MATRIX TRANSFORMATIONS BETWEEN THE SPACES OF CESÀRO SEQUENCES AND INVARIANT MEANS

M. MURSALEEN, E. SAVAŞ, M. AIYUB, AND S. A. MOHIUDDINE

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The main purpose of this paper is to characterize the classes of matrices $(\operatorname{ces}[(p),(q)],c^\sigma)$ and $(\operatorname{ces}[(p),(q)],l_\infty^\sigma)$, where c^σ is the space of all bounded sequences all of whose σ-means are equal, l_∞^σ is the space of σ- bounded sequences, and $\operatorname{ces}[(p),(q)]$ is the generalized Cesàro sequence space.

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1. Introduction

Let ω be the space of all sequences, real or complex, and let l_∞ and c, respectively, be the Banach spaces of bounded and convergent sequences $x=(x_n)$ with norm $\|x\| = \sup_{k\geq 0}|x_k|$. Let σ be a mapping of the set of positive integers into itself. A continuous linear functional ϕ on l_∞ is said to be an invariant mean or a σ- mean if and only if (i) $\phi(x)\geq 0$, when the sequence $x=(x_n)$ has $x_n \geq 0$ for each n; (ii) $\phi(e)=1$, where $e=(1,1,1,\dots)$; and (iii) $\phi((x_{\sigma(n)}))=\phi(x)$, $x\in l_\infty$.

For certain kinds of mappings, every σ- mean extends the limit functional ϕ on c in the sense that $\phi(x)=\lim x$ for $x\in c$ (see [2, 15]). Consequently, $c\subset c^\sigma$, where c^σ is the set of bounded sequences, all of whose invariant means are equal (see [1, 9, 10]). When σ is translation, the σ- means are classical Banach limits on l_∞ (see [2]) and c^σ is the set of almost convergent sequences $\hat{c}$ (see [7]). Almost convergence for double sequences was introduced and studied by Móricz and Rhoades [8] and further by Mursaleen and Savaş [13], Mursaleen and Edely [12], and Mursaleen [11].

If $x=(x_n)$, write $Tx=(Tx_n)=(x_{\sigma(n)})$, then

$$c^\sigma = \Big\{x\in l_\infty : \lim_{m\to\infty} t_{m,n}(x)=L, \text{ uniformly in } n,\ L=\sigma-\lim x\Big\}, \tag{1.1}$$

where

$$t_{m,n}(x)=\frac{1}{m+1}\sum_{j=0}^{m} T^j x_n \quad \text{with } T^j x_n = x_{\sigma^j(n)},\ t_{-1,n}(x)=0. \tag{1.2}$$

We define l_∞^σ the space of σ- bounded sequences (Ahmad et al. [2]) in the following way.

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Let $x_n = z_0 + z_1 + z_2 + \cdots + z_n$ and

$$l_\infty^\sigma = \left\{ z \in \omega : \sup_{m,n} \left| \psi_{m,n}(z) \right| < \infty \right\}, \tag{1.3}$$

where

$$\begin{aligned} \psi_{m,n}(z) &= t_{m,n}(x) - t_{m-1,n}(x) \\ &= \frac{1}{m(m+1)} \sum_{j=1}^{m} j \sum_{i=h_{j-1}+1}^{h_j} z_i, \quad h_j = \sigma^j(n). \end{aligned} \tag{1.4}$$

If $\sigma(n) = (n+1)$, then l_∞^σ is the set of almost bounded sequences $\widehat{l_\infty}$ (see [14]).

Let $A = (a_{nk})$ be an infinite matrix of complex numbers a_{nk} $(n,k = 1,2,\dots)$ and X, Y two subsets of ω. We say that the matrix A defines a matrix transformation from X into Y if for every sequence $x = (x_k) \in X$ the sequence $A(x) = (A_n(x)) \in Y$, where $A_n(x) = \sum_k a_{nk} x_k$ converges for each n. We denote the class of matrix transformations from X into Y by (X,Y).

The main purpose of this paper is to characterize the classes $(\text{ces}[(p),(q)], c^\sigma)$ and $(\text{ces}[(p),(q)], l_\infty^\sigma)$ and deduce some known and unknown interesting results as corollaries.

The classes $(\text{ces}[(p),(q)], c^\sigma)$ and $(\text{ces}[(p),(q)], l_\infty^\sigma)$ are due to Khan and Rahman [4].

If $\{q_n\}$ is a sequence of positive real numbers, then for $p = (p_r)$ with $\inf p_r > 0$, we define the space $\text{ces}[(p),(q)]$ by

$$\text{ces}[(p),(q)] = \left\{ x \in \omega : \sum_{r=0}^{\infty} \left(\frac{1}{Q_{2^r}} \sum_r q_k |x_k| \right)^{p_r} < \infty \right\}, \tag{1.5}$$

where $Q_{2^r} = q_{2^r} + q_{2^r+1} + \cdots + q_{2^{r+1}-1}$ and $\sum_r$ denotes a sum over the range $2^r \le k < 2^{r+1}$.

Remark 1.1. If $q_n = 1$ for all n, then $\text{ces}[(p),(q)]$ reduces to $\text{ces}(p)$ studied by Lim [6]. Also, if $p_n = p$ for all n and $q_n = 1$ for all n, then $\text{ces}[(p),(q)]$ reduces to ces_p studied by Lim [5].

For any bounded sequence p, the space $\text{ces}[(p),(q)]$ is a paranormed space with the paranorm given by (see [4])

$$g(x) = \left(\sum_{r=0}^{\infty} \left(\frac{1}{Q_{2^r}} \sum_r q_k |x_k| \right)^{p_r} \right)^{1/M} \tag{1.6}$$

if $H = \sup_r p_r < \infty$ and $M = \max(1,H)$.

2. Sequence-to-sequence transformations

In this section, we characterize the classes $(\text{ces}[(p),(q)],c^\sigma)$ and $(\text{ces}[(p),(q)],l_\infty^\sigma)$.

We write $a(n,k)$ to denote the elements a_{nk} of the matrix A, and for all integers $n,m\geq 1$, we write

$$\begin{aligned} t_{mn}(Ax) &= \frac{Ax_n+TAx_n+\cdots+T^mAx_n}{m+1} \\ &= \sum_k t(n,k,m)x_k, \end{aligned} \tag{2.1}$$

where $t(n,k,m)=1/(m+1)\sum_{j=0}^m a(\sigma^j(n),k)$.

We also define the spaces of σ- convergent series and σ- bounded series, respectively, as follows:

$$\begin{aligned} c_s^\sigma &= \left\{x:\sum_{i=1}^m\left(\frac{1}{i+1}\sum_{j=0}^i x_{\sigma^j(n)}\right)\text{ is convergent uniformly in } n,\text{ as } m\longrightarrow\infty\right\}, \\ b_s^\sigma &= \left\{x:\sup_{n,m}\sum_{i=1}^m\left(\frac{1}{i+1}\sum_{j=0}^i x_{\sigma^j(n)}\right)<\infty\right\}. \end{aligned} \tag{2.2}$$

If we take $\sigma(n)=n+1$, c_s^σ and b_s^σ reduce to $\widehat{c}_s$ and $\widehat{b}_s$, as defined below:

$$\begin{aligned} \widehat{c}_s &= \left\{x:\sum_{i=1}^m\left(\frac{1}{i+1}\sum_{j=0}^i x_{j+n}\right)\text{ is convergent uniformly in } n,\text{ as } m\longrightarrow\infty\right\}, \\ \widehat{b}_s &= \left\{x:\sup_{n,m}\sum_{i=1}^m\left(\frac{1}{i+1}\sum_{j=0}^i x_{j+n}\right)<\infty\right\}. \end{aligned} \tag{2.3}$$

Now we prove the following theorem.

Theorem 2.1. *Let $1<p_r\leq\sup_r p_r<\infty$. Then $A\in(\text{ces}[(p),(q)],c^\sigma)$ if and only if*

(i) *there exists an integer $E>1$ such that for all n,*

$$U(E)=\sup_m\sum_{r=0}^\infty\left(Q_{2^r}\max_r\left(\frac{|t(n,k,m)|}{q_k}\right)\right)^{t_r}E^{-t_r}<\infty, \tag{2.4}$$

where $1/p_r+1/t_r=1$, $r=0,1,2,\ldots$, and $\max_r$ means maximum over $2^r\leq k<2^{r+1}$;

(ii) *$a_{(k)}=(a_{nk})_{n=1}^\infty\in c^\sigma$ for each k, that is, $\lim_m t(n,k,m)=u_k$ uniformly in n, for each k.*

In this case, σ-limit of Ax is $\sum_{k=1}^\infty u_kx_k$.

Proof

Neccessity. Suppose that $A\in(\text{ces}[(p),(q)],c^\sigma)$. Now $\sum_{k=1}^\infty t(n,k,m)x_k$ exists for each m and n and $x\in\text{ces}[(p),(q)]$, whence $\{t(n,k,m)\}_k\in\text{ces}^*[(p),(q)]$ for each m and n, (see F. M. Khan and M. A. Khan [3] for Köthe-Toeplitz and continuous duals of $\text{ces}[(p),(q)]$).

Therefore, it follows that each $\{f_{m,n}\}_m$ defined by

$$f_{m,n}(x) = t_{m,n}(Ax) \tag{2.5}$$

is an element of ces*$[(p),(q)]$. Since ces$[(p),(q)]$ is complete and further for each n, $\sup_m |t_{m,n}(Ax)| < \infty$ on ces$[(p),(q)]$. Now arguing with the uniform boundedness principle, we have condition (i). Since $e_k \in$ ces$[(p),(q)]$, condition (ii) follows.

Sufficiency. Suppose that the conditions hold. Fix $n \in \mathbb{N}$. For every integer $s \geq 1$, from (i) we have

$$\sum_{r=0}^{s}\Big(Q_{2^r}\max_r\big(q_k^{-1}|t(n,k,m)|\big)\Big)^{t_r}E^{-t_r} \leq \sup_m\sum_{r=0}^{\infty}\Big(Q_{2^r}\max_r\big(q_k^{-1}|t(n,k,m)|\big)\Big)^{t_r}E^{-t_r}. \tag{2.6}$$

Now letting $s \to \infty$, we obtain

$$lim_{m\to\infty}\sum_{r=0}^{\infty}\Big(Q_{2^r}\max_r\big(q_k^{-1}|t(n,k,m)|\big)\Big)^{t_r}E^{-t_r} \leq \sup_m\sum_{r=0}^{\infty}\Big(Q_{2^r}\max_r\big(q_k^{-1}|t(n,k,m)|\big)\Big)^{t_r}E^{-t_r}. \tag{2.7}$$

Therefore, from (ii) we have

$$\sum_{r=0}^{\infty}\Big(Q_{2^r}\max_r\big(q_k^{-1}|u_k|\big)\Big)^{t_r}E^{-t_r} \leq \sup_m\sum_{r=0}^{\infty}\Big(Q_{2^r}\max_r\big(q_k^{-1}|t(n,k,m)|\big)\Big)^{t_r}E^{-t_r} < \infty. \tag{2.8}$$

Hence $(u_k)_k$ and $\{t(n,k,m)\}_k \in$ ces*$[(p),(q)]$, therefore the series $\sum_{k=1}^{\infty} t(n,k,m)x_k$ and $\sum_{k=1}^{\infty} u_k x_k$ converge for each m and n and $x \in$ ces$[(p),(q)]$. For given $\epsilon > 0$ and $x \in$ ces$[(p),(q)]$, choose s such that

$$\left(\sum_{r=s+1}^{\infty}\left(\frac{1}{Q_{2^r}}\sum_r q_k|x_k|\right)^{p_r}\right)^{1/M} < \epsilon. \tag{2.9}$$

Since (ii) holds, there exists m_0 such that

$$\left|\sum_{k=1}^{s} t(n,k,m) - u_k\right| < \epsilon \quad \forall m > m_0. \tag{2.10}$$

Since (i) holds, it follows that

$$\left|\sum_{k=s+1}^{\infty} t(n,k,m) - u_k\right| \text{ is arbitrary small.} \tag{2.11}$$

Therefore,

$$\lim_{m} \sum_{k=1}^{\infty} t(n,k,m)x_k = \sum_{k=1}^{\infty} u_k x_k, \quad \text{uniformly in } n. \tag{2.12}$$

This completes the proof. □

Remark 2.1. For different choices of p, q, and σ, we can deduce many corollaries from the above theorem to characterize the matrix classes, for example, $(\text{ces}(p), c^{\sigma})$, $(\text{ces}_p, c^{\sigma})$, $(\text{ces}_p(q), c^{\sigma})$, $(\text{ces}[(p),(q)], \hat{c})$, and so forth. The class $(\text{ces}(p), \hat{c})$ was characterized by F. M. Khan and M. A. Khan [3] which we can obtain directly from our theorem by taking $q_n = 1$ for all n and $\sigma(n) = n+1$.

We write (see [2])

$$\begin{aligned} x_0 &= z_0 + z_1 + \cdots + z_n, \\ \psi_{m,n}(Az) &= \sum_k \alpha(n,k,m) z_k, \end{aligned} \tag{2.13}$$

where

$$\alpha(n,k,m) = \frac{1}{m(m+1)} \sum_{j=1}^{m} j \left[\sum_{i=h_{j-1}+1}^{h_j} a_{ik} \right], \quad h_j = \sigma^j(n). \tag{2.14}$$

Now we prove the following theorem.

Theorem 2.2. *Let $1 < p_r \leq \sup_r p_r < \infty$. Then $A \in (\text{ces}[(p),(q)], l_\infty^{\sigma})$ if and only if*

$$\sup_{m,n} \sum_{r=0}^{\infty} \left(Q_{2^r} \max_r \left(q_k^{-1} \left| \alpha(n,k,m) \right| \right) \right)^{t_r} E^{-t_r} < \infty, \tag{2.15}$$

where E is an integer greater than 1 and $1/p_r + 1/t_r = 1$, $r = 0,1,2,\ldots$.

Proof

Necessity. Suppose that $A \in (\text{ces}[(p),(q)], l_\infty^{\sigma})$. Now $\sum_{k=1}^{\infty} \alpha(n,k,m) z_k$ exists for each m and n and $z \in \text{ces}[(p),(q)]$, whence $\{\alpha(n,k,m)\}_k \in \text{ces}^*[(p),(q)]$ for each m and n. Therefore, it follows that $\{f_{m,n}\}$ defined by

$$f_{m,n}(x) = \psi_{m,n}(Az) \tag{2.16}$$

is an element of $\text{ces}^*[(p),(q)]$. Since $\text{ces}[(p),(q)]$ is complete and further $\sup_{m,n} |\psi_{m,n}(Az)| < \infty$ on $\text{ces}[(p),(q)]$, so by arguing with uniform boundedness principle, we have the condition.

Sufficiency. Suppose that condition (2.15) holds. Fix $n \in \mathbb{N}$. For every integer $s \geq 1$ we have

$$\sum_{r=0}^{s} \Big(Q_{2^r} \max_r \big(q_k^{-1} \,|\, \alpha(n,k,m) \,|\, \big) \Big)^{t_r} E^{-t_r} \leq \sup_{m,n} \sum_{r=0}^{\infty} \Big(Q_{2^r} \max_r \big(q_k^{-1} \,|\, \alpha(n,k,m) \,|\, \big) \Big)^{t_r} E^{-t_r}. \tag{2.17}$$

So

$$\begin{aligned} &\lim_{s\to\infty} \sum_{r=0}^{s} \Big(Q_{2^r} \max_r \big(q_k^{-1} \,|\, \alpha(n,k,m) \,|\, \big) \Big)^{t_r} E^{-t_r} \\ &\quad \leq \sup_{m,n} \sum_{r=0}^{\infty} \Big(Q_{2^r} \max_r \big(q_k^{-1} \,|\, \alpha(n,k,m) \,|\, \big) \Big)^{t_r} E^{-t_r} < \infty. \end{aligned} \tag{2.18}$$

Hence $\{\alpha(n,k,m)\} \in \text{ces}^*[(p),(q)]$. Therefore, the series $\sum_{k=1}^{\infty} \alpha(n,k,m) z_k$ converges for each m and n and $z \in \text{ces}[(p),(q)]$.

This completes the proof. □

Remark 2.2. The matrix class $(\text{ces}(p), \widehat{l_\infty})$, was characterized by F. M. Khan and M. A. Khan [3] which we can obtain directly from the above theorem by letting $q_n = 1$ for all n and $\sigma(n) = n+1$. Besides, we can further deduce many corollaries for different choices of p, q, and σ.

3. Sequence-to-series transformations

For all integers $m, n \geq 1$, we write

$$t^*_{mn}(Ax) = \sum_{i=1}^{m} t_{in}(Ax) = \sum_k \sum_{i=1}^{m} \frac{1}{i+1} \sum_{j=0}^{i} a(\sigma^j(n),k) x_k = \sum_k t^*(m,n,k) x_k, \tag{3.1}$$

where

$$t^*(m,n,k) = \sum_{i=1}^{m} \frac{1}{i+1} \sum_{j=0}^{i} a(\sigma^j(n),k). \tag{3.2}$$

Theorem 3.1. *Let $1 < p_r \leq \sup_r p_r < \infty$. Then $A \in (\text{ces}[(p),(q)], c_s^\sigma)$ if and only if*

(i) *there exists an integer $E > 1$ such that for all n,*

$$U(E) = \sup_m \sum_{r=0}^{\infty} \left(Q_{2^r} \max_r \left(\frac{|\, t^*(n,k,m) \,|}{q_k} \right) \right)^{t_r} E^{-t_r} < \infty, \tag{3.3}$$

where $1/p_r + 1/t_r = 1$, $r = 0,1,2,\ldots$, and $\max_r$ means maximum over $2^r \leq k \leq 2^{r+1}$;

(ii) $a_{(k)} = \{a_{nk}\}_{n=1}^{\infty} \in c_s^{\sigma}$ *for each* k*, that is,* $\lim_m t^*(n,k,m) = u_k$ *uniformly in* n*, for each* k*.*

In this case, the σ*-limit of* Ax *is* $\sum_{k=1}^{\infty} u_k x_k$*.*

Theorem 3.2. *Let* $1 < p_r \leq \sup_r p_r < \infty$*. Then* $A \in (\mathrm{ces}[(p),(q)], b_{\infty}^{\sigma})$ *if and only if*

$$\sup_{m,n} \sum_{r=0}^{\infty} \Big(Q_{2^r} \max_r \big(q_k^{-1} \,|\, t^*(n,k,m) \,|\, \big) \Big)^{t_r} E^{-t_r} < \infty, \tag{3.4}$$

where E *is an integer greater than* 1 *and* $1/p_r + 1/t_r = 1$*,* $r = 0,1,2,\ldots$*.*

Proofs of Theorems 3.1 and 3.2 are similar to those of Theorems 2.1 and 2.2, respectively.

Remark 3.1. If σ is translation, then Theorems 3.1 and 3.2 give the characterization for the classes $(\mathrm{ces}[(p),(q)], \widehat{c}_s)$ and $(\mathrm{ces}[(p),(q)], \widehat{b}_s)$. As Remarks 2.1 and 2.2, for different choices of p, q, and σ, we can deduce many corollaries.

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M. Mursaleen: Department of Mathematics, Faculty of Science, King Abdul-Aziz University, P.O. Box 80203, Jeddah, The Kingdom of Saudi Arabia
E-mail address: mursaleen@math.com

E. Savaş: Department of Mathematics, Y. Y. University, 65080 Van, Turkey
E-mail address: ekremsavas@yahoo.com

M. Aiyub: Department of Mathematics, Aligarh Muslim University (AMU), Aligarh 202002, India
E-mail address: mohdaiyub@rediffmail.com

S. A. Mohiuddine: Department of Mathematics, Aligarh Muslim University (AMU), Aligarh 202002, India
E-mail address: mohiuddine@math.com