AN EXTENSION AND A REFINEMENT OF VAN DER CORPUT'S INEQUALITY

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van der Corput's inequality is extended and refined by using Euler-Maclaurin formula and other analytic techniques.

1. Introduction

Let $S_n = \sum_{k=1}^{n}(1/k)$ and $a_n \geq 0$ for $n \in \mathbb{N}$ such that $0 < \sum_{n=1}^{\infty} a_n < \infty$. The famous van der Corput inequality [10] reads that

$$\sum_{n=1}^{\infty}\left(\prod_{k=1}^{n} a_k^{1/k}\right)^{1/S_n} < e^{1+\gamma}\sum_{n=1}^{\infty}(n+1)a_n, \tag{1.1}$$

where $\gamma = 0.57721566\ldots$ stands for Euler-Mascheroni constant. The constant $e^{1+\gamma}$ in (1.1) is the best possible.

Hu [5] gave a strengthened version of (1.1) as

$$\sum_{n=1}^{\infty}\left(\prod_{k=1}^{n} a_k^{1/k}\right)^{1/S_n} < e^{1+\gamma}\sum_{n=1}^{\infty}\left(n - \frac{\ln n}{4}\right)a_n. \tag{1.2}$$

Yang [14] established a relation between Carleman's inequality and van der Corput's inequality and presented the following:

$$\sum_{n=1}^{\infty}\left(\prod_{k=1}^{n} a_k^{1/k^{\alpha}}\right)^{1/S_n(\alpha)} < e\sum_{n=1}^{\infty} e^{\alpha n^{\alpha-1}S_n(\alpha)}a_n, \tag{1.3}$$

where $S_n(\alpha) = \sum_{k=1}^{n}(1/k^{\alpha})$ and $\alpha \in [0,1]$.

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In a recent paper [15], Yang has obtained another extension of (1.1) as follows:

$$\sum_{n=1}^{\infty}\left(\prod_{k=1}^{n} a_k^{1/(k+\beta)}\right)^{1/S_n(\beta)} < e^{1+\gamma_1(\beta)}\sum_{n=1}^{\infty}\left(n+\frac{1}{2}+\beta\right)a_n, \tag{1.4}$$

where $\beta \in (-1,\infty)$, $S_n(\beta) = \sum_{k=1}^{n}(1/(k+\beta))$, and

$$\gamma_1(\beta) = \lim_{n\to\infty}\left[\sum_{k=1}^{n}\frac{1}{k+\beta} - \ln(n+\beta)\right]. \tag{1.5}$$

Applying $\beta = 0$ in (1.4) leads to

$$\sum_{n=1}^{\infty}\left(\prod_{k=1}^{n} a_k^{1/k}\right)^{1/S_n} < e^{1+\gamma}\sum_{n=1}^{\infty}\left(n+\frac{1}{2}\right)a_n, \tag{1.6}$$

which improved inequality (1.1) clearly.

For more information about van der Corput's inequality, please refer to [2, 5, 10, 14, 15] and the references therein.

The aim of this paper is to further extend and refine van der Corput's inequality by using Euler-Maclaurin formula and other analytic techniques.

Our main results are the following two theorems.

Theorem 1.1. *Let* $a_n \geq 0$ *for* $n \in \mathbb{N}$ *such that* $0 \leq \sum_{n=1}^{\infty} a_n < \infty$. *Then*

$$\sum_{n=1}^{\infty}\left[\prod_{k=1}^{n} a_k^{1/\sqrt{k(k+\lambda)}}\right]^{1/S_n(\lambda)} < e^{1+(1+\lambda/3)\gamma(\lambda)}\sum_{n=1}^{\infty}(n+1)^{\lambda/3}\left[1-\frac{\ln(n+1)}{4(n+1+\lambda/2)}\right]a_n, \tag{1.7}$$

where $\lambda \in [0,\infty)$,

$$S_n(\lambda) = \sum_{k=1}^{n}\frac{1}{\sqrt{k(k+\lambda)}}, \tag{1.8}$$

$$\gamma(\lambda) = \lim_{n\to\infty}\left[S_n(\lambda) - 2\ln\frac{\sqrt{n}+\sqrt{n+\lambda}}{1+\sqrt{1+\lambda}}\right]. \tag{1.9}$$

Theorem 1.2. *Let* $a_n \geq 0$ *for* $n \in \mathbb{N}$ *such that* $0 \leq \sum_{n=1}^{\infty} a_n < \infty$. *Then*

$$\sum_{n=1}^{\infty}\left(\prod_{k=1}^{n} a_k^{1/k}\right)^{1/S_n} < e^{1+\gamma}\sum_{n=1}^{\infty} n\left(1-\frac{\ln n}{3n-1/4}\right)a_n. \tag{1.10}$$

Remark 1.3. It is easy to see that inequality (1.10) refines inequalities (1.1), (1.2), and (1.6).

2. Lemmas

To prove our main results, the following lemmas are necessary.

Recall [7, 9] that a function f is called completely monotonic on an interval I if f has derivatives of all orders on I and $0 < (-1)^k f^{(k)}(x) < \infty$ for all $k \geq 0$ on I. The background information and an extensive bibliography about the theory of completely monotonic function can be found in the recent papers [4, 7, 8].

Lemma 2.1. *The function* $f(x) = 1/\sqrt{x(x+\lambda)}$ *for* $\lambda \in [0,\infty)$ *is completely monotonic in* $(0,\infty)$ *and* $\lim_{x\to\infty} f^{(i)}(x) = 0$ *for any nonnegative integer i.*

Proof. It is not difficult to verify that the functions $1/\sqrt{x}$ and $1/\sqrt{x+\lambda}$ are completely monotonic in $x \in (0,\infty)$. Since the product of any finite completely monotonic functions is also strictly completely monotonic (see [11]), then the function $f(x)$ is strictly completely monotonic in $(0,\infty)$.

By induction, it is easy to verify that $\lim_{x\to\infty} f^{(i)}(x) = 0$ holds for any nonnegative integer i. The proof of Lemma 2.1 is complete. □

Recall that Euler-Maclaurin formula (see [1, pages 617–623] and [6, 12, 14]) states

$$\sum_{k=1}^{n} f(k) = \int_1^n f(x)\,dx + \frac{1}{2}[f(n)+f(1)] + \int_1^n \rho_1(x) f'(x)\,dx, \tag{2.1}$$

where $\rho_1(x) = x - [x] + 1/2$ is Bernoulli's function and $f \in C^1[1,\infty)$. Furthermore, if $(-1)^i f^{(i)}(x) > 0$ and $\lim_{x\to\infty} f^{(i)}(x) = 0$ for $i = 1,2,3$, then

$$\int_n^\infty \rho_1(x) f'(x)\,dx = -\frac{1}{12} f'(n)\epsilon, \quad 0 < \epsilon < 1. \tag{2.2}$$

Lemma 2.2. *For* $n \in \mathbb{N}$ *and* $\lambda \in [0,\infty)$,

$$S_n(\lambda) < \ln(n+1) + \gamma(\lambda), \tag{2.3}$$

where $S_n(\lambda)$ *and* $\gamma(\lambda)$ *are defined by (1.8) and (1.9), respectively.*

Proof. It is clear that Lemma 2.1 allows us to apply Euler-Maclaurin formula (2.1) and formula (2.2) to $f(x) = 1/\sqrt{x(x+\lambda)}$. From this, it follows that

$$\begin{aligned} S_n(\lambda) &= 2\ln\frac{\sqrt{n}+\sqrt{n+\lambda}}{1+\sqrt{1+\lambda}} + \frac{1}{2}\left[\frac{1}{\sqrt{1+\lambda}} + \frac{1}{\sqrt{n(n+\lambda)}}\right] \\ &\quad + \int_1^n \rho_1(x)\left[\frac{1}{\sqrt{x(x+\lambda)}}\right]' dx, \end{aligned} \tag{2.4}$$

$$\int_n^\infty \rho_1(x)\left[\frac{1}{\sqrt{x(x+\lambda)}}\right]' dx = -\frac{1}{12}\left[\frac{1}{\sqrt{n(n+\lambda)}}\right]' \epsilon = \frac{(2n+\lambda)\epsilon}{24[n(n+\lambda)]^{3/2}},$$

where $0 < \epsilon < 1$, and

$$\gamma(\lambda) = \frac{1}{2\sqrt{1+\lambda}} + \int_1^\infty \rho_1(x)\left[\frac{1}{\sqrt{x(x+\lambda)}}\right]' dx. \tag{2.5}$$

Therefore,

$$\begin{aligned} S_n(\lambda) &= 2\ln\frac{\sqrt{n}+\sqrt{n+\lambda}}{1+\sqrt{1+\lambda}} + \gamma(\lambda) - \frac{(2n+\lambda)\epsilon}{24\left[n(n+\lambda)\right]^{3/2}} + \frac{1}{2\sqrt{n(n+\lambda)}} \\ &< \ln n + \gamma(\lambda) + \frac{1}{2\sqrt{n(n+\lambda)}}, \end{aligned} \tag{2.6}$$

and then

$$\begin{aligned} S_n(\lambda) &= \sum_{k=1}^{n+1}\frac{1}{\sqrt{k(k+\lambda)}} - \frac{1}{\sqrt{(n+1)(n+1+\lambda)}} \\ &< \ln(n+1) + \gamma(\lambda) - \frac{1}{2\sqrt{(n+1)(n+1+\lambda)}} < \ln(n+1) + \gamma(\lambda). \end{aligned} \tag{2.7}$$

The proof of Lemma 2.2 is complete. □

Lemma 2.3. *For $k \in \mathbb{N}$ and $\lambda \in [0,\infty)$,*

$$\sqrt{\frac{k(k+\lambda)}{(k+1)(k+1+\lambda)}} \le \frac{k+\lambda/2}{k+1+\lambda/2}, \tag{2.8}$$

$$\sqrt{(k+1)(k+1+\lambda)} - \sqrt{k(k+\lambda)} \le 1 + \frac{\lambda}{3}. \tag{2.9}$$

Proof. Inequality (2.8) is equivalent to

$$\left(k+\frac{\lambda}{2}\right)^2(k+1)(k+1+\lambda) \ge k(k+\lambda)\left(k+1+\frac{\lambda}{2}\right)^2. \tag{2.10}$$

The difference between both sides of (2.10) equals

$$\begin{aligned} &\left[k^4 + 2k^3(\lambda+1) + k^2\left(\frac{5}{4}\lambda^2 + 3\lambda + 1\right) + k\left(\frac{\lambda^3}{4} + \frac{3}{2}\lambda^2 + \lambda\right) + \frac{\lambda+1}{4}\lambda^2\right] \\ &\quad - \left[k^4 + 2k^3(\lambda+1) + k^2\left(\frac{5}{4}\lambda^2 + 3\lambda + 1\right) + k\left(\frac{\lambda^3}{4} + \lambda^2 + \lambda\right)\right] \\ &= \frac{k\lambda^2}{2} + \frac{\lambda^2}{4} + \frac{\lambda^3}{4} \ge 0. \end{aligned} \tag{2.11}$$

Inequality (2.9) can be deduced straightforwardly from

$$\begin{aligned} &\sqrt{(k+1)(k+1+\lambda)} - \sqrt{k(k+\lambda)} \\ &\qquad = \frac{2k+\lambda+1}{\sqrt{(k+1)(k+1+\lambda)} + \sqrt{k(k+\lambda)}} \le 1 + \frac{\lambda}{3}. \end{aligned} \tag{2.12}$$

The proof of Lemma 2.3 is complete. □

Lemma 2.4. *For* $x \in (0,\infty)$ *and* $\lambda \in [0,\infty)$,

$$\left[1 - \frac{1}{2(x+1+\lambda/2)}\right]^{\ln(x+1)} < 1 - \frac{\ln(x+1)}{4(x+1+\lambda/2)}. \tag{2.13}$$

Proof. Let $u(x,\lambda) = 2(x+1+\lambda/2) - \ln(x+1)$ for $x \in (0,\infty)$ and $\lambda \in [0,\infty)$. Then $\partial u(x,\lambda)/\partial x = 2 - 1/(x+1) > 0$ and $u(0,\lambda) = 2+\lambda > 0$. Thus,

$$\frac{\ln^2(x+1)}{8(x+1+\lambda/2)^2} < \frac{\ln(x+1)}{4(x+1+\lambda/2)}. \tag{2.14}$$

As a result, by

$$\left(1 - \frac{1}{t}\right)^{-t} > e \tag{2.15}$$

for $t > 1$ and

$$e^t < 1 + t + \frac{t^2}{2} \tag{2.16}$$

for $t < 0$, it follows that

$$\begin{aligned}\left[1 - \frac{1}{2(x+1+\lambda/2)}\right]^{\ln(x+1)} &< \left(\frac{1}{e}\right)^{\ln(x+1)/2(x+1+\lambda/2)} \\ &< 1 - \frac{\ln(x+1)}{2(x+1+\lambda/2)} + \frac{\ln^2(x+1)}{8(x+1+\lambda/2)^2} < 1 - \frac{\ln(x+1)}{4(x+1+\lambda/2)}.\end{aligned} \tag{2.17}$$

The proof of Lemma 2.4 is complete. □

Lemma 2.5. *For* $k \in \mathbb{N}$ *and* $\lambda \in [0,\infty)$,

$$\begin{aligned}B_k(\lambda) &\triangleq \left[\frac{\sqrt{(k+1)(k+1+\lambda)}\,S_{k+1}(\lambda)}{\sqrt{k(k+\lambda)}\,S_k(\lambda)}\right]^{\sqrt{k(k+\lambda)}S_k(\lambda)} \\ &\leq e^{1+(1+\lambda/3)\gamma(\lambda)}(k+1)^{1+\lambda/3}\left[1 - \frac{\ln(k+1)}{4(k+1+\lambda/2)}\right].\end{aligned} \tag{2.18}$$

Proof. For $k \in \mathbb{N}$,

$$B_k(\lambda) = \left\{1 + \frac{1 + \left[\sqrt{(k+1)(k+1+\lambda)} - \sqrt{k(k+\lambda)}\right]S_k(\lambda)}{\sqrt{k(k+\lambda)}\,S_k(\lambda)}\right\}^{\sqrt{k(k+\lambda)}S_k(\lambda)} \triangleq C_k^{h(k,\lambda)}, \tag{2.19}$$

where

$$\begin{aligned}C_k &= \left[1 + \frac{1}{g(k,\lambda)}\right]^{g(k,\lambda)}, \quad g(k,\lambda) = \frac{\sqrt{k(k+\lambda)}\,S_k(\lambda)}{h(k,\lambda)}, \\ h(k,\lambda) &= 1 + \left[\sqrt{(k+1)(k+1+\lambda)} - \sqrt{k(k+\lambda)}\right]S_k(\lambda).\end{aligned} \tag{2.20}$$

It is easy to see that

$$\begin{aligned} g(k,\lambda)+1 &= \frac{1+\sqrt{(k+1)(k+1+\lambda)}S_k(\lambda)}{1+\left[\sqrt{(k+1)(k+1+\lambda)}-\sqrt{k(k+\lambda)}\right]S_k(\lambda)} \\ &\le \frac{\sqrt{(k+1)(k+1+\lambda)}}{\sqrt{(k+1)(k+1+\lambda)}-\sqrt{k(k+\lambda)}}. \end{aligned} \tag{2.21}$$

By using the inequality $(1+1/x)^x < e[1-1/2(x+1)]$ obtained in [13], inequalities (2.21) and (2.8) in Lemma 2.3, it is deduced that

$$\begin{aligned} C_k = \left[1+\frac{1}{g(k,\lambda)}\right]^{g(k,\lambda)} &\le e\left\{1-\frac{1}{2[g(k,\lambda)+1]}\right\} \\ \le e\left[\frac{1}{2}+\frac{\sqrt{k(k+\lambda)}}{2\sqrt{(k+1)(k+1+\lambda)}}\right] &\le e\left[1-\frac{1}{2(k+1+\lambda/2)}\right]. \end{aligned} \tag{2.22}$$

Hence, from inequalities (2.3), (2.9), (2.22) in Lemma 2.2, and (2.13) in Lemma 2.4, it is shown that

$$\begin{aligned} B_k(\lambda) &\le \left\{e\left[1-\frac{1}{2(k+1+\lambda/2)}\right]\right\}^{h(k,\lambda)} \\ &\le \left\{e\left[1-\frac{1}{2(k+1+\lambda/2)}\right]\right\}^{1+(1+\lambda/3)[\ln(k+1)+\gamma(\lambda)]} \\ &\le e^{1+(1+\lambda/3)\gamma(\lambda)}(k+1)^{1+\lambda/3}\left[1-\frac{1}{2(k+1+\lambda/2)}\right]^{\ln(k+1)} \\ &\le e^{1+(1+\lambda/3)\gamma(\lambda)}(k+1)^{1+\lambda/3}\left[1-\frac{\ln(k+1)}{4(k+1+\lambda/2)}\right]. \end{aligned} \tag{2.23}$$

The proof of Lemma 2.5 is complete. □

LEMMA 2.6. *For $n \in \mathbb{N}$,*

$$\left(1-\frac{1}{2n+11/6}\right)^{\ln n} \le 1-\frac{\ln n}{3n-1/4}. \tag{2.24}$$

Proof. For $n=1$, inequality (2.24) holds clearly. For $n=2$,

$$1-\frac{\ln 2}{6-1/4}-\left(1-\frac{1}{4+11/6}\right)^{\ln 2} = 0.0016626\ldots > 0, \tag{2.25}$$

inequality (2.24) holds also.

For $n \geq 3$, by using (2.15) and (2.16), it is shown that

$$\left(1-\frac{1}{2n+11/6}\right)^{\ln n} < e^{-\ln n/(2n+11/6)} < 1-\frac{\ln n}{2n+11/6}+\frac{\ln^2 n}{2(2n+11/6)^2}. \tag{2.26}$$

So, it is sufficient to prove that

$$1-\frac{\ln n}{2n+11/6}+\frac{\ln^2 n}{2(2n+11/6)^2} < 1-\frac{\ln n}{3n-1/4} \tag{2.27}$$

for $n \geq 3$. For this purpose, let $m = n - 1/12$, then inequality (2.27) can be rearranged as

$$g(m) \triangleq \frac{4m}{3}-\frac{4}{3}-\frac{8}{3m}-\ln\left(m+\frac{1}{12}\right) > 0. \tag{2.28}$$

Differentiation of $g(x)$ gives

$$g'(x) = \frac{4}{3}+\frac{8}{3x^2}-\frac{1}{x+1/12} > 0. \tag{2.29}$$

This means that $g(m)$ is increasing. Further, since

$$g\left(3-\frac{1}{12}\right) = \frac{4}{3}\left(3-\frac{1}{12}\right)-\frac{4}{3}-\frac{8}{3(3-1/12)}-\ln 3 = 0.5426575526\ldots > 0, \tag{2.30}$$

then $g(m)$ is positive for $m \geq 3$. The proof of Lemma 2.6 is complete. □

3. Proofs of theorems

Proof of Theorem 1.1. Setting $c_k > 0$ for $1 \leq k \leq n$ and letting

$$\left[\prod_{k=1}^{n} c_k^{1/\sqrt{k(k+\lambda)}}\right]^{-1/S_n(\lambda)} = \frac{1}{\sqrt{(n+1)(n+1+\lambda)}S_{n+1}(\lambda)}, \tag{3.1}$$

then

$$c_k = \frac{\left[\sqrt{(k+1)(k+1+\lambda)}S_{k+1}(\lambda)\right]^{\sqrt{k(k+\lambda)}S_k(\lambda)}}{\left[\sqrt{k(k+\lambda)}S_k(\lambda)\right]^{\sqrt{k(k+\lambda)}S_{k-1}(\lambda)}}. \tag{3.2}$$

Using the discrete weighted arithmetic-geometric mean inequality and (3.2) and interchanging the order of summation yield

$$
\begin{aligned}
&\sum_{n=1}^{\infty}\left(\prod_{k=1}^{n} a_k^{1/\sqrt{k(k+\lambda)}}\right)^{1/S_n(\lambda)} \\
&\quad=\sum_{n=1}^{\infty}\left(\prod_{k=1}^{n}\left(c_k a_k\right)^{1/\sqrt{k(k+\lambda)}}\right)^{1/S_n(\lambda)}\left(\prod_{k=1}^{n} c_k^{1/\sqrt{k(k+\lambda)}}\right)^{-1/S_n(\lambda)} \\
&\quad\leq\sum_{n=1}^{\infty}\sum_{k=1}^{n}\frac{1}{\sqrt{k(k+\lambda)}S_n(\lambda)}c_k a_k\frac{1}{\sqrt{(n+1)(n+1+\lambda)}S_{n+1}(\lambda)} \\
&\quad=\sum_{k=1}^{\infty}\frac{1}{\sqrt{k(k+\lambda)}}c_k a_k\sum_{n=k}^{\infty}\frac{1}{\sqrt{(n+1)(n+1+\lambda)}S_n(\lambda)S_{n+1}(\lambda)} \\
&\quad=\sum_{k=1}^{\infty}\frac{1}{\sqrt{k(k+\lambda)}}c_k a_k\sum_{n=k}^{\infty}\left[\frac{1}{S_n(\lambda)}-\frac{1}{S_{n+1}(\lambda)}\right] \\
&\quad=\sum_{k=1}^{\infty}\frac{1}{\sqrt{k(k+\lambda)}}c_k a_k\frac{1}{S_k(\lambda)} \\
&\quad=\sum_{k=1}^{\infty}\left[\frac{\sqrt{(k+1)(k+1+\lambda)}S_{k+1}(\lambda)}{\sqrt{k(k+\lambda)}S_k(\lambda)}\right]^{\sqrt{k(k+\lambda)}S_k(\lambda)}a_k.
\end{aligned}
\tag{3.3}
$$

Applying inequality (2.18) in the final line of (3.3) gives inequality (1.7). The proof of Theorem 1.1 is complete. □

Proof of Theorem 1.2. It is easy to see that

$$
B_1(0)=3<e^{1+\gamma},\qquad B_2(0)=\left(\frac{11}{6}\right)^3<e^{1+\gamma}\cdot 2\left(1-\frac{\ln 2}{6-1/4}\right). \tag{3.4}
$$

For $n\geq 3$, inequality

$$
e^{1/2n}\left(1-\frac{1}{2n+11/6}\right)^{1+\gamma}<e^{1/2n}e^{-(1+\gamma)/(2n+11/6)}<1 \tag{3.5}
$$

follows from using an inequality

$$
\left(1+\frac{1}{x}\right)^x<e\left(1-\frac{1}{2x+11/6}\right) \tag{3.6}
$$

in [3]. By (2.3), (3.6), Lemma 2.6, and inequality (3.5),

$$
\begin{aligned}
B_n(0) &= \left[\frac{(n+1)S_n+1}{nS_n}\right]^{nS_n} = \left\{\left[1+\frac{1}{nS_n/(S_n+1)}\right]^{nS_n/(S_n+1)}\right\}^{S_n+1} \\
&< \left\{e\left[1-\frac{1}{2nS_n/(S_n+1)+11/6}\right]\right\}^{S_n+1} \\
&< \left\{e\left[1-\frac{1}{2nS_n/(S_n+1)+11/6}\right]\right\}^{1+\ln n+\gamma+1/2n} \\
&< \left[e\left(1-\frac{1}{2n+11/6}\right)\right]^{1+\ln n+\gamma+1/2n} \\
&< e^{1+\gamma}ne^{1/2n}\left(1-\frac{1}{2n+11/6}\right)^{1+\ln n+\gamma} \\
&< e^{1+\gamma}n\left(1-\frac{\ln n}{3n-1/4}\right).
\end{aligned}
\tag{3.7}
$$

Taking $\lambda = 0$ in inequality (3.3) yields

$$
\begin{aligned}
\sum_{n=1}^{\infty}\left(\prod_{k=1}^{n} a_k^{1/k}\right)^{1/S_n} &\le \sum_{n=1}^{\infty}\left[\frac{(n+1)S_{n+1}}{nS_n}\right]^{nS_n} a_n \\
&= \sum_{n=1}^{\infty} B_n(0)a_n < e^{1+\gamma}\sum_{n=1}^{\infty} n\left(1-\frac{\ln n}{3n-1/4}\right)a_n.
\end{aligned}
\tag{3.8}
$$

The proof of Theorem 1.2 is complete. □

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