} \rangle \psi_{j,k}(t)$ , for  $j \in \mathbb{N}$ , then the error of the approximation is  $O(2^{-J/2})$ .*

*Proof.* The error of the approximation in  $L_p(\mathbb{R})$  is

$$\begin{aligned} \|f - g\|_{L_p} &= \left\| f - \sum_{j=0}^{J-1} \sum_{k=0}^{2^j-1} \langle f, \psi_{j,k} \rangle \psi_{j,k}(t) \right\|_{L_p} = \left\| \sum_{j=J}^{\infty} \sum_{k=0}^{2^j-1} \langle f, \psi_{j,k} \rangle \psi_{j,k}(t) \right\|_{L_p} \\ &\approx \left( \sum_{j=J}^{\infty} \sum_{k=0}^{2^j-1} |\langle f, \psi_{j,k} \rangle|^p \right)^{1/p} \approx \left( \sum_{j=J}^{\infty} 2^{-jp/2} \right)^{1/p} \approx 2^{-J/2} = O(2^{-J/2}). \end{aligned} \quad (3.3)$$

□

#### 3.4. Approximation in $\text{Lip}_M(\alpha, L_p)$ spaces.

**THEOREM 3.2.** *If  $f \in \text{Lip}_M(\alpha, L_p)$ ,  $0 < \alpha \leq 1$ ,  $1 < p \leq \infty$ ,  $M > 0$ , and  $g(t) = \sum_{j=0}^{J-1} \sum_{k=0}^{2^j-1} \langle f, \psi_{j,k} \rangle \psi_{j,k}(t)$  is the Haar wavelet series of  $f$  for some  $J \in \mathbb{N}$ , then the error of the approximation in  $\text{Lip}_M(\alpha, L_p)$  is  $O(2^{-J\alpha})$ .*

*Proof.* From DeVore [3] we have if  $f \in \text{Lip}_M(\alpha, L_p)$ ,  $0 < \alpha \leq 1$ ,  $1 < p \leq \infty$ ,  $\text{dist}(f, S_N)_p \leq \inf_{g \in S_N} \|f - g\|_p \leq C_p \|f\|_{\text{Lip}(\alpha, L_p)} \delta^\alpha$ , where  $\delta = \max_{0 \leq k < N} |t_{k+1} - t_k|$ .

So the error of the approximation in  $\text{Lip}_M(\alpha, L_p)$  is

$$\begin{aligned} \|f - g\|_{L_p} &= \left\| f - \sum_{j=0}^{J-1} \sum_{k=0}^{2^j-1} \langle f, \psi_{j,k} \rangle \psi_{j,k}(t) \right\|_{L_p} \\ &\leq C_p |f|_{\text{Lip}(\alpha, L_p)} (2^{-J})^\alpha \\ &\leq M (2^{-J})^\alpha = O(2^{-J\alpha}), \end{aligned} \quad (3.4)$$

where  $C_p$  is depending on  $p$ . □

### 3.5. Approximation in Sobolev spaces $H^m(\mathbb{R})$ .

**THEOREM 3.3.** *If  $f \in H^m(\mathbb{R})$  and  $g(t) = \sum_{j=0}^{J-1} \sum_{k=0}^{N-1} \langle f, \psi_{j,k} \rangle \psi_{j,k}(t)$  is the finite Haar wavelet series of  $f$  for some  $J \in N$ , then the error of the approximation is  $O(2^{-mN/2})$ , where  $N = 2^J$ .*

*Proof.* The error of the approximation is

$$\begin{aligned} \|f - g\|_{L_2} &= \left\| \sum_{j=J}^{\infty} \sum_{k=0}^{N-1} \langle f, \psi_{j,k} \rangle \psi_{j,k}(t) \right\|_{L_2} \\ &\leq \left( \sum_{j=J}^{\infty} \sum_{k=0}^{2^j-1} |\langle f, \psi_{j,k} \rangle|^2 \right)^{1/2} \leq \left( \sum_{j=J}^{\infty} \sum_{k=0}^{N-1} \frac{2^{mk}}{2^{mN}} |\langle f, \psi_{j,k} \rangle|^2 \right)^{1/2} \\ &\leq \left( 2^{-mN} \left( \sum_{j=J}^{\infty} \sum_{k=0}^{N-1} 2^{mk} |\langle f, \psi_{j,k} \rangle|^2 \right) \right)^{1/2}. \end{aligned} \quad (3.5)$$

By using the properties of Besov space we have

$$\|f\|_{H^m(L_2(\mathbb{R}))} \cong \left( \left( \sum_{j=J}^{\infty} \sum_{k=0}^{N-1} 2^{mk} |\langle f, \psi_{j,k} \rangle|^2 \right)^{1/2} \right). \quad (3.6)$$

Therefore  $\|f - g\|_{L_2} \leq 2^{-mN/2} \|f\|_{H^m(L_2(\mathbb{R}))} = O(2^{-mN/2})$ . □

### 3.6. Approximation in Besov space $B_q^\alpha(L_p(\mathbb{R}))$ .

**THEOREM 3.4.** *If  $f \in B_q^\alpha(L_q(\mathbb{R}))$ ,  $\alpha > 0$ ,  $0 < q \leq \infty$ , and  $g(t) = \sum_{j=0}^{J-1} \sum_{k=0}^{N-1} \langle f, \psi_{j,k} \rangle \psi_{j,k}(t)$  is the finite Haar wavelet series of  $f$  for some  $J \in N$ , then the error of the approximation is  $O(2^{-\alpha N/2})$ , where  $N = 2^J$ .*

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*Proof.* The error of the approximation is

$$\begin{aligned}
 \|f - g\|_{L_q} &= \left\| \sum_{j=J}^{\infty} \sum_{k=0}^{N-1} \langle f, \psi_{j,k} \rangle \psi_{j,k}(t) \right\|_{L_q} \\
 &\leq \left( \sum_{j=J}^{\infty} \sum_{k=0}^{N-1} |\langle f, \psi_{j,k} \rangle|^q \right)^{1/q} \leq \left( \sum_{j=J}^{\infty} \sum_{k=0}^{N-1} \frac{2^{\alpha k}}{2^{\alpha N}} |\langle f, \psi_{j,k} \rangle|^q \right)^{1/q} \\
 &\leq \left( 2^{-\alpha v/q} \left( \sum_{j=J}^{\infty} \sum_{k=0}^{N-1} 2^{\alpha k} |\langle f, \psi_{j,k} \rangle|^q \right)^{1/q} \right).
 \end{aligned} \tag{3.7}$$

By using the properties of Besov space we have

$$\|f\|_{B_q^{\alpha}(L_q(\mathbb{R}))} \cong \left( \left( \sum_{j=J}^{\infty} \sum_{k=0}^{N-1} 2^{\alpha k} |\langle f, \psi_{j,k} \rangle|^q \right)^{1/q} \right). \tag{3.8}$$

Therefore  $\|f - g\|_{L_q(\mathbb{R})} \leq 2^{-\alpha N/q} \|f\|_{B_q^{\alpha}(L_q(\mathbb{R}))} = O(2^{-\alpha N/q})$ , where  $1/q = \alpha/2 + 1/2$ .  $\square$

*Conclusion.* The above theorem shows that the approximation order will improve if the smoothness of the approximation spaces is improved.

### 4. Nonlinear approximation by Haar wavelet

Our previous discussion is finite linear approximation by Haar wavelet. Now we consider nonlinear approximation via Haar wavelet. We have seen that for each level  $j$ , exactly one Haar coefficient is nonzero. One can see Christensen and Christensen [1] and Walnut [8].

If we can calculate  $N = 2^j$  biggest Haar coefficients, in that case the approximation error is

$$\sigma_N(f)_X = \text{dist} \cdot (f, S_N)_X = \inf_{g \in \Sigma_N} \|f - g\|_X, \tag{4.1}$$

where  $\Sigma_N$  and  $\sigma_N(f)$  denote the set of wavelets and approximation error, respectively, in the nonlinear spaces.

#### 4.1. Nonlinear approximation in $L_p(\mathbb{R})$ .

**THEOREM 4.1.** *If  $f \in L_p(\mathbb{R})$  and the partial sum of the Haar wavelet series of  $f$  is  $g = \sum_{j=0}^{N-1} \sum_{k=0}^{N-1} \langle f, \psi_{j,k} \rangle \psi_{j,k}(t)$ , for  $j \in N$ , then the error of the nonlinear approximation is  $O(2^{-N/p})$ .*

*Proof.* The error of the nonlinear approximation in  $L_p(\mathbb{R})$  is

$$\begin{aligned} \|f - g\|_{L_p} &= \left\| f - \sum_{j=0}^{N-1} \sum_{k=0}^{N-1} \langle f, \psi_{j,k} \rangle \psi_{j,k}(t) \right\|_{L_p} = \left\| \sum_{j=N+1}^{\infty} \sum_{k=0}^{N-1} \langle f, \psi_{j,k} \rangle \psi_{j,k}(t) \right\|_{L_p} \\ &\leq \left( \sum_{j=N+1}^{\infty} \sum_{k=0}^{N-1} |\langle f, \psi_{j,k} \rangle|^p \right)^{1/p} \approx \left( \sum_{j=N+1}^{\infty} 2^{-Np/2} \right)^{1/p} \approx 2^{-N/2} = O(2^{-N/2}). \end{aligned} \quad (4.2)$$

□

*Conclusion.* From the above discussion we have seen that in the case of linear approximation the approximation order depends on the order of smoothness of the function space. But in the case of nonlinear approximation there is a significant improvement in the approximation order compared to that in linear approximation.

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M. R. Islam: Mathematics Discipline, Khulna University, Khulna 9208, Bangladesh  
*E-mail address:* mrislam66@gmail.com

S. F. Ahemmed: Mathematics Discipline, Khulna University, Khulna 9208, Bangladesh  
*E-mail address:* sfahemmed@yahoo.com

S. M. A. Rahman: Mathematics Discipline, Khulna University, Khulna 9208, Bangladesh  
*E-mail address:* arahmanku@yahoo.com